A SEMI-INFINITE ELASTIC STRIP SUPPORTING A THIN HEAVY OVERHANGING BEAM

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(Communicated by F. C. Auluck, F.N.I.)

(*Received 19 May 1967*)

Assuming displacements are small, the elastic problem is solved using the Airy stress function and a Laplace transform, for which two unknown boundary conditions are expressed in terms of infinite Fourier series. The equation for the heavy beam is solved separately and then, assuming that the beam and the elastic strip remain in contact everywhere, the reaction between the two can be found in principle. An infinity of simultaneous equations is obtained for the unknown Fourier coefficients and, after assigning suitable values to the parameters, numerical results are found by truncation, the convergence being particularly good.

1. Introduction

A simple two-dimensional model consists of a thin heavy beam ($-l \leqslant x < \infty$) resting on an elastic foundation ($0 \leqslant x < \infty$) which is such that the reaction is proportional to the vertical displacement v, measured downwards. Applying the usual boundary conditions and solving the beam equation (Timoshenko 1952) it is found that for $0 \leqslant x < \infty$

$$v = \frac{W}{k}\{1+\beta l e^{-\beta x}[(2+\beta l)\cos\beta x - \beta l \sin\beta x]\} \quad \ldots \quad \ldots \quad (1.1)$$

where β, k and W are constants determining respectively the rigidity of the beam, the reaction of the elastic foundation and the weight per unit length of the beam. It is easily observed that v and, therefore, the vertical reaction have a maximum value at $x = 0$.

The more complicated situation which will be considered here in detail consists of the same heavy beam resting on an elastic strip $0 \leqslant x < \infty$, $0 \leqslant y \leqslant h$ with y and the vertical displacement v now measured upwards. The elastic plane strain problem is solved using the airy stress function ψ and a Laplace transform with respect to x as was done by Bentham (1963), who expressed the unknown boundary conditions on $\frac{\partial^2\psi}{\partial x^2}$ and $\frac{\partial^3\psi}{\partial x^3}$ at $x = 0$ as infinite Fourier series in y. Requiring that there be no vertical displacement at the base of the elastic strip, and neglecting the weight of the strip in comparison with that of the beam, it is then possible to treat the problem as one half of a situation symmetric about the x-axis with consequent simplification of ψ. The vertical displacement v, at $y = h$ is found in terms of the vertical stress $(W+f(x))$ there, and assuming the heavy beam and the elastic strip remain in contact for all $x \geqslant 0$, the equation of the beam enables the displacement term to be eliminated and an expression found for $\bar{f}(p)$, the Laplace transform of $f(x)$, the required change of stress due to the overhanging portion of the beam.

$f(x)$ is then the sum of an infinite number of residues at the poles of $k(ph+\sin ph \cos ph)+2p^3 \sin^2 ph = 0$ in the half plane re$p \leqslant 0$; these contain the Fourier coefficients which in turn are given by the infinity of simultaneous equations obtained by equating to zero the residues of $\bar{f}(p)$ at the poles in the half-plane re$p > 0$.

Numerical solutions are found by assigning suitable values to the parameters kh^3 and l/h and truncating the Fourier series; comparison is made of the results for $f(x)$ and the convergence $-v(x, h)$, obtained by solving 8 and 12 simultaneous equations on a digital computer and the agreement on the dominant term is extremely good. The maximum value of $f(x)$ does not occur at $x = 0$ as in the simple model.

With the elements of the infinite determinant satisfying the necessary convergence conditions (Riesz 1913), a unique solution for the Fourier coefficients is assured. Thus $f(x)$ is determined as a sum of terms of the form

$$e^{-p_r x}(A \cos p_i x + B \sin p_i x)$$

where (p_r+ip_i) is a solution of $k(ph+\sin ph \cos ph)+2p^3 \sin^2 ph = 0$; since this is true whatever particular boundary conditions are imposed at $x = 0$, $y = 0$ and $y = h$, one can only deduce that the above set of functions are in some sense a complete set. However, an attempt to obviate inverting the infinite matrix by expanding $f(x)$ in terms of these eigenfunctions and solving for the coefficients fails because there are no suitable orthogonality relations.

This same difficulty has faced previous authors who have considered the semi-infinite elastic strip with stress-free sides and an arbitrary end-load (the problem considered by Bentham). With ψ an even function of x, the eigenvalue equation is $\sin 2k+2k = 0$ with eigenfunctions $\Phi_k\, e^{kx}$ where

$$\Phi_k = [(\cos 2k-1) \cos ky - 2ky \sin ky] \quad (\text{re}\, k < 0).$$

Their properties have been studied in two papers by Buchwald (1964 and 1965); the most rapidly converging numerical technique has been developed by Gaydon and Shepherd (1964) who expanded the Φ_k in terms of an orthogonal set of functions and solved a set of 40 equations for 20 unknowns by a method of least squares. With the elements of the truncated infinite determinant tabulated, they have reduced the problem to a routine computation.

2. The Beam Equation

The equation for displacement of the beam in the overhanging section $-l \leqslant x \leqslant 0$ is

$$EI\frac{d^4v}{dx^4} = -W \quad \quad (2.1)$$

(*see* Timoshenko 1952), where E is the modulus of elasticity in tension or compression, I the moment of inertia about the z-axis, W the weight per unit length and v is measured upwards. The boundary conditions to be imposed are that there shall be zero bending moment and shearing stress at the free end, i.e.

$$\frac{d^2v}{dx^2} = \frac{d^3v}{dx^3} = 0 \text{ at } x = -l.$$

Then the solution of (2.1) is

$$v = \frac{W}{EI}(A+Bx-\tfrac{1}{4}l^2x^2-\tfrac{1}{6}lx^3-\tfrac{1}{24}x^4) \quad \quad (2.2)$$

where A, B are constants still to be determined. For the rest of the beam $(x > 0)$ the equation corresponding to (2.1) is

$$EI\frac{d^4v}{dx^4} = f(x)$$

where, since $f(x)$ is the change in vertical stress due to the overhanging portion of the beam, the upward reaction of the elastic strip is $W+f(x)$. Applying a Laplace transform with respect to x and using (2.2) to make

$$v,\ \frac{dv}{dx},\ \frac{d^2v}{dx^2} \text{ and } \frac{d^3v}{dx^3}$$

continuous at $x = 0$, it follows that

$$p^4\bar{v} = \frac{W}{EI}(Ap^3+Bp^2-\tfrac{1}{2}l^2p-l)+\frac{1}{EI}\bar{f}(p). \quad \quad (2.3)$$

After solving the elastic strip problem in the next section, this equation will be used to eliminate the vertical displacement term.

3. The Elastic Strip: Setting up of the Problem

The two-dimensional static equations of isotropic elasticity in the strip $0 \leqslant y \leqslant h$, $x \geqslant 0$ under the influence of gravity are

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau}{\partial y} = 0$$

$$\frac{\partial\tau}{\partial x}+\frac{\partial\sigma_y}{\partial y} = \rho g \quad \quad (3.1)$$

where σ_x, σ_y are the stresses in the x, y directions respectively, τ is the shear stress, ρ is the density and g the gravitational acceleration. The equations relating to stresses with the displacements u, v in the x, y directions respectively are

$$\left.\begin{aligned} \sigma_x &= (\lambda+2\mu)\frac{\partial u}{\partial x}+\lambda\frac{\partial v}{\partial y} \\ \sigma_y &= \lambda\frac{\partial u}{\partial x}+(\lambda+2\mu)\frac{\partial v}{\partial y} \\ \tau &= \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \end{aligned}\right\} \quad \ldots \quad (3.2)$$

where (*see* Muskhelishvili 1953) λ and μ are the Lamé constants.

The boundary conditions to be imposed are

(i) at $y = 0$, $\tau = 0$ and $v = 0$,

(ii) at $x = 0$, $\sigma_x = \tau = 0$,

(iii) at $y = h$, $\sigma_y = -[W+f(x)]$, $\tau = 0$.

(Alternative conditions at $y = 0$ and $y = h$ could be imposed, but would not radically affect the solution).

Then, since

$$\sigma_x = \frac{\rho g\lambda y}{\lambda+2\mu},\ \sigma_y = \rho g y,\ \tau = 0,\ u = 0,\ v = \frac{\rho g y^2}{2(\lambda+2\mu)}$$

is a particular solution of (3.1) and (3.2), it is possible to introduce the Airy stress function ψ by writing

$$\left.\begin{aligned} \sigma_x &= \frac{\partial^2\psi}{\partial y^2}-\frac{\lambda}{\lambda+2\mu}(W-\rho g y) \\ \sigma_y &= \frac{\partial^2\psi}{\partial x^2}-W+\rho g y \\ \tau &= -\frac{\partial^2\psi}{\partial x\,\partial y} \end{aligned}\right\}. \quad \ldots \quad (3.3)$$

In this way, ψ will measure the change of stress due to the overhanging section of the beam and the stress-free end of the elastic strip. The boundary conditions (i), (ii) and (iii) are easily shown by suitable choice of the three arbitrary constants to be equivalent to

(i) at $y = 0$, $$\frac{\partial\psi}{\partial y} = 0 \quad \ldots \quad (3.4)$$

$$\frac{\partial^3\psi}{\partial y^3} = 0 \quad \ldots \quad (3.5)$$

(ii) at $x = 0$, $$\psi = \frac{\lambda y^2}{2(\lambda+2\mu)}(W-\tfrac{1}{3}\rho g y) \quad \ldots \quad (3.6)$$

$$\frac{\partial\psi}{\partial x} = 0 \quad \ldots \quad (3.7)$$

(iii) at $y = h$, $$\frac{\partial\psi}{\partial y} = \frac{\lambda h}{\lambda+2\mu}(W-\tfrac{1}{2}\rho g h) \quad \ldots \quad (3.8)$$

$$\frac{\partial^2\psi}{\partial x^2} = -f(x)-\rho g h. \quad \ldots \quad (3.9)$$

Since σ_x, σ_y and τ can all be expressed in terms of u and v, the horizontal and vertical displacements appearing in eqn. (3.1), a further equation must exist and this is the compatibility condition (Muskhelishvili 1953, eqn. (27.6)), which on substituting from (3.3) becomes

$$\nabla^4\psi = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (3.10)$$

The problem has now been set up in such a way that the Laplace transform methods of Bentham (1963) can be used. As in his case, there can be no stress singularities at the corners of the strip. However, it should first be observed that the equations and boundary conditions on ψ are, except for the gravity term in (3.6), such as to make ψ an even function of y. It is here assumed that the weight of the beam is much greater than that of the strip and so the gravity terms will be neglected.

With ψ now even in y, the problem can now be treated as one half of a situation symmetrical about the x-axis, with $\frac{\partial\psi}{\partial y} = -\frac{\lambda Wh}{\lambda+2\mu}$ and $\frac{\partial^2\psi}{\partial x^2} = -f(x)$ at $y = -h$. When a Laplace transform with respect to x is made, the values of $\frac{\partial^2\psi}{\partial x^2}$ and $\frac{\partial^3\psi}{\partial x^3}$ at $x = 0$ will be required and these will be expressed in Fourier cosine series with unknown coefficients. Now the period of ψ with respect to y is $4h$ and in the interval $(h, 3h)$, for instance, all stresses are reversed. Hence in the Fourier expansion of $(\psi)_{x=0}$, only terms of the form $\cos (m+\frac{1}{2})\frac{\pi}{h}y$ appear, and thus at $x = 0$,

$$\left.\begin{aligned} \frac{\partial^2\psi}{\partial x^2} &= \sum_{m=0}^{\infty} A_m \cos\,(m+\tfrac{1}{2})\,\frac{\pi}{h}\,y \\ \frac{\partial^3\psi}{\partial x^3} &= \sum_{m=0}^{\infty} B_m \cos\,(m+\tfrac{1}{2})\,\frac{\pi}{h}\,y \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (3.11)$$

where the coefficients A_m, $B_m (m \geqslant 0)$ are to be determined. It is to be expected that either all odd or all even terms disappear as the entire set is not orthogonal in $(-h, h)$.

4. The Elastic Strip: Solution by Laplace Transform

When a Laplace transform with respect to x is applied to eqn. (3.10) the end conditions at $x = 0$ are given by (3.6), (3.7) and (3.11) and the result is

$$\frac{d^4\bar{\psi}}{dy^4}+2p^2\frac{d^2\bar{\psi}}{dy^2}+p^4\bar{\psi} = p^3\cdot\frac{\lambda W y^2}{2(\lambda+2\mu)}+\sum_{m=0}^{\infty}(pA_m+B_m)\cos\,(m+\tfrac{1}{2})\,\frac{\pi}{h}\,y+2p\cdot\frac{\lambda W}{\lambda+2\mu}$$

where

$$\bar{\psi}(p) = \int_0^{\infty} \psi(x)e^{-px}\,dx.$$

Hence

$$\bar{\psi} = \frac{\lambda W y^2}{2p(\lambda+2\mu)} + \sum_{m=0}^{\infty} \frac{(pA_m+B_m)}{\left[p^2-(m+\frac{1}{2})^2\frac{\pi^2}{h^2}\right]^2} \cos\,(m+\tfrac{1}{2})\frac{\pi}{h}y + C\cos py + Dy\sin py \qquad (4.1)$$

choosing only the two complementary functions which are even in y. The new boundary conditions are obtained by transforming eqns. (3.8) and (3.9) and are

$$\left.\begin{aligned} \frac{d\bar{\psi}}{dy} &= \pm\frac{\lambda W h}{p(\lambda+2\mu)} \text{ at } y = \pm h \\ \bar{\psi} &= -\frac{\bar{f}(p)}{p^2} + \frac{\lambda W h^2}{2p(\lambda+2\mu)} \text{ at } y = \pm h \end{aligned}\right] \qquad (4.2)$$

As was mentioned in the Introduction, the aim of this section is to find an expression for $(\bar{v})_{y=h}$, the transform of the vertical displacement at $y = h$, and use it with (2.3) to find $f(p)$. With $\bar{\psi}$ now known, $\bar{v}$ can be found fairly easily; from (3.2) and (3.3) with $\rho = 0$

$$4\mu(\mu+\lambda)\frac{d\bar{v}}{dy} = (\lambda+2\mu)\left[p^2\bar{\psi} - p\frac{\lambda W y^2}{2(\lambda+2\mu)}\right] - \lambda\frac{d^2\bar{\psi}}{dy^2} - \frac{4\mu(\mu+\lambda)}{p(\lambda+2\mu)}W$$

after applying a Laplace transform and using the end conditions (3.6) and (3.7). This can be integrated to give

$$\frac{4\mu(\mu+\lambda)}{(\lambda+2\mu)}\bar{v} = \sum_{m=0}^{\infty} \frac{(pA_m+B_m)}{\left[p^2-(m+\frac{1}{2})^2\frac{\pi^2}{h^2}\right]^2}\frac{hp^2}{(m+\frac{1}{2})\pi}\sin\,(m+\tfrac{1}{2})\frac{\pi}{h}y + Cp\sin py$$
$$+D\,(\sin py - py\cos py) - \frac{\lambda}{\lambda+2\mu}\cdot\frac{d\bar{\psi}}{dy} - \frac{4\mu(\mu+\lambda)}{(\lambda+2\mu)^2}\cdot\frac{Wy}{p}.$$

The constant of integration is zero in order to make $\bar{v}$ an odd function of y and C, D are determined from (4.1) and (4.2).

Hence

$$\frac{4\mu(\mu+\lambda)}{(\lambda+2\mu)}(\bar{v})_{y=h} = \sum_{m=0}^{\infty}\frac{(pA_m+B_m)(-1)^m}{\left[p^2-(m+\frac{1}{2})^2\frac{\pi^2}{h^2}\right]^2}\left\{\frac{hp^2}{(m+\frac{1}{2})\pi} + \left(\frac{\sin ph\cos ph - ph}{\sin ph\cos ph + ph}\right)\frac{(m+\frac{1}{2})\pi}{h}\right\}$$
$$-\frac{2\bar{f}(p)\sin^2 ph}{p(ph+\sin ph\cos ph)} - \frac{Wh}{p}. \qquad (4.3)$$

The last term represents the constant displacement due to the weight of the beam and would not have appeared if only the change in displacement due to the overhang of the beam and to the boundary conditions at $x = 0$ had been considered. However, a corresponding term will appear in the constant A in the beam eqn. (2.3) and so, when $(\bar{v})_{y=h}$ is eliminated, the net result is the same.

5. Inverting the Transform

Assuming that the heavy beam and elastic strip remain in contact everywhere, then $(\bar{v})_{y=h}$ in (4.3) can be equated to $\bar{v}$ in (2.3). Further the constants A, B can be evaluated from the condition that $\bar{f}(p) \to 0$ as $p \to \infty$ through real values since it is a Laplace transform. Thus

$$\frac{4\mu(\mu+\lambda)}{(\lambda+2\mu)} \cdot \frac{W}{EI}(Ap+B) \to -Whp + \sum_{m=0}^{\infty} \frac{(pA_m+B_m)(-1)^m h}{(m+\frac{1}{2})\pi}.$$

Putting $k = \dfrac{4\mu(\mu+\lambda)}{(\lambda+2\mu)EI}$ and equating coefficients, it follows that

$$kWA = -Wh + \sum_{m=0}^{\infty} \frac{(-1)^m A_m h}{(m+\frac{1}{2})\pi}$$

$$kWB = \sum_{m=0}^{\infty} \frac{(-1)^m B_m h}{(m+\frac{1}{2})\pi}.$$

These values of A, B could have been obtained directly by working out v and $\dfrac{\partial v}{\partial x}$ at the corner $x = 0$, $y = h$. Hence (2.3) becomes

$$\frac{4\mu(\mu+\lambda)}{(\lambda+2\mu)} p^4(\bar{v})_{y=h} = -Whp^3 + \frac{hp^2}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m(pA_m+B_m)}{(m+\frac{1}{2})} - kWl(\tfrac{1}{2}lp+1) + k\bar{f}(p).$$

Then, combining (2.3) and (4.3) as described above, the equation for $\bar{f}(p)$ is

$$\bar{f}(p)[k(ph+\sin ph\cos ph)+2p^3\sin^2 ph]$$

$$= \frac{p^2\pi}{h} \sum_{m=0}^{\infty} \frac{(-1)^m(pA_m+B_m)(m+\frac{1}{2})}{\left[p^2-(m+\frac{1}{2})^2\dfrac{\pi^2}{h^2}\right]} \left\{ph+\sin ph\cos ph + \frac{2p^2\sin ph\cos ph}{\left[p^2-(m+\frac{1}{2})^2\dfrac{\pi^2}{h^2}\right]}\right\}$$

$$+kWl(\tfrac{1}{2}lp+1)(ph+\sin ph\cos ph). \qquad (5.1)$$

The inverse transform is given by

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{f}(p)e^{px}\,dp$$

(Doetsch 1961) where c is real and such that $\bar{f}(p)$ is regular in re $p > c$. By inspection, $\bar{f}(p)$ has no poles at the points $p = 0$ and $p = \pm(m+\frac{1}{2})\dfrac{\pi}{h}$ but has, for arbitrary values of A_m, B_m, poles at the zeros $(p \neq 0)$ of $k(ph+\sin ph\cos ph)+2p^3\sin^2 ph$. Clearly if p is a zero of this function, then so also is $-p$, $\bar{p}$ and $-\bar{p}$; thus let $p_j (j = 1, 2, \ldots)$ denote the zeros in the first quadrant.

Now, in order that $f(x)$ shall tend to zero as $x \to \infty$, the residues of $\bar{f}(p)$ at all poles in the half plane re $p > 0$ must be zero because of the exponential terms in the integral, i.e.

$$\frac{p_j^2 \pi}{h} \sum_{m=0}^{\infty} \frac{(-1)^m (p_j A_m + B_m)(m+\frac{1}{2})}{\left[p_j^2 - (m+\frac{1}{2})^2 \frac{\pi^2}{h^2}\right]} \left\{ p_j h + \sin p_j h \cos p_j h + \frac{2p_j^2 \sin p_j h \cos p_j h}{\left[p_j^2 - (m+\frac{1}{2})^2 \frac{\pi^2}{h^2}\right]} \right\}$$
$$+ kWl(\tfrac{1}{2} l p_j + 1)(p_j h + \sin p_j h \cos p_j h) = 0 \qquad (5.2)$$

for all p_j. Taking real and imaginary parts, this gives two equations for each p_j and the same pairs would be obtained by considering the residues at the poles $\bar{p}_j$. The infinite set of equations (5.2) is, in theory, sufficient to solve for the infinity of unknowns A_m, B_m. However, in practice, only a finite number of terms of each series can be retained.

The residue of $\bar{f}(p) e^{px}$ at $-p_j$ is

$$\tfrac{1}{2}\left[kh \cos^2 p_j h + 3p_j^2 \sin^2 p_j h + 2p_j^3 h \sin p_j h \cos p_j h\right]^{-1} e^{-p_j x}$$

$$\times \left[\frac{p_j^2 h}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m (p_j A_m - B_m)(m+\frac{1}{2})}{\left[p_j^2 - (m+\frac{1}{2})^2 \frac{\pi^2}{h^2}\right]} \left\{ p_j h + \sin p_j h \cos p_j h + \frac{2p_j^2 \sin p_j h \cos p_j h}{\left[p_j^2 - (m+\frac{1}{2})^2 \frac{\pi^2}{h^2}\right]} \right\} \right.$$
$$\left. + kWl(\tfrac{1}{2} l p_j - 1)(p_j h + \sin p_j h \cos p_j h) \right]$$

which in virtue of (5.2) reduces to

$$-\left[kh \cos^2 p_j h + 3p_j^2 \sin^2 p_j h + 2p_j^3 h \sin p_j h \cos p_j h\right]^{-1} e^{-p_j x}$$
$$\times \left[\frac{p_j^2 h}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m B_m (m+\frac{1}{2})}{\left[p_j^2 - (m+\frac{1}{2})^2 \frac{\pi^2}{h^2}\right]} \left\{ p_j h + \sin p_j h \cos p_j h + \frac{2p_j^2 \sin p_j h \cos p_j h}{\left[p_j^2 - (m+\frac{1}{2})^2 \frac{\pi^2}{h^2}\right]} \right\} \right.$$
$$\left. + kWl(p_j h + \sin p_j h \cos p_j h) \right].$$

Both the equations (5.2) and the residues above can be further simplified by using the fact that

$$k(p_j h + \sin p_j h \cos p_j h) + 2p_j^3 \sin^2 p_j h = 0 \qquad (j = 1, 2, \ldots).$$

The residue of $\bar{f}(p) e^{px}$ at $-\bar{p}_j$ is clearly the complex conjugate of that at $-p_j$ and hence their sum is given by twice the real part of the above expression. The inverse transform $f(x)$ is then the sum of the residues at the poles in the left-hand half-plane re $p < 0$.

Further, to find the vertical displacement v at $y = h$, elimination of $\bar{f}(p)$ from (2.3) and (4.3) shows that, after the equations (5.2) have been satisfied, $\bar{v}(p, h)$ only has poles at $0, -p_j, -\bar{p}_j$ ($j = 1, 2, \ldots$).
It is easily shown that

(i) the residue of $\bar{v}(p, h)$ at $-p_j$ is $(p_j^4 EI)^{-1}$ times the residue of $\bar{f}(p)$ at $-p_j$;

(ii) the residue of $\bar{v}(p, h)$ at 0 is $-\dfrac{Wh(\lambda+2\mu)}{4\mu(\mu+\lambda)} = -\dfrac{Wh}{kEI}$, the constant displacement existing without the overhanging section of the beam and the conditions at $x = 0$ (§4).

Thus the final expressions for $f(x)$ and $v(x, h)$ are

$$\left.\begin{aligned} f(x) &= 2 \operatorname{re}\left[\sum_j \operatorname{res} \bar{f}(-p_j) e^{-p_j x}\right] \\ v(x, h) &= \frac{2}{EI} \operatorname{re}\left[\sum_j \frac{\operatorname{res} \bar{f}(-p_j)}{p_j^4} e^{-p_j x}\right] - \frac{Wh}{kEI} \end{aligned}\right\} \quad (5.3)$$

An immediate observation here is that for small displacements, EI must be large with $kEI \gg W$.

6. Numerical Work

The roots of the equation

$$k(ph+\sin ph \cos ph)+2p^3 \sin^2 ph = 0$$

were calculated by Newton's formula. Three values of the dimensionless parameters kh^3 were considered and in each case the four roots in the first quadrant with the lowest real parts were found. The values of ph were as follows:

$kh^3 = 0{\cdot}1$	$kh^3 = 1$	$kh^3 = 10$
$0{\cdot}39744+i0{\cdot}39740$	$0{\cdot}70404+i0{\cdot}70263$	$1{\cdot}23275+i1{\cdot}17759$
$3{\cdot}14241+i0{\cdot}07110$	$3{\cdot}14922+i0{\cdot}22315$	$3{\cdot}19454+i0{\cdot}67105$
$6{\cdot}2833\ +i0{\cdot}03557$	$6{\cdot}28418+i0{\cdot}11230$	$6{\cdot}29180+i0{\cdot}34883$
$9{\cdot}4246\ +i0{\cdot}0237$	$9{\cdot}42507+i0{\cdot}07495$	$9{\cdot}42755+i0{\cdot}23508$

An idea of where the roots lie in the complex plane can be obtained by order of magnitude considerations. When $(kh^3)^{\frac{1}{4}}$ is of order 1, there is a root approximating to $p = \left(\dfrac{k}{h}\right)^{\frac{1}{4}} e^{i\frac{\pi}{4}}$. As $|ph|$ increases, sin ph must become small in order to compensate for the $(ph)^3$ factor, i.e. the roots are close to the multiples of π. This suggests the possibility of a series approximation to the roots in powers of $\frac{1}{n\pi}$ and it is easy to obtain by the usual methods

$$ph = n\pi + \frac{i}{n\pi}\sqrt{\frac{kh^3}{2}} + \frac{kh^3}{4n^3\pi^3} - \frac{i}{12\sqrt{2}}\left(\frac{\sqrt{kh^3}}{n\pi}\right)^3.$$

For $n = 1, 2, 3$, this formula gives

$kh^3 = 0{\cdot}1$	$kh^3 = 1$	$kh^3 = 10$
$3{\cdot}14240+i0{\cdot}07116$	$3{\cdot}14965+i0{\cdot}22318$	$3{\cdot}22222+i0{\cdot}65166$
$6{\cdot}28328+i0{\cdot}03558$	$6{\cdot}28419+i0{\cdot}11230$	$6{\cdot}29326+i0{\cdot}34837$
$9{\cdot}42480+i0{\cdot}02372$	$9{\cdot}42507+i0{\cdot}07496$	$9{\cdot}42776+i0{\cdot}23502$

The agreement with the known roots is remarkable, the only big errors being in the case $n = 1$, $kh^3 = 10$ when the series for ph does not converge quickly. Certainly, further roots can be written down with ease.

It will be seen that, in each case, for the first root the factor $(p_j h)^4$ in the terms of $v(x, h)$ (eqn. (5.3)) is close to the negative real axis, whilst for the remaining roots it is near the positive real axis. The contributions to the change of stress $f(x)$ and the convergence $-v(x, h)$ from the residues at the first pole are, because of the exponential factor, dominant except near $x = 0$ and, since the first pole is approximately $\left(\frac{k}{h}\right)^{\frac{1}{4}} e^{i\frac{\pi}{4}}$, are almost related by the factor h/kEI, the same as that appearing in the convergence term due to the heavy beam along (eqn. (5.3)). Thus these leading terms are almost the same as those obtained in the simple model outlined in the introduction in which an elastic foundation is considered. The small deviation from direct proportionality and the existence of further terms, in which corresponding terms in $f(x)$ and the divergence $v(x, h)$ are almost proportional, is the result of using the full equations of elasticity and imposing the end conditions on the elastic strip, something which the simple model cannot do.

For numerical calculations, only a finite number of equations in the system (5.2) can be used with a corresponding number of unknowns A_m, B_m. The calculation of the coefficients is laborious and can only practically be done on a desk calculating machine. Taking the parameter kh^3 to be unity and the ratio l/h to be 1, 2 and 4 in turn, 12 equations of (5.2) were solved simultaneously on a digital computer with the following results:

$$\frac{f(x)}{W} = e^{-0.70404x/h}\left[\begin{Bmatrix}1.802\\4.570\\13.007\end{Bmatrix}\cos 0.70263x/h - \begin{Bmatrix}0.101\\1.176\\6.248\end{Bmatrix}\sin 0.70263x/h\right]$$
$$-e^{-3.14922x/h}\left[\begin{Bmatrix}0.637\\1.642\\5.102\end{Bmatrix}\cos 0.22315x/h - \begin{Bmatrix}4.241\\9.954\\28.162\end{Bmatrix}\sin 0.22315x/h\right]$$
$$-e^{-6.28418x/h}\left[\begin{Bmatrix}0.389\\0.840\\2.657\end{Bmatrix}\cos 0.11230x/h - \begin{Bmatrix}14.854\\35.836\\104.75\end{Bmatrix}\sin 0.11230x/h\right]$$
$$+e^{-9.42507x/h}\left[\begin{Bmatrix}0.015\\0.053\\0.196\end{Bmatrix}\cos 0.07495x/h - \begin{Bmatrix}1.732\\6.302\\24.017\end{Bmatrix}\sin 0.07495x/h\right]$$
$$+e^{-12.56687x/h}\left[\begin{Bmatrix}0.008\\0.029\\-0.107\end{Bmatrix}\cos 0.05615x/h - \begin{Bmatrix}1.718\\6.050\\-24.87\end{Bmatrix}\sin 0.05615x/h\right]$$
$$+e^{-15.70803x/h}\left[\begin{Bmatrix}0.005\\0.018\\0.067\end{Bmatrix}\cos 0.04500x/h - \begin{Bmatrix}1.503\\5.798\\22.88\end{Bmatrix}\sin 0.04500x/h\right]$$

where the choice of three in each bracket corresponds to the values 1, 2 and 4 respectively of the ratio l/h. Also solved on the computer to test the convergence of the process were eight equations of (5.2), giving

$$\frac{f(x)}{W} = e^{-0.70404x/h}\left[\begin{Bmatrix}1.803\\4.572\\13.010\end{Bmatrix}\cos 0.70263x/h - \begin{Bmatrix}0.101\\1.176\\6.2475\end{Bmatrix}\sin 0.70263x/h\right]$$
$$-e^{-3.14922x/h}\left[\begin{Bmatrix}0.728\\1.883\\5.478\end{Bmatrix}\cos 0.22315x/h - \begin{Bmatrix}4.869\\11.619\\30.76\end{Bmatrix}\sin 0.22315x/h\right]$$
$$-e^{-6.28418x/h}\left[\begin{Bmatrix}0.326\\0.798\\2.102\end{Bmatrix}\cos 0.11230x/h - \begin{Bmatrix}18.46\\44.91\\121.77\end{Bmatrix}\sin 0.11230x/h\right]$$
$$+e^{-9.42507x/h}\left[\begin{Bmatrix}0.015\\0.052\\0.193\end{Bmatrix}\cos 0.07495x/h - \begin{Bmatrix}1.715\\6.268\\23.89\end{Bmatrix}\sin 0.07495x/h\right].$$

In both cases, the computations were made to 6 sig. fig. and the results have been rounded off; the exponential factors are given more accurately as they were not involved in the calculations. It will be noticed that agreement between the first terms is, for each value of l/h, well within 0·1% whereas, in the second and third terms, discrepancies of up to 20% occur. However, due to the much lower exponential decay, the first term is dominant in $f(x)$ except near $x = 0$ and, therefore, it would appear that the stress distribution can be accurately determined from a small number of equations of (5.2), unless the stress near $x = 0$ is required in which case certainly more than 12 equations of (5.2) must be retained in the computations. The assumptions of plane strain and small displacements probably break down in this latter region anyway.

The corresponding results for the convergence $-v(x, h)$ are obtained from (5.3); the decay of successive terms is heightened by the factor p_j^4 and with $kh^3 = 1$, the condition $kEI \gg W$ becomes $\left(\frac{Wh^3}{EI}\right) \ll 1$. Thus, with 12 equations of (5.2)

$$\frac{v(x, h)}{h}\left(\frac{Wh^3}{EI}\right)^{-1}$$

$$= -1-e^{-0\cdot70404x/h}\left[\begin{Bmatrix} 1\cdot841 \\ 4\cdot674 \\ 13\cdot314 \end{Bmatrix} \cos 0\cdot70263x/h - \begin{Bmatrix} 0\cdot096 \\ 1\cdot183 \\ 6\cdot330 \end{Bmatrix} \sin 0\cdot70263x/h\right]$$

$$+\frac{1}{10}e^{-3\cdot14922x/h}\left[\begin{Bmatrix} 0\cdot058 \\ 0\cdot121 \\ 0\cdot298 \end{Bmatrix} \cos 0\cdot22315x/h + \begin{Bmatrix} 0\cdot428 \\ 1\cdot008 \\ 2\cdot865 \end{Bmatrix} \sin 0\cdot22315x/h\right]$$

$$+\frac{1}{100}e^{-6\cdot28418x/h}\left[\begin{Bmatrix} 0\cdot043 \\ 0\cdot110 \\ 0\cdot3095 \end{Bmatrix} \cos 0\cdot11230x/h + \begin{Bmatrix} 0\cdot951 \\ 2\cdot294 \\ 6\cdot707 \end{Bmatrix} \sin 0\cdot11230x/h\right]$$

$$-\frac{1}{1000}e^{-9\cdot42507x/h}\left[\begin{Bmatrix} 0\cdot005 \\ 0\cdot019 \\ 0\cdot072 \end{Bmatrix} \cos 0\cdot07495x/h + \begin{Bmatrix} 0\cdot203 \\ 0\cdot798 \\ 3\cdot042 \end{Bmatrix} \sin 0\cdot07495x/h\right]$$

$$-\frac{1}{10000}e^{-12\cdot56687x/h}\left[\begin{Bmatrix} 0\cdot009 \\ 0\cdot032 \\ -0\cdot221 \end{Bmatrix} \cos 0\cdot05615x/h + \begin{Bmatrix} 0\cdot689 \\ 2\cdot426 \\ -9\cdot890 \end{Bmatrix} \sin 0\cdot05615x/h\right]$$

$$-\frac{1}{100000}e^{-15\cdot70803x/h}\left[\begin{Bmatrix} 0\cdot020 \\ 0\cdot080 \\ 0\cdot320 \end{Bmatrix} \cos 0\cdot04500x/h + \begin{Bmatrix} 2\cdot469 \\ 9\cdot524 \\ 37\cdot577 \end{Bmatrix} \sin 0\cdot04500x/h\right].$$

With 8 equations of (5.2), the result is

$$\frac{v(x, h)}{h}\left(\frac{Wh^3}{EI}\right)^{-1}$$

$$= -1-e^{-0\cdot70404x/h}\left[\begin{Bmatrix} 1\cdot842 \\ 4\cdot676 \\ 13\cdot317 \end{Bmatrix} \cos 0\cdot70263x/h - \begin{Bmatrix} 0\cdot096 \\ 1\cdot183 \\ 6\cdot329 \end{Bmatrix} \sin 0\cdot70263x/h\right]$$

$$+\frac{1}{10}e^{-3\cdot14922x/h}\left[\begin{Bmatrix} 0\cdot066 \\ 0\cdot144 \\ 0\cdot335 \end{Bmatrix} \cos 0\cdot22315x/h + \begin{Bmatrix} 0\cdot491 \\ 1\cdot176 \\ 3\cdot127 \end{Bmatrix} \sin 0\cdot22315x/h\right]$$

$$+\frac{1}{100}e^{-6\cdot28418x/h}\left[\begin{Bmatrix} 0\cdot064 \\ 0\cdot155 \\ 0\cdot423 \end{Bmatrix} \cos 0\cdot11230x/h + \begin{Bmatrix} 1\cdot181 \\ 2\cdot874 \\ 7\cdot793 \end{Bmatrix} \sin 0\cdot11230x/h\right]$$

$$-\frac{1}{1000}e^{-9\cdot42507x/h}\left[\begin{Bmatrix} 0\cdot005 \\ 0\cdot019 \\ 0\cdot072 \end{Bmatrix} \cos 0\cdot07495x/h + \begin{Bmatrix} 0\cdot217 \\ 0\cdot794 \\ 3\cdot026 \end{Bmatrix} \sin 0\cdot07495x/h\right].$$

The discrepancies between corresponding terms are the same as for $f(x)$; however, the factors 10^{-1} and 10^{-2} in the second and third terms mean that the percentage variations in the total displacement are correspondingly smaller, whilst the factors 10^{-4} and 10^{-5} in the fifth and sixth terms reduce them to within the accuracy of the computations and they, therefore, cannot be justifiably retained.

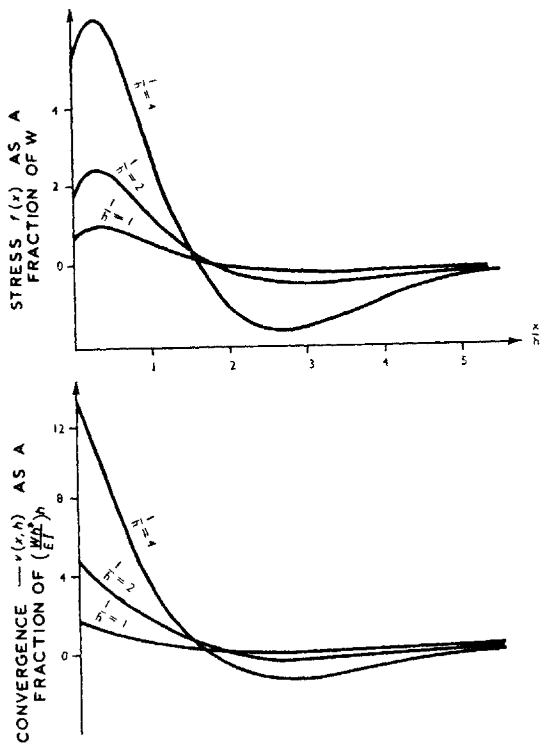

FIG. 1

The results are shown graphically in Fig. 1, from which it can be observed that

(i) the maximum stress $f(x)$ does not occur at $x = 0$ as in the simpler model;

(ii) for $l/h = 1, 2$, $f(x) > -W$ for all x and, therefore, for the particular value of kh^3 chosen, the beam and strip remain in contact as assumed. However, the assumption breaks down when $l/h = 4$, indicating that a less rigid beam, i.e. higher value of kh^3, is then appropriate.

Considering the time taken to do the above computations, it did not seem worth while to repeat them for other values of kh^3.

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