ASSOCIATED LEGENDRE FUNCTIONS AND HEAT PRODUCTION IN A CYLINDER

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In this paper we have employed associated Legendre functions to solve the fundamental differential equation of the heat production in a cylinder.

§ 1. In this paper we have considered the diffusion of heat in a cylinder of radius a when there are sources of heat within it which lead to an axially symmetrical temperature distribution. The fundamental differential equation is then of the form (Sneddon 1951)

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \theta(r, t) \qquad \ldots \quad \ldots \quad \ldots \quad (1.1)$$

if we assume that the rate of generation of heat is independent of temperature and that the cylinder is infinitely long, so that the variation of z may be neglected. We shall in addition suppose that the surface $r = a$ is maintained at zero temperature and the initial distribution of temperature is also zero.

In particular we suppose that

$$\theta(r, t) = \frac{k}{K} f(r) g(t) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.2)$$

where k is the diffusivity and K the conductivity of the material.

In this paper we have considered six sets of values for $f(r)$ and $g(t)$. It will be observed that the single function $f(r)$ can represent both sources and sinks embedded in the system. Whenever the product $f(r)g(t)$ gives a negative value, it should be treated as a sink.

In section 2, we have established two integrals involving Meijer's G-function and associated Legendre function which are required in the proofs of proceeding sections.

Cases, in which heat is produced in solids, are becoming increasingly important in technical applications (Carslaw and Jaeger 1959). Space research and nuclear reactors also give rise to different problems of heat transfer.

§ 2. The integrals to be established are

$$\int_0^1 x^{\lambda-1}(1-x^2)^{\mu/2}P_\nu^\mu(x)G_{p,q}^{m,n}\left[zx^{2\delta}\middle|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right]dx$$

$$=\frac{(2\delta)^{-\mu-1}(-1)^\mu\Gamma(1+\mu+\nu)}{\Gamma(1-\mu+\nu)}G_{p+2\delta,q+2\delta}^{m,n+2\delta}\left[z\middle|\begin{matrix}\Delta(2\delta,1-\lambda), & a_1,\ldots,a_p\\ b_1,\ldots,b_q, & \Delta\left(\delta,\frac{1-\lambda-\mu+\nu}{2}\right), \quad \Delta\left(\delta,-\frac{\lambda+\mu+\nu}{2}\right)\end{matrix}\right] \quad \ldots (2.1)$$

where

$$2(m+n)>p+q,\ |\arg z|<(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,\ \mathrm{Re}\ (\lambda+2\delta b_j)>0,$$
$$j=1,2,\ldots,m;\ \mu=0,1,2,\ldots$$

Another set of conditions of validity: $p<q$ (or $p\leqslant q$ and $|z|<1$), Re $(\lambda+2\delta b_j)>0$, $j=1,2,\ldots,m$; $\mu=0,1,2,\ldots$

$$\int_0^1 x^{\lambda-1}(1-x^2)^{-\mu/2}P_\nu^\mu(x)G_{p,q}^{m,n}\left[zx^{2\delta}\middle|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right]dx$$

$$=(2\delta)^{-\mu-1}G_{p+2\delta,q+2\delta}^{m,n+2\delta}\left[z\middle|\begin{matrix}\Delta(2\delta,1-\lambda), & a_1,\ldots,a_p\\ b_1,\ldots,b_q, & \Delta\left(\delta,\frac{1-\lambda+\mu+\nu}{2}\right), \quad \Delta\left(\delta,-\frac{\lambda-\mu+\nu}{2}\right)\end{matrix}\right]. \quad \ldots (2.2)$$

where

$$2(m+n)>p+q,\ |\arg z|<(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,\ \mathrm{Re}(\lambda+2\delta b_j)>0,$$
$$j=1,2,\ldots,m;\ \mu<1.$$

Another set of conditions of validity: $p<q$ (or $p\leqslant q$ and $|z|<1$), Re $(\lambda+2\delta b_j)>0$, $j=1,2,\ldots,m$; $\mu<1$.

In (2.1) and (2.2) δ is a positive integer and the symbol $\Delta(\delta,\alpha)$ represents the set of parameters $\frac{\alpha}{\delta},\frac{\alpha+1}{\delta},\ldots,\frac{\alpha+\delta-1}{\delta}$.

PROOF: To establish (2.1), expressing the G-function as Mellin-Barnes type integral (Erdelyi 1953, p. 207, eqn. 1) and interchanging the order of integrations which is justifiable due to the absolute convergence of the integrals involved in the process, we have

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m\Gamma(b_j-s)\prod_{j=1}^n\Gamma(1-a_j+s)}{\prod_{j=m+1}^q\Gamma(1-b_j+s)\prod_{j=n+1}^p\Gamma(a_j-s)}z^s\int_0^1 x^{\lambda+2s\delta-1}(1-x^2)^{\mu/2}P_\nu^\mu(x)dx\,ds.$$

Evaluating the inner integral with the help of formula (4) of (Erdelyi 1954, p. 313) and using multiplication formula (11) for gamma-function (Erdelyi 1953, p. 4), we have

$$\frac{(2\delta)^{-\mu-1}(-1)^{\mu}\Gamma(1+\mu+\nu)}{\Gamma(1-\mu+\nu)}\cdot\frac{1}{2\pi i}$$

$$\times\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)\prod_{i=0}^{2\delta-1}\Gamma\left(1-\frac{1-\lambda+i}{2\delta}+s\right)z^s}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)\prod_{i=0}^{\delta-1}\Gamma\left(1-\frac{\frac{1-\lambda-\mu+\nu}{2}+i}{\delta}+s\right)\prod_{i=0}^{\delta-1}\Gamma\left(1-\frac{-\frac{\lambda+\mu+\nu}{2}+i}{\delta}+s\right)}\,ds.$$

Now on applying eqn. (1) of (Erdelyi 1953, p. 207) the integral (2.1) is established.

Proceeding in the same way and using eqn. (4) (Erdelyi 1954, p. 314) the integral (2.2) is obtained.

§ 3. *Finite Hankel transforms.*—Let the finite Hankel transform of $f(r)$ be (Sneddon 1951, p. 83)

$$J[f(r)]=\int_0^a rf(r)J_0(r\xi_i)\,dr=f_J(\xi_i) \qquad \ldots \quad \ldots \quad (3.1)$$

then in (2.1) and (2.2), putting $m=1$, $n=p=0$, $q=2$, $b_1=b_2=\sigma$, $\delta=1$, $z=\frac{\xi_i^2a^2}{4}$, $x=\frac{r}{a}$ and using eqn. (3) (Erdelyi 1954, p. 434) and eqn. (3) (Erdelyi 1954, p. 439) we obtain

$$J\left[r^{2\sigma+\lambda-2}\{a^2-r^2\}^{\mu/2}P_\nu^\mu\left(\frac{r}{a}\right)\right]$$

$$=\frac{a^{2\sigma+\lambda+\mu}(-1)^{\mu}\Gamma(1+\mu+\nu)}{\Gamma(1-\mu+\nu)}\cdot\frac{\Gamma\left(\frac{1+2\sigma+\lambda}{2}\right)\Gamma\left(\frac{2\sigma+\lambda}{2}\right)}{\Gamma\left(\frac{1+2\sigma+\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{2+2\sigma+\lambda+\mu+\nu}{2}\right)}$$

$$\times{}_2F_3\left[\begin{matrix}(1+2\sigma+\lambda)/2,\ (2\sigma+\lambda)/2\\ 1,\ (1+2\sigma+\lambda+\mu-\nu)/2,\ (2+2\sigma+\lambda+\mu+\nu)/2\end{matrix};\ -\frac{\xi_i^2a^2}{4}\right] \qquad \ldots \quad (3.2)$$

where

$$\text{Re}\,(\lambda+2\sigma)>0,\ \mu=0,1,2,\ldots,$$

and

$$J\left[r^{2\sigma+\lambda-2}(a^2-r^2)^{-\mu/2}P_\nu^\mu\left(\frac{r}{a}\right)\right]$$

$$=a^{2\sigma+\lambda-\mu}\frac{\Gamma\left(\frac{1+2\sigma+\lambda}{2}\right)\Gamma\left(\frac{2\sigma+\lambda}{2}\right)}{\Gamma\left(\frac{1+2\sigma+\lambda-\mu-\nu}{2}\right)\Gamma\left(\frac{2+2\sigma+\lambda-\mu+\nu}{2}\right)}$$

$$\times{}_2F_3\left[\begin{matrix}(1+2\sigma+\lambda)/2,\ (2\sigma+\lambda)/2\\ 1,\ (1+2\sigma+\lambda-\mu-\nu)/2,\ (2+2\sigma+\lambda-\mu+\nu)/2\end{matrix};\ -\frac{\xi_i^2a^2}{4}\right]$$

$$\ldots \quad (3.3)$$

where

$$\text{Re}\,(\lambda+2\sigma) > 0,\ \mu < 1,$$

and ξ_i is the root of the transcendental equation,

$$J(a\xi_i) = 0. \qquad \text{(3.4)}$$

By virtue of the inversion theorem (Sneddon 1951, p. 83), we have

$$r^{2\sigma+\lambda-2}\{a^2-r^2\}^{\mu/2}P_\nu^\mu\left(\frac{r}{a}\right)$$

$$= 2a^{2\sigma+\lambda+\mu-2}(-1)^\mu \frac{\Gamma(1+\mu+\nu)\,\Gamma\left(\frac{1+2\sigma+\lambda}{2}\right)\Gamma\left(\frac{2\sigma+\lambda}{2}\right)}{\Gamma(1-\mu+\nu)\,\Gamma\left(\frac{1+2\sigma+\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{2+2\sigma+\lambda+\mu+\nu}{2}\right)}$$

$$\times \sum_i {}_2F_3\left[\begin{matrix}(1+2\sigma+\lambda)/2,\ (2\sigma+\lambda)/2\\ 1,\ (1+2\sigma+\lambda+\mu-\nu)/2,\ (2+2\sigma+\lambda+\mu-\nu)/2\end{matrix};\ -\frac{\xi_i^2a^2}{4}\right]\frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2} \qquad \text{(3.5)}$$

and

$$r^{2\sigma+\lambda-2}\{a^2-r^2\}^{-\mu/2}P_\nu^\mu\left(\frac{r}{a}\right)$$

$$= 2a^{2\sigma+\lambda+\mu-2}\frac{\Gamma\left(\frac{1+2\sigma+\lambda}{2}\right)\Gamma\left(\frac{2\sigma+\lambda}{2}\right)}{\Gamma\left(\frac{1+2\sigma+\lambda-\mu-\nu}{2}\right)\Gamma\left(\frac{2+2\sigma+\lambda-\mu+\nu}{2}\right)}$$

$$\times \sum_i {}_2F_3\left[\begin{matrix}(1+2\sigma+\lambda)/2,\ (2\sigma+\lambda)/2\\ 1,\ (1+2\sigma+\lambda-\mu-\nu)/2,\ (2+2\sigma+\lambda-\mu+\nu)/2\end{matrix};\ -\frac{\xi_i^2a^2}{4}\right]\frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2} \qquad \text{(3.6)}$$

where the sum is taken over all the positive roots of eqn. (3.4).

The results (3.5) and (3.6) will prove useful in the verification of the solutions.

§ 4. *Solutions of the problem.*—We apply finite Hankel transforms (3.2) and (3.3) to obtain the solutions of (1.1). Its solutions obtained as (Sneddon 1951, p. 203) are

$$u(r,t) = 2a^{2\sigma+\lambda+\mu-2}(-1)^\mu \frac{\Gamma(1+\mu+\nu)\Gamma\left(\frac{1+2\sigma+\lambda}{2}\right)\Gamma\left(\frac{2\sigma+\lambda}{2}\right)}{\Gamma(1-\mu+\nu)\Gamma\left(\frac{1+2\sigma+\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{2+2\sigma+\lambda+\mu+\nu}{2}\right)}$$

$$\times\frac{k}{K}\sum_i {}_2F_3\left[\begin{matrix}(1+2\sigma+\lambda)/2,\ (2\sigma+\lambda)/2\\ 1,\ (1+2\sigma+\lambda+\mu-\nu)/2,\ (2+2\sigma+\lambda+\mu+\nu)/2\end{matrix};\ -\frac{a^2\xi_i^2}{4}\right]\frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2}h(\xi_i,t) \qquad \text{(4.1)}$$

and

$$u(r,t) = 2a^{2\sigma+\lambda-\mu-2} \frac{\Gamma\left(\frac{1+2\sigma+\lambda}{2}\right)\Gamma\left(\frac{2\sigma+\lambda}{2}\right)}{\Gamma\left(\frac{1+2\sigma+\lambda-\mu-\nu}{2}\right)\Gamma\left(\frac{2+2\sigma+\lambda-\mu+\nu}{2}\right)} \cdot \frac{k}{K}$$

$$\times \sum_i {}_2F_3\left[\begin{matrix}(1+2\sigma+\lambda)/2,\ (2\sigma+\lambda)/2 \\ 1,\ (1+2\sigma+\lambda-\mu-\nu)/2,\ (2+2\sigma+\lambda-\mu+\nu)/2\end{matrix};\ -\frac{a^2\xi_i^2}{4}\right] \frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2} h(\xi_i, t) \quad \text{.. (4.2)}$$

where the sum is taken over all the positive roots of the equation

$$J_0(a\xi_i) = 0 \quad \text{.. .. .. .. (4.3)}$$

and

$$h(\xi_i, t) = \int_0^t g(\tau) e^{-k\xi_i^2(t-\tau)}\, d\tau. \quad \text{.. .. .. (4.4)}$$

We shall evaluate (4.4) for some heat sources of important character.

§ 5. *Verification of the solutions.*—From (4.1), and (5.2.4) and (5.2.5) of Lebedev (1965, p. 100), we have

$$\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = -\frac{2a^{2\sigma+\lambda+\mu-2}(-1)^\mu \Gamma(1+\mu+\nu)\Gamma\left(\frac{1+2\sigma+\lambda}{2}\right)\Gamma\left(\frac{2\sigma+\lambda}{2}\right)}{\Gamma(1-\mu+\nu)\Gamma\left(\frac{1+2\sigma+\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{2+2\sigma+\lambda+\mu+\nu}{2}\right)}$$

$$\times \frac{k^2}{K} \sum_i {}_2F_3\left[\begin{matrix}(1+2\sigma+\lambda)/2,\ (2\sigma+\lambda)/2 \\ 1,\ (1+2\sigma+\lambda+\mu-\nu)/2,\ (2+2\sigma+\lambda+\mu+\nu)/2\end{matrix};\ -\frac{\xi_i^2 a^2}{4}\right] \frac{\xi_i^2 J_0(r\xi_i)}{[J_1(a\xi_i)]^2}$$

$$\times \int_0^t g(\tau) e^{-k\xi_i^2(t-\tau)}\, d\tau. \quad \text{.. .. .. (5.1)}$$

From (1.2) and (3.5), we obtain

$$\theta(r,t) = \frac{2k}{K} a^{2\sigma+\lambda+\mu-2}(-1)^\mu \frac{\Gamma(1+\mu+\nu)\Gamma\left(\frac{1+2\sigma+\lambda}{2}\right)\Gamma\left(\frac{2\sigma+\lambda}{2}\right)}{\Gamma(1-\mu+\nu)\Gamma\left(\frac{1+2\sigma+\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{2+2\sigma+\lambda+\mu+\nu}{2}\right)}$$

$$\times \sum_i {}_2F_3\left[\begin{matrix}(1+2\sigma+\lambda)/2,\ (2\sigma+\lambda)/2 \\ 1,\ (1+2\sigma+\lambda+\mu-\nu)/2,\ (2+2\sigma+\lambda+\mu+\nu)/2\end{matrix};\ -\frac{\xi_i^2 a^2}{4}\right] \frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2} g(t). \quad \text{.. (5.2)}$$

And from (4.1), we get

$$\frac{\partial u}{\partial t}=\frac{2a^{2\sigma+\lambda+\mu-2}(-1)^{\mu}\Gamma(1+\mu+\nu)\Gamma\left(\frac{1+2\sigma+\lambda}{2}\right)\Gamma\left(\frac{2\sigma+\lambda}{2}\right)}{\Gamma(1-\mu+\nu)\Gamma\left(\frac{1+2\sigma+\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{2+2\sigma+\lambda+\mu+\nu}{2}\right)}$$

$$\times\frac{k}{K}\sum_i {}_2F_3\left[\begin{matrix}(1+2\sigma+\lambda)/2,\ (2\sigma+\lambda)/2\\ 1,\ (1+2\sigma+\lambda+\mu-\nu)/2,\ (2+2\sigma+\lambda+\mu+\nu)/2\end{matrix};\ -\frac{a^2\xi_i^2}{4}\right]\frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2}$$

$$\times\left[g(t)-k\xi_i^2\int_0^t g(\tau)e^{-k\xi_i^2(t-\tau)}\,d\tau\right]. \quad \ldots \quad (5.3)$$

Substituting the above values in (1.1), the equation is satisfied.

The boundary condition $u(a, t) = 0$ is satisfied, because $J_0(a\xi_i)$ which is present in every term of $u(a, t)$ is zero. The initial condition is satisfied because $h(\xi_i, 0) = 0$.

We see that (4.1) converges uniformly when $t > 0$, and so the function $u(r, t)$ represented by it is continuous when $0 \leqslant r \leqslant a$.

The term-by-term differentiations are justified because (5.1) and (5.3) are uniformly convergent, when $t > 0$ and $0 \leqslant r \leqslant a$.

Similarly solution (4.2) can be verified easily.

§ 6. *Heat source of general character.*—Let

$$g(\tau) = g_0 e^{-\tau z}\tau^{\gamma-1}(t-\tau)^{\rho-1}{}_2F_1\left[\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix};\ \tau/t\right] \quad \ldots \quad (6.1)$$

then using (8) (Erdelyi 1954, p. 400), we have

$$h(\xi_i, t) = g_0\frac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\Gamma(\gamma+\rho-\beta)}e^{-zt}t^{\gamma+\rho-1}$$

$$\times {}_2F_2\left[\begin{matrix}\rho,\ \gamma+\rho-\alpha-\beta\\ \gamma+\rho-\alpha,\ \gamma+\rho-\beta\end{matrix};\ \left(z-k\xi_i^2\right)t\right] \quad \ldots \quad (6.2)$$

where

$$\operatorname{Re}\gamma > 0,\ \operatorname{Re}\rho > 0,\ \operatorname{Re}(\gamma+\rho-\alpha-\beta) > 0.$$

Substituting in (4.1) and (4.2) from (6.2), we obtain

$$u(r, t) = 2g_0\frac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\Gamma(\gamma+\rho-\beta)}e^{-zt}t^{\gamma+\rho-1}$$

$$\times(-1)^{\mu}a^{2\sigma+\lambda+\mu-2}\frac{\Gamma(1+\mu+\nu)\Gamma\left(\frac{1+2\sigma+\lambda}{2}\right)\Gamma\left(\frac{2\sigma+\lambda}{2}\right)}{\Gamma(1-\mu+\nu)\Gamma\left(\frac{1+2\sigma+\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{2+2\sigma+\lambda+\mu+\nu}{2}\right)}$$

$$\times\frac{k}{K}\sum_i {}_2F_3\left[\begin{matrix}(1+2\sigma+\lambda)/2,\ (2\sigma+\lambda)/2\\ 1,\ (1+2\sigma+\lambda+\mu-\nu)/2,\ (2+2\sigma+\lambda+\mu+\nu)/2\end{matrix};\ -\frac{a^2\xi_i^2}{4}\right]\frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2}$$

$$\times {}_2F_2\left[\begin{matrix}\rho,\ \gamma+\rho-\alpha-\beta\\ \gamma+\rho-\alpha,\ \gamma+\rho-\beta\end{matrix};\ \left(z-k\xi_i^2\right)t\right] \quad \ldots \quad (6.3)$$

and

$$u(r,t) = 2g_0 \frac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\Gamma(\gamma+\rho-\beta)} e^{-zt} t^{\gamma+\rho-1}$$

$$\times a^{2\sigma+\lambda-\mu-2} \frac{\Gamma\left(\frac{1+2\sigma+\lambda}{2}\right)\Gamma\left(\frac{2\sigma+\lambda}{2}\right)}{\Gamma\left(\frac{1+2\sigma+\lambda-\mu-\nu}{2}\right)\Gamma\left(\frac{2+2\sigma+\lambda-\mu+\nu}{2}\right)}$$

$$\times \frac{k}{K}\sum_i {}_2F_3\left[\begin{matrix}(1+2\sigma+\lambda)/2,\ (2\sigma+\lambda)/2\\ 1,\ (1+2\sigma+\lambda-\mu-\nu)/2,\ (2+2\sigma+\lambda-\mu+\nu)/2\end{matrix};\ -\frac{a^2\xi_i^2}{4}\right]\frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2}$$

$$\times {}_2F_2\left[\begin{matrix}\rho,\ \gamma+\rho-\alpha-\beta\\ \gamma+\rho-\alpha,\ \gamma+\rho-\beta\end{matrix};\ (z-k\xi_i^2)t\right]. \quad \ldots \quad (6.4)$$

Obviously $u(r, 0) = 0$.

7. *Heat source of polynomial character.*—In (6.1) and (6.2), replacing α by $-n$, β by $1+\alpha+\beta+n$, γ by $1+\alpha$ and using (Erdelyi 1954, p. 268), we have

$$g(\tau) = g_0 e^{-\tau z} \tau^{\alpha}(t-\tau)^{\rho-1} \frac{n!}{(1+\alpha)_n} P_n^{(\alpha,\beta)}\left\{1-\frac{2\tau}{t}\right\} \quad \ldots \quad (7.1)$$

and

$$h(\xi_i, t) = g_0 \frac{\Gamma(1+\alpha)\Gamma(\rho)\Gamma(\rho-\beta)}{\Gamma(1+\alpha+\rho+n)\Gamma(\rho-\beta-n)} e^{-zt} t^{\alpha+\rho}$$

$$\times {}_2F_2\left[\begin{matrix}\rho,\ \rho-\beta\\ 1+\alpha+\rho+n,\ \rho-\beta-n\end{matrix};\ (z-k\xi_i^2)t\right] \quad \ldots \quad (7.2)$$

where

$$\text{Re}\ \alpha > -1,\ \text{Re}\ \rho > 0,\ \text{Re}\ (\rho-\beta) > 0.$$

From (4.1), (4.2) and (7.2), we have

$$u(r,t) = 2g_0 \frac{\Gamma(1+\alpha)\Gamma(\rho)\Gamma(\rho-\beta)}{\Gamma(1+\alpha+\rho+n)\Gamma(\rho-\beta-n)} e^{zt} t^{\rho+\alpha}$$

$$\times (-1)^{\mu} a^{2\sigma+\lambda+\mu-2} \frac{\Gamma(1+\mu+\nu)\Gamma\left(\frac{1+2\sigma+\lambda}{2}\right)\Gamma\left(\frac{2\sigma+\lambda}{2}\right)}{\Gamma(1-\mu+\nu)\Gamma\left(\frac{1+2\sigma+\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{2+2\sigma+\lambda+\mu+\nu}{2}\right)}$$

$$\times \frac{k}{K}\sum_i {}_2F_3\left[\begin{matrix}(1+2\sigma+\lambda)/2,\ (2\sigma+\lambda)/2\\ 1,\ (1+2\sigma+\lambda+\mu-\nu)/2,\ (2+2\sigma+\lambda+\mu+\nu)/2\end{matrix};\ -\frac{a^2\xi_i^2}{4}\right]\frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2}$$

$$\times {}_2F_2\left[\begin{matrix}\rho,\ \rho-\beta\\ 1+\alpha+\rho+n,\ \rho-\beta-n\end{matrix};\ (z-k\xi_i^2)t\right] \quad \ldots \quad (7.3)$$

and

$$u(r, t) = 2g_0 \frac{\Gamma(1+\alpha)\Gamma(\rho)\Gamma(\rho-\beta)}{\Gamma(1+\alpha+\rho+n)\Gamma(\rho-\beta-n)} e^{-zt}t^{\rho+\alpha}$$

$$\times a^{2\sigma+\lambda-\mu-2} \frac{\Gamma\left(\frac{1+2\sigma+\lambda}{2}\right)\Gamma\left(\frac{2\sigma+\lambda}{2}\right)}{\Gamma\left(\frac{1+2\sigma+\lambda-\mu-\nu}{2}\right)\Gamma\left(\frac{2+2\sigma+\lambda-\mu+\nu}{2}\right)}$$

$$\times \sum_i {}_2F_3\left[\begin{matrix}(1+2\sigma+\lambda)/2,\ (2\sigma+\lambda)/2\\ 1,\ (1+2\sigma+\lambda-\mu-\nu)/2,\ (2+2\sigma+\lambda-\mu+\nu)\end{matrix};\ -\frac{a^2\xi_i^2}{4}\right]\cdot\frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2}$$

$$\times {}_2F_2\left[\begin{matrix}\rho,\ \rho-\beta\\ 1+\alpha+\rho+n,\ \rho-\beta-n\end{matrix};\ (z-k\xi_i^2)t\right]. \qquad \ldots \quad \ldots \quad \ldots \quad (7.4)$$

Obviously $u(r, 0) = 0$.

In view of the expansion property of orthogonal polynomials (Szegö 1959) the heat source of this character may yield several cases of interest.

§ 8. *Heat source of exponential character.*—In (6.1) and (6.2), putting $\alpha = \beta = 0$, we get

$$g(\tau) = g_0 e^{-z\tau}\tau^{\gamma-1}(t-\tau)^{\rho-1} \qquad \ldots \quad \ldots \quad \ldots \quad (8.1)$$

and

$$h(\xi_i, t) = g_0 \frac{\Gamma(\gamma)\Gamma(\rho)}{\Gamma(\gamma+\rho)} e^{-zt}t^{\gamma+\rho-1}{}_1F_1\left[{}^{\rho}_{\gamma+\rho};\ (z-k\xi_i^2)t\right] \quad \ldots \quad \ldots \quad (8.2)$$

where

$$\text{Re}\ \gamma > 0,\ \text{Re}\ \rho > 0.$$

In (8.1) and (8.2), taking $\rho = 1$, we get the heat source obtained by Bhonsle (1966) by applying convolution theorem of Laplace transformation.

Further putting $\gamma = 1$ and $z = 0$, using (Lebedev 1965, p. 271), viz. ${}_1F_1[{}^1_2; z] = \frac{e^z-1}{z}$, we get the heat source acting for a finite interval of time obtained by Bhonsle (1966).

Substituting in (4.1) and (4.2) from (8.2), we have

$$u(r, t) = 2g_0 \frac{\Gamma(\gamma)\Gamma(\rho)}{\Gamma(\gamma+\rho)} e^{-zt}t^{\gamma+\rho-1}$$

$$\times (-1)^{\mu} a^{2\sigma+\lambda+\mu-2} \frac{\Gamma(1+\mu+\nu)\Gamma\left(\frac{1+2\sigma+\lambda}{2}\right)\Gamma\left(\frac{2\sigma+\lambda}{2}\right)}{\Gamma(1-\mu+\nu)\Gamma\left(\frac{1+2\sigma+\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{2+2\sigma+\lambda+\mu+\nu}{2}\right)}$$

$$\times \frac{k}{K}\sum_i {}_2F_3\left[\begin{matrix}(1+2\sigma+\lambda)/2,\ (2\sigma+\lambda)/2\\ 1,\ (1+2\sigma+\lambda+\mu-\nu)/2,\ (2+2\sigma+\lambda+\mu+\nu)/2\end{matrix};\ -\frac{a^2\xi_i^2}{4}\right]$$

$$\times \frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2}{}_1F_1\left[{}^{\rho}_{\gamma+\rho};\ (z-k\xi_i^2)t\right] \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8.3)$$

and

$$u(r, t) = 2g_0 \frac{\Gamma(\gamma)\Gamma(\rho)}{\Gamma(\gamma+\rho)} e^{-zt} t^{\gamma+\rho-1} a^{2\sigma+\lambda-\mu-2} \frac{\Gamma\left(\frac{1+2\sigma+\lambda}{2}\right)\Gamma\left(\frac{2\sigma+\lambda}{2}\right)}{\Gamma\left(\frac{1+2\sigma+\lambda-\mu-\nu}{2}\right)\Gamma\left(\frac{2+2\sigma+\lambda-\mu+\nu}{2}\right)}$$

$$\times \frac{k}{K}\sum_i {}_2F_3\left[\begin{matrix}(1+2\sigma+\lambda)/2,\ (2\sigma+\lambda)/2 \\ 1,\ (1+2\sigma+\lambda-\mu-\nu)/2,\ (2+2\sigma+\lambda-\mu+\nu)/2\end{matrix};\ -\frac{a^2\xi_i^2}{4}\right]$$

$$\times \frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2}\, {}_1F_1\left[\begin{matrix}\rho \\ \gamma+\rho\end{matrix};\ (z-k\xi_i^2)t\right]. \qquad (8.4)$$

Obviously $u(r, 0) = 0$.

§ 9. *The behaviour of $f(r)$*.—From (3.2) we have

$$f(r) = r^{2\sigma+\lambda-2}\{a^2-r^2\}^{\mu/2} P_\nu^\mu\left(\frac{r}{a}\right)$$

$$= \frac{r^{2\sigma+\lambda-2}(-1)^\mu(a+r)^\mu}{\Gamma(1-\mu)}\, {}_2F_1\left[\begin{matrix}-\nu,\ 1+\nu \\ 1-\mu\end{matrix};\ \tfrac{1}{2}-\tfrac{1}{2}\frac{r}{a}\right]. \qquad (9.1)$$

Let $\mu = 0$, $\nu = 1$, then

$$f(r) = \frac{r^{2\sigma+\lambda-1}}{a}. \qquad (9.2)$$

Now we have

$$f(r) = 0 \text{ when } r = 0.$$

Similarly the behaviour of $f(r)$ from (3.3) may be discussed.

If $g(t) > 0$, then the inner cylinder will enclose sources, while the volume between two concentric cylinders will contain sinks. If $g(t) < 0$, then the sources and sinks will interchange their roles. From (6.1), (7.1) and (8.1), we notice that $g(t) = 0$, if $\rho \neq 1$ and $g(t) \neq 0$, if $\rho = 1$.

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