A NOTE ON A CURIOUS SOLUTION OF EINSTEIN'S FIELD EQUATIONS

by A. R. PRASANNA, *Department of Mathematics and Statistics, University of Poona, Poona 7*

(Communicated by V. V. Narlikar, F.N.I.)

(*Received 6 September 1967*)

It is shown that Møller's energy-momentum complex vanishes for a well-known Riemannian metric for which $R_{ij} = 0$ and the usual pseudo-energy tensor also vanishes. The space-torsion does not vanish for the metric and only two of the fourteen scalar invariants are non-zero. The crucial role of the constant k, which has to be non-zero, is brought out in several contexts. The space-time singularity $1+kt = 0$ is without an analogue in non-relativistic physics.

1. INTRODUCTION

The physical situations of special relativity are marked by the conditions of flat space-time, viz. $R_{hijk} = 0$, as distinguished from those of general relativity wherein we have $R_{hijk} \neq 0$. The usual sources of gravitation are T^{ij}, the cosmological constant Λ and certain types of field singularities. Are there any situations of physical interest wherein we have $T^{ij} = 0$, $\Lambda = 0$, and no singularity, and yet $R_{hijk} \neq 0$? The only known metric solution for which all these conditions are satisfied and the usual pseudo-energy tensor vanishes seems to be of the type presented by Narlikar and Karmarkar (1946), viz.

$$ds^2 = dt^2-(1+kt)^p dx^2-(1+kt)^q dy^2-(1+kt)^r dz^2 \qquad (1.1)$$

where

$$p+q+r = 2, \quad pq+qr+rp = 0, \quad t \gtrless -1/k. \qquad (1.2)$$

The singularity $1+kt = 0$ appears to be without a Newtonian analogue. It is more a space-time singularity than a familiar field singularity; and yet when $k = 0$ the space-time is flat. What could be the source of the field producing gravitation? Could there be an energy distribution which is not revealed by the usual T^{ij}? We have to understand the full physical implications of the fact, $k \neq 0$.

In answer to the above question we consider the new scheme of the energy-momentum complex discussed by Møller (1961). Møller remarks that the components of the fundamental tensor g_{ij} are not the true field variables and considers the tetrad formulation. He has constructed a superpotential U_i^{jk} (which is a true tensor density) from a tetrad of suitably oriented orthonormal vectors and defines the energy-momentum complex as

$$\tau_i^j = U_{i\;\;,k}^{jk} \qquad (1.3)$$

in the usual notation. We have examined the solution (1.1) in the light of this formalism. Our investigations show that even the new energy-momentum complex fails to identify any energy content in the field.

We have looked into some other aspects of the line-element (1.1). We find that the space-torsion is not zero. We have obtained the tracks of a test particle in this field, and also get an expression for the relative mass m of a particle of proper mass m_0. In the present case only two of the 14 Narlikar-Karmarkar scalar invariants (1949) are found to be non-zero.

During the course of the above investigation, the author's attention was drawn to the line-element

$$ds^2 = dt^2 - \frac{e^{2\sqrt{\Lambda/3}t}}{c^2}(dx^2+dy^2+dz^2) \quad \ldots \quad \ldots \quad \ldots \quad (1.4)$$

given by McVitte (1964). (We shall put $c = 1$). Actually this line-element falls into the category where the cosmological constant Λ acts as a source; because both the pseudo-energy-momentum tensor of Einstein t^{ij} and Møller's energy-momentum complex τ_i^j, which are non-zero, vanish if $\Lambda = 0$.

2. Møller's Tetrad Formalism

As a mathematical preliminary we first give the scheme of the new energy-momentum complex. Møller builds up the complex by first considering an orthonormal tetrad $\lambda^i_{(a)}$ defined by

$$\lambda_{(a)i} = g_{ij}\lambda^j_{(a)}, \; \lambda^i_{(a)}\lambda^{(a)j} = g^{ij}, \quad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

from which he constructs the tensor γ_{ikl}.

$$\gamma_{ikl} = \lambda^{(a)}_i \lambda_{(a)k\,;\,l} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

which is a local tetrad tensor such that its components along the directions of the tetrad are the 'Ricci coefficients of rotation',

$$\gamma_{(abc)} = \gamma_{ikl}\lambda^i_{(a)}\lambda^k_{(b)}\lambda^l_{(c)}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.3)$$

Here and in what follows a semicolon before a suffix indicates a covariant differentiation.

Though the $\lambda^i_{(a)}$ given by (2.1) determine the metric tensor uniquely, for a given $g_{ik}(x)$ there is certain freedom in the choice of the tetrads $\lambda^i_{(a)}$ corresponding to arbitrary independent rotations of the four-unit vectors of the tetrad. In order to get the $\tau^j_i(x)$ which is a unique function of the coordinates for a given physical system one has to specify the relative orientation at different points of the tetrads entering the expression τ^j_i. Hence to single out the suitably oriented tetrads Møller gives the supplementary conditions

$$\gamma^{\;\;l}_{ik;\,l} + \gamma_{ikl}\phi^l = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.4)$$

which have to be satisfied. Here ϕ^l follows from

$$\phi_k = \gamma^{\,i}_{ki}. \qquad (2.5)$$

The energy-momentum complex τ^j_i is now defined as

$$\tau^j_i = U^{jk}_{i,\,k}$$

where

$$U^{jk}_i = U^{[jk]}_i = \pm \frac{\sqrt{-g}}{2\kappa}\{\lambda^j_{(a)}\lambda^{(a)k}_{\;;\,i} - \lambda^k_{(a)}\lambda^{(a)j}_{\;;\,i} + 2(\delta^j_i\lambda^{(a)k} - \delta^k_i\lambda^{(a)j})\lambda^s_{(a);\,s}\} \qquad (2.6)$$

$g = \det |g_{ij}|$
and κ is the gravitational constant (for all our purposes we choose $\kappa = 1$). The total energy content in any finite volume V is given by

$$E = -\iiint \tau^4_4 dx\, dy\, dz \qquad (2.7)$$

and this is independent of the spatial coordinates used.

Consider the line-element (1.1). Before going into the energy considerations we note that the possible values of p, q, r satisfying conditions (1.2) can be expressed as

$$p,\ (1-\tfrac{1}{2}p) \pm (1+p-\tfrac{3}{4}p^2)^{\frac{1}{2}}. \qquad (2.8)$$

This general case does not seem to have been noticed by Petrov (1962). If the roots are to be real, we must have each satisfying a relation of the type

$$-\tfrac{2}{3} \leqslant p \leqslant 2. \qquad (2.9)$$

Since the product $pqr < 0$, we find that one and only one of them must be negative.

While presenting the solution (1.1) Narlikar and Karmarkar have observed that the line-element, besides satisfying the field equations $R_{ij} = 0$, has all the components of the pseudo-energy-momentum tensor t^{ij} vanishing.

We construct the following tetrad at each world point, which satisfies the conditions (2.4)

$$\left.\begin{array}{llll}
\lambda^i_{(1)} = \{(1+kt)^{-p/2}, & 0, & 0, & 0\} \\
\lambda^i_{(2)} = \{0, & (1+kt)^{-q/2}, & 0, & 0\} \\
\lambda^i_{(3)} = \{0, & 0, & (1+kt)^{-r/2}, & 0\} \\
\lambda^i_{(4)} = \{0, & 0, & 0, & 1\}
\end{array}\right] \qquad (2.10)$$

Using (2.6) direct computation leads us to the fact that out of the 24 independent components of U^{jk}_i only three are non-vanishing and they are given by

$$U^{14}_1 = -k\left(\frac{q+r}{2}\right), \quad U^{24}_2 = -k\left(\frac{r+p}{2}\right), \quad U^{34}_3 = -k\left(\frac{p+q}{2}\right). \qquad (2.11)$$

Since k, p, q and r are all constants we find that the derivatives of U_i^{jk} all vanish identically. Hence we get

$$\tau_i^j = 0. \qquad \ldots \quad (2.12)$$

The total energy E as given by (2.7) is, therefore, zero.

3. Some Other Aspects of the Line-element

We shall next consider the torsion of the 4-space defined by the line-element 1.1. In general the torsion is given by γ_{ikl} defined in (2.2). Using $\overset{i}{\lambda}_{(a)}$ given in (2.10) we find that there are only three independent non-vanishing components of the torsion coefficients, viz.

$$\gamma_{141} = \frac{kp(1+kt)^{p-1}}{2}, \quad \gamma_{242} = \frac{kq(1+kt)^{q-1}}{2}, \quad \gamma_{343} = \frac{kr(1+kt)^{r-1}}{2}. \qquad (3.1)$$

Further the Ricci rotation coefficients are given by

$$\gamma_{(141)} = \frac{kp}{2(1+kt)}, \gamma_{(242)} = \frac{kq}{2(1+kt)}, \gamma_{(343)} = \frac{kr}{2(1+kt)}. \qquad \ldots \quad (3.2)$$

Møller (1965) remarks that for a completely empty space we have to assume that, besides the curvature, the torsion of the tetrad lattice also to be zero. But here we seem to have a situation wherein we are getting a 'completely empty space-time devoid of all energy but still some components of the curvature and torsion are non-zero'. The curvature and torsion components are mutually related as given by

$$R_{hijk} = \gamma_{hij;\,k} - \gamma_{hik;\,j} + \gamma_{sik}\gamma^{s}_{hj} - \gamma_{sij}\gamma^{s}_{hk}. \qquad \ldots \quad (3.3)$$

Here we place on record the values of the six independent components of R_{hijk} which are non-zero for the line-element (1.1).

$$\left.\begin{aligned} R_{1212} &= -\frac{k^2pq(1+kt)^{p+q-2}}{4}, & R_{1414} &= \frac{k^2p(p-2)(1+kt)^{p-2}}{4} \\ R_{1313} &= -\frac{k^2pr(1+kt)^{p+r-2}}{4}, & R_{2424} &= \frac{k^2q(q-2)(1+kt)^{q-2}}{4} \\ R_{2323} &= -\frac{k^2qr(1+kt)^{q+r-2}}{4}, & R_{3434} &= \frac{k^2r(r-2)(1+kt)^{r-2}}{4}. \end{aligned}\right\} \qquad \ldots \quad (3.4)$$

Geodesics—The geodesic equation for a test particle in this field has the first integrals as given by

$$\left.\begin{aligned} &\frac{dx}{ds} = \frac{c_1}{(1+kt)^p}, \quad \frac{dy}{ds} = \frac{c_2}{(1+kt)^q}, \quad \frac{dz}{ds} = \frac{c_3}{(1+kt)^r} \\ &\frac{dt}{ds} = \left\{1 + \frac{c_1^2}{(1+kt)^p} + \frac{c_2^2}{(1+kt)^q} + \frac{c_3^2}{(1+kt)^r}\right\}^{\frac{1}{2}}. \end{aligned}\right\} \qquad \ldots \quad (3.5)$$

where c_1, c_2, c_3 are constants. In order to get the tracks we shall assume that $k = 1/T$, where T is sufficiently large so that $(t/T)^2$ and higher powers may be neglected. Then the parametric equations of the tracks are

$$\left.\begin{aligned} x &= \frac{c_1 t}{\sqrt{1+c^2}} - \frac{c_1}{2T\sqrt{1+c^2}}\left(p - \frac{Q}{2(1+c^2)}\right)t^2 \\ y &= \frac{c_2 t}{\sqrt{1+c^2}} - \frac{c_2}{2T\sqrt{1+c^2}}\left(q - \frac{Q}{2(1+c^2)}\right)t^2 \\ z &= \frac{c_3 t}{\sqrt{1+c^2}} - \frac{c_3}{2T\sqrt{1+c^2}}\left(r - \frac{Q}{2(1+c^2)}\right)t^2 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (3.6)$$

where

$$c^2 = c_1^2+c_2^2+c_3^2, \quad Q = pc_1^2+qc_2^2+rc_3^2.$$

Considering the projection on the xy plane we get the parabola

$$\{2(c_2xq-c_1yp)(1+c^2)-Q(c_2x-c_1y)\}^2 = 8Tc_1c_2(1+c^2)^{3/2}(p-q)(c_2x-c_1y). \quad (3.7)$$

If we make the assumption $p = q$, (3.7) reduces to the pair of coincident straight lines

$$(c_2x-c_1y)^2 = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.8)$$

On the other hand transforming T to the other side and making it to tend to infinity we get the straight line

$$(c_2x-c_1y) = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.9)$$

The last case is obvious, for in this case the line-element (1.1) reduces to Minkowski's form.

A test particle at rest initially will continue to be at rest. It is clear from (3.6) that the force terms responsible for the parabolic path automatically disappear if the velocity constants c_1, c_2 and c_3 vanish. Since it is forbidden to have $p = q = r$ the force terms persist when $c_1, c_2, c_3 \neq 0$.

Consider now a particle of proper mass m_0 at the point (x, y, z) at $t = 0$ with velocity $\dot{x} = u \equiv$ (constant), $\dot{y} = \dot{z} = 0$. Its relative mass m is then given by

$$\left.\begin{aligned} m &= m_0\frac{dt}{ds} = m_0\left(1+\frac{u^2}{1-u^2}\right)^{\frac{1}{2}} \\ m &= m_0(1-u^2)^{-\frac{1}{2}}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (3.10)$$

At any arbitrary time t the relative mass for a similar motion is given by

$$m = m_0\left\{1+\frac{u^2}{(1-u^2)(1+kt)^p}\right\}^{\frac{1}{2}}. \quad \ldots \quad \ldots \quad \ldots \quad (3.11)$$

Thus the relative mass decreases or increases as time progresses, according as $p > 0$, or < 0.

Narlikar-Karmarkar scalar invariants—It is well known that there are only 14 scalar differential invariants of order two. In the present case we find that only two of the 14 in the notation of Narlikar and Karmarkar (1949) are non-zero. They are

$$J_1 = A_{hijk} g^{hj} g^{ik}, \; J_2 = B_{hijk} g^{hj} g^{ik}$$

where

$$\left.\begin{aligned} A_{hijk} &= C_{hipq} C_{rsjk} g^{pr} g^{qs} \\ B_{hijk} &= C_{hipq} A_{rsjk} g^{pr} g^{qs} \\ C_{hijk} &= R_{hijk} - \tfrac{1}{2}(R_{ij} g_{hk} + R_{hk} g_{ij} - R_{hj} g_{ik} - R_{ik} g_{hj}) + \frac{R}{\delta 6}(g_{ij} g_{hk} - g_{hj} g_{ik}). \end{aligned}\right\} \quad (3.12)$$

Using the components of R_{hijk} given in (3.4) we get

$$J_1 = -\frac{2k^4 pqr}{(1+kt)^4}, \; J_2 = -\frac{3k^6 p^2 q^2 r^2}{4(1+kt)^6} \quad \ldots \quad (3.13)$$

$pqr = 0$ leads to flat space. In the case of interest $pqr < 0$, so that

$$J_1 > 0, \quad J_2 < 0.$$

4. McVitte's Example

Before we conclude this paper we consider the line-element (1.3) for energy considerations. McVitte has shown that this line-element has non-zero curvature components but all the components of T^{ij} vanish. As we have remarked earlier T^{ij} itself is not the complete source for a gravitational field. In fact for this case we find that the pseudo-energy-momentum tensor t^{ij} has the following non-vanishing components:

$$\left.\begin{aligned} t^{11} &= t^{22} = t^{33} = -\tfrac{7}{3} \Lambda e^{-2\sqrt{\Lambda/3}\, t} \\ t^{44} &= -\Lambda, \; t^{ij} = 0 (i \neq j). \end{aligned}\right\} \quad \ldots \quad (4.1)$$

To calculate the Møller's complex we choose the tetrad,

$$\left.\begin{aligned} \lambda^i_{(a)} &= e^{-\sqrt{\Lambda/3}\, t} \delta^i_{(a)}, \; a = 1, 2, 3 \\ \lambda^i_{(4)} &= \delta^i_{(4)}. \end{aligned}\right\} \quad \ldots \quad (4.2)$$

Ensuring that this tetrad satisfies the condition (2.4) we find that there are three non-zero independent components of U_i^{jk} as given by

$$U_1^{14} = U_2^{24} = U_3^{34} = -2e^{\sqrt{3\Lambda}\, t}. \quad \ldots \quad (4.3)$$

Hence we get

$$\left.\begin{aligned} \tau^1_1 &= \tau^2_2 = \tau^3_3 = -2\sqrt{3\Lambda} e^{\sqrt{3\Lambda}\, t} \\ \tau^4_4 &= 0, \; \tau^j_i = 0 (i \neq j). \end{aligned}\right\} \quad \ldots \quad (4.4)$$

The total energy content in any finite volume is zero but there are certain non-vanishing stress components which have no Newtonian analogue, depending entirely on the cosmological constant Λ.

5. Concluding Remarks

In McVitte's example we find that the expressions for pseudo-energy-momentum tensor all vanish when $\Lambda = 0$. Similarly there is a crucial part played by the constant k in the solution (1.1), as is clear from the expressions (3.1), (3.4), (3.6) and (3.13). But the important point is that the presence of k does in no way show any energy distribution in the field, even with the new definition of Møller. We considered Synge's recent formulation (1967) for the pseudo-energy tensor, but found that it does not help for the cases when $T^{ij} = 0$. One would have expected an energy field to co-exist with a curved space-time. But the above example makes one doubt such a co-existence and brings out clearly that there is a role of space-time singularity in general relativity which is quite distinct from the field singularity which may exist even when the curvatures are negligible as in the case of electrodynamics in flat space-time. The physical significance of the singularity condition $(1+kt) = 0$ deserves attention, particularly because of the current interest in Mach's principle and the role of singularities in general relativity and cosmology. The existence of a metric like (1.1) is contrary to the spirit of Mach's principle or of a field theory according to which we equate geometry and physics. However, it is interesting to put $k = 1/T$ in the above example, T being large like H^{-1} (H, Hubble's constant) or the age of the universe. The results may then assume some physical significance for a portion of space-time excluding $1+kt = 0$ which is linked with metrics for other portions of occupied space-time by appropriate boundary conditions. The material field in these external regions may then be considered responsible for the odd properties of the metric (1.1).

Acknowledgements

The author wishes to place on record his sincere thanks to Professor V. V. Narlikar, Lokamanya Tilak Professor of Applied Mathematics, University of Poona, for his guidance throughout the preparation of this paper. His thanks are also due to the Government of India for having awarded him a research training scholarship which facilitated this work.

References

McVitte, G. C. (1964). General Relativity and Cosmology, Vol. 4. Chapman and Hall Ltd., London, 76-77.

Møller, C. (1961). Conservation laws and absolute parallelism in general relativity. *Mat. Fys. Skr. Dan. Vid. Selsk.*, **1** (10).

——— (1965). Energy and momentum carried by gravitational Waves. *Atti Conv. Relativita generale*, **3**.

Narlikar, V. V., and Karmarkar, K. R. (1946). On a curious solution of relativistic field equations. *Curr. Sci.*, **3**, 69.

——— (1949). The Scalar invariants of a general gravitational metric. *Proc. Indian Acad. Sci.*, **129 A**, 91–97.

Petrov, A. Z. (1962). Gravitational Field Geometry as the Geometry of Automorphisms: Recent Developments in General Relativity. Pergamon Press, 383.

Synge, J. L. (1967). A new pseudo-tensor with vanishing divergence. *Nature, Lond.*, **215**, 5096, 102.