A CLASS OF INTEGRAL TRANSFORMS

by P. ANANDANI, *Department of Mathematics, Holkar Science College, Indore*

(Communicated by P. L. Bhatnagar, F.N.I.)

(*Received 24 October 1967*)

In this paper we discuss a new class of integral transforms and their inversion formula. The kernel in the transform is H-function and integration is performed with respect to the argument of that function. In the inversion formula, the kernel is likewise H-function, but there integration is performed with respect to a parameter.

§ 1. Fox (1961, p. 408) introduced the H-function in the form of Mellin-Barnes type integral, which has been symbolically denoted by Gupta and Jain (*unpublished data*),

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_T\frac{\prod\limits_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod\limits_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\xi)\prod\limits_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)}\,x^{\xi}\,d\xi \qquad (1.1)$$

where $\{(f_r,\gamma_r)\}$ stands for the set of the parameters $(f_1,\gamma_1),\ldots,(f_r,\gamma_r)$; x is not equal to zero and empty product is interpreted as unity; p, q, m and n are integers satisfying $1\leqslant m\leqslant q$, $0\leqslant n\leqslant p$; α_j $(j=1,2,\ldots,p)$, β_j $(j=1,2,\ldots,q)$ are positive numbers and a_j $(j=1,2,\ldots,p)$; b_j $(j=1,2,\ldots,q)$ are complex numbers such that no pole of $\Gamma(b_h-\beta_h s)$ $(h=1,2,\ldots,m)$ coincides with any pole of $\Gamma(1-a_i+\alpha_i s)$ $(i=1,2,\ldots,n)$, i.e.

$$\alpha_i(b_h+\gamma)\neq\beta_h(a_i-\eta-1)$$
$$(\gamma,\eta=0,1,\ldots;\ h=1,2,\ldots m;\ i=1,2,\ldots,n). \qquad \ldots \quad (1.2)$$

According to Braaksma (1963, p. 278), H-function makes sense and defines an analytic function of x in the following two cases:

(a) If $\mu>0$, $x\neq 0$ where

$$\mu=\sum_1^q(\beta_j)-\sum_1^p(\alpha_j). \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.3)$$

(b) If $\mu=0$, $0<|x|<\delta^{-1}$ where $\delta=\prod\limits_{j=1}^{p}(\alpha_j)^{\alpha_j}\prod\limits_{j=1}^{q}(\beta_j)^{-\beta_j}$.

From eqn. (6.5), given by Braaksma (1963), we have

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right.\right]=O(|x|^{\alpha})\ \text{ for small } x, \qquad \ldots \quad \ldots \quad (1.4)$$

where

$$\mu \geqslant 0 \text{ and } \alpha = \text{Re}\left(\frac{b_h}{\beta_h}\right) \quad (h = 1, 2, \ldots, m).$$

From (2.16), given by Braaksma (1963), we get

$$H_{p,q}^{m,n}\left[x \middle| \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix}\right] = O(|x|^{\rho}) \text{ for large } x, \qquad \ldots \quad (1.5)$$

where

$$\mu > 0, \; \sum_1^n (\alpha_j) - \sum_{n+1}^p (\alpha_j) + \sum_1^m (\beta_j) - \sum_{m+1}^q (\beta_j) \equiv \lambda > 0,$$

$$|\arg x| < \tfrac{1}{2}\lambda\pi \text{ and } \rho = \text{Re}\left(\frac{a_i - 1}{\alpha_i}\right) \quad (i = 1, 2, \ldots, n).$$

The Mellin transform

$$g(s) = \mathcal{M}_s\{f(x)\} = \int_0^\infty x^{s-1} f(x)\, dx \qquad \ldots \quad (1.6)$$

and its inversion formula

$$f(x) = \mathcal{M}_x^{-1}\{g(s)\} = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} x^{-s} g(s)\, ds \qquad \ldots \quad (1.7)$$

will be used constantly.

§ 2. In this section, we establish two lemmas which are required in the development of the present work.

Lemma 1—Let

(i) $1 \leqslant m \leqslant q, \; 0 \leqslant n \leqslant p,$

(ii) $\text{Re}\left(s + \frac{b_j}{\beta_j}\right) > 0 \; (j = 1, 2, \ldots, m), \; \text{Re}\left(s + \frac{a_i - 1}{\alpha_i}\right) < 0 \; (i = 1, 2, \ldots, n),$

$\text{Re}\left(s - \frac{b+c}{2\beta}\right) < 0,$

(iii) $\sum_1^m (\beta_j) - \sum_{m+1}^q (\beta_j) + \sum_1^n (\alpha_j) - \sum_{n+1}^p (\alpha_j) + 2\beta \equiv \lambda > 0, \; |\arg x| < \tfrac{1}{2}\lambda\pi,$

then

$$\mathcal{M}_s\left\{H_{p+2,q}^{m,n+2}\left[x \middle| \begin{matrix} \left(1 - \frac{b+c}{2}, \beta\right), \left(\frac{1}{2} - \frac{b+c}{2}, \beta\right), \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix}\right]\right\}$$

$$= \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j s) \prod_{j=1}^n \Gamma(1 - a_j - \alpha_j s)\Gamma\left(\frac{b+c}{2} - \beta s\right)\Gamma\left(\frac{1}{2} + \frac{b+c}{2} - \beta s\right)}{\prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j s) \prod_{j=n+1}^p \Gamma(a_j + \alpha_j s)} \quad (2.1)$$

which follows easily from Fox (1961, p. 408).

Lemma 2—Let

(i) $1 \leqslant m \leqslant q,\ 0 \leqslant n \leqslant p,\ \sum_{1}^{m}(\beta_j) - \sum_{m+1}^{q}(\beta_j) + \sum_{1}^{n}(\alpha_j) - \sum_{n+1}^{p}(\alpha_j) + 2\beta \equiv \lambda > 0,$

(ii) $|\arg z| < \frac{1}{2}\lambda\pi,$

(iii) $\operatorname{Re}\left(\frac{b+c}{2}\right) > -\delta,\ \operatorname{Re}\beta\left(\frac{a_i-1}{\alpha_i}\right) < \delta,\ (i = 1, 2, \ldots, n),$

$\operatorname{Re}\beta(b_j/\beta_j) > \delta\ \ (j = 1, 2, \ldots, m),\ \operatorname{Re}(\nu+s) > -\delta,$

$\operatorname{Re}(b+c-\nu-s) > -\delta$ where $\beta\sigma = \delta$ and $\beta > 0$,

then

$$\frac{2^{b+c-1}}{\sqrt{\pi}} \mathcal{M}_s \left\{ \frac{x^\nu}{(1+x)^{b+c}} H^{m,\,n+2}_{p+2,\,q}\left[\frac{z(4x)^\beta}{(1+x)^{2\beta}} \middle| \begin{matrix} \left(1-\frac{b+c}{2}, \beta\right), \left(\frac{1}{2}-\frac{b+c}{2}, \beta\right), \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix} \right] \right\}$$

$$= H^{m,\,n+2}_{p+2,\,q}\left[z \middle| \begin{matrix} (1-\nu-s, \beta), (1-b-c+\nu+s, \beta), \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix} \right]. \quad \ldots \quad (2.2)$$

Proof—On the left-hand side using the representations (1.6) and (1.1); interchanging the order of integration, which is permissible due to the absolute convergence of the integrals involved in the process, evaluating the inner integral with the help of (8) of Erdelyi *et al.* (1954, p. 233), with $y = 1$ and then using Legendre's duplication formula (Rainville 1960, p. 26) and (1.1), the definition of the H-function, we get the result.

§ 3. The transform pair to be established is:

$$g(w) = \int_0^\infty H^{m,\,n+2}_{p+2,\,q}\left[t \middle| \begin{matrix} 1-\nu-iw, \beta), (1-b-c+\nu+iw, \beta), \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix} \right] f(t)\, dt, \quad \ldots \quad (3.1)$$

$$f(w) = \frac{\beta e^{\pi i(\beta-\nu-1)}}{\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{-\pi i u}\, u\, g(-iu)\, \Omega(we^{\beta\pi i})\, du$$

$$+ \frac{(b+c-2\nu)\beta e^{\pi i(\beta-\nu)}}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{-\pi i u} g(-iu)\, \Omega(we^{\beta\pi i})\, du \quad \ldots \quad \ldots \quad (3.2)$$

where

$$\Omega(we^{\beta\pi i}) = H^{q-m,\,p-n+1}_{p+2,\,q}\left[we^{\beta\pi i} \middle| \begin{matrix} (1-a_{n+1}-\alpha_{n+1}, \alpha_{n+1}), \ldots, (1-a_p-\alpha_p, \alpha_p), \\ (1-b_{m+1}-\beta_{m+1}, \beta_{m+1}), \ldots, (1-b_q-\beta_q, \beta_q), \end{matrix} \right.$$
$$\left. \begin{matrix} (1-\beta+\nu+u, \beta), (1-a_1-\alpha_1, \alpha_1), \ldots, (1-a_n-\alpha_n, \alpha_n), (1-\beta+b+c-\nu-u, \beta) \\ (1-b_1-\beta_1, \beta_1), \ldots, (1-b_m-\beta_m, \beta_m) \end{matrix} \right] \quad \ldots \quad (3.3)$$

and the conditions on the parameters are those of Lemma 2.

PROOF—In (3.1), putting $w = -iu$, multiplying both the sides by x^{-u} and integrating with respect to u from $\sigma - i\infty$ to $\sigma + i\infty$, we have

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} g(-iu)x^{-u}\,du$$

$$= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left\{\int_0^\infty H^{m,\,n+2}_{p+2,\,q}\left[t\,\middle|\,\begin{matrix}(1-\nu-u,\beta),\,(1-b-c+\nu+u,\beta),\,\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right] f(t)\,dt\right\} x^{-u}\,du \quad \ldots \quad (3.4)$$

on the right-hand side of (3.4), changing the order of integration, evaluating the inner integral with the help of (2.2) and taking

$$\frac{(4x)^\beta}{(1+x)^{2\beta}} = y. \text{ i.e. } x = y^{-\frac{1}{\beta}}\left\{1-\left(1-y^{\frac{1}{\beta}}\right)^{\frac{1}{2}2}\right\}, \quad \ldots \quad \ldots \quad (3.5)$$

we have

$$\frac{2\sqrt{\pi}\,y^{\frac{1}{\beta}(\nu-b-c)}}{\left[1-\left(1-y^{\frac{1}{\beta}}\right)^{\frac{1}{2}}\right]^{2\nu-b-c}}\bar{g}(x)$$

$$= \int_0^\infty H^{m,\,n+2}_{p+2,\,q}\left[yt\,\middle|\,\begin{matrix}\left(1-\frac{b+c}{2},\beta\right),\left(\frac{1}{2}-\frac{b+c}{2},\beta\right),\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right] f(t)\,dt, \qquad (3.6)$$

where

$$\bar{g}(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} g(-iu)\,x^{-u}\,du. \quad \ldots \quad \ldots \quad \ldots \quad (3.7)$$

Using the notation of Titchmarsh (1962, p. 315), we have

$$\mathscr{G}(s) = 2\sqrt{\pi}\int_0^\infty \frac{y^{\frac{1}{\beta}(\nu-b-c)+s-1}}{\left[1-\left(1-y^{\frac{1}{\beta}}\right)^{\frac{1}{2}}\right]^{2\nu-b-c}}\bar{g}(x)\,dy$$

$$= \frac{\beta\sqrt{\pi}}{2^{b+c-1-2\beta s}}\int_0^{-1}(1-x)\,x^{\beta s-\nu-1}(1+x)^{b+c-1-2\beta s}\,\bar{g}(x)\,dx. \quad \ldots \quad (3.8)$$

Now substituting the value of $\bar{g}(x)$ from (3.7), changing the order of integration, replacing s by $1-s$ and evaluating the inner integral (Erdelyi *et al.* 1953, p. 9), we have

$$\mathcal{G}(1-s) = \frac{\beta e^{\pi i(\beta-\beta s-\nu-1)}\Gamma\left(\frac{b+c}{2}-\beta+\beta s\right)\Gamma\left(\frac{1}{2}+\frac{b+c}{2}-\beta-\beta s\right)}{\pi i}$$

$$\times \int_{\sigma-i\infty}^{\sigma+i\infty} e^{-\pi i u}\, u\, g(-iu) \frac{\Gamma(\beta-\beta s-\nu-u)}{\Gamma(b+c-\nu-\beta-u+\beta s+1)}\, du$$

$$+\frac{\beta e^{\pi i(\beta-\beta s-\nu)}\Gamma\left(\frac{b+c}{2}-\beta+\beta s\right)\Gamma\left(\frac{1}{2}+\frac{b+c}{2}-\beta+\beta s\right)\Gamma(b+c-2\nu)}{2\pi i}$$

$$\times \int_{\sigma-i\infty}^{\sigma+i\infty} e^{-\pi i u}\, g(-iu) \frac{\Gamma(\beta-\beta s-\nu-u)}{\Gamma(b+c-\nu-u-\beta+1+\beta s)}\, du. \quad \ldots \quad (3.9)$$

Also from (2.1)

$$\mathcal{R}(1-s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j+\alpha_j s)\Gamma\left(\frac{b+c}{2}-\beta+\beta s\right)\Gamma\left(\frac{1}{2}+\frac{b+c}{2}-\beta+\beta s\right)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j-\alpha_j s)}. \quad \ldots \quad (3.10)$$

Now, by using the notation of Titchmarsh (1962, p. 316),

$$f(w) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\mathcal{G}(1-s)}{\mathcal{R}(1-s)}\, w^{-s}\, ds. \quad \ldots \quad \ldots \quad (3.11)$$

Using (3.9), (3.10) and then changing the order of integrations, we have

$$f(w) = \frac{\beta e^{\pi i(\beta-\nu-1)}}{\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{-\pi i u}\, u\, g(-iu)$$

$$\times \mathcal{M}^{-1}_{we^{\beta\pi i}}\left\{\frac{\prod_{m+1}^{q}\Gamma(1-b_j-\beta_j+\beta_j s)\prod_{n+1}^{p}\Gamma(a_j+\alpha_j-\alpha_j s)\Gamma(\beta-\beta s-\nu-u)}{\prod_{1}^{m}\Gamma(b_j+\beta_j-\beta_j s)\prod_{1}^{n}\Gamma(1-a_j-\alpha_j+\alpha_j s)\Gamma(b+c-\nu-u-\beta+1+\beta s)}\right\} du$$

$$+\frac{(b+c-2\nu)\beta e^{\pi i(\beta-\nu)}}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{-\pi i u}\, g(-iu)$$

$$\times \mathcal{M}^{-1}_{we^{\beta\pi i}}\left\{\frac{\prod_{m+1}^{q}\Gamma(1-b_j-\beta_j+\beta_j s)\prod_{n+1}^{p}\Gamma(a_j+\alpha_j-\alpha_j s)\Gamma(\beta-\beta s-\nu-u)}{\prod_{1}^{m}\Gamma(b_j+\beta_j-\beta_j s)\prod_{1}^{n}\Gamma(1-a_j-\alpha_j+\alpha_j s)\Gamma(b+c-\nu-u-\beta+1+\beta s)}\right\} du. \quad \ldots \quad (3.12)$$

Hence, using (1.7) and notation of Fox (1961, p. 408), we get the result (3.2).

§ 4. *Applications*—(i) Setting $\alpha_j = \beta_h = \beta$ ($j = 1, 2, \ldots, p$; $h = 1, 2, \ldots, q$) in (3.1) and (3.2) and using the relation

$$H^{m,n}_{p,q}\left[x\,\middle|\,\begin{matrix}\{(a_p, \beta)\}\\ \{(b_q, \beta)\}\end{matrix}\right] = \frac{1}{\beta} G^{m,n}_{p,q}\left(x^{\frac{1}{\beta}}\,\middle|\,\begin{matrix}a_1, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\right) \quad \ldots \quad \ldots \quad (4.1)$$

where $\beta > 0$, we get the transform pair:

$$g(w) = \frac{1}{\beta}\int_0^\infty G^{m,\, n+2}_{p+2,\, q}\left(t^{\frac{1}{\beta}}\middle|\begin{matrix}1-\nu-iw,\ 1-b-c+\nu+iw,\ a_1,\ \ldots,\ a_p\\ b_1,\ \ldots,\ b_q\end{matrix}\right)f(t)\,dt \quad .. \quad (4.2)$$

$$f(w) = \frac{e^{\pi i(\beta-\nu-1)}}{\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{-\pi iu}\, u\, g(-iu) G^{q-m,\, p-n+1}_{p+2,\, q}\left(w^{\frac{1}{\beta}}e^{\pi i}\middle|\begin{matrix}1-a_{n+1}-\beta,\ \ldots,\ 1-a_p-\beta,\\ 1-b_{m+1}-\beta,\ \ldots,\ 1-b_q-\beta,\end{matrix}\right.$$
$$\left.\begin{matrix}1-\beta+\nu+u,\ 1-a_1-\beta,\ldots,\ 1-a_n-\beta,\ 1-\beta+b+c-\nu-u\\ 1-b_1-\beta,\ \ldots,\ 1-b_m-\beta\end{matrix}\right)du$$
$$+\frac{(b+c-2\nu)e^{\pi i(\beta-\nu)}}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{-\pi iu} g(-iu) G^{q-m,\, p-n+1}_{p+2,\, q}\left(w^{\frac{1}{\beta}}e^{\pi i}\middle|\begin{matrix}1-a_{n+1}-\beta,\ \ldots,\ 1-a_p-\beta,\\ 1-b_{m+1}-\beta,\ \ldots,\ 1-b_q-\beta,\end{matrix}\right.$$
$$\left.\begin{matrix}1-\beta+\nu+u,\ 1-a_1-\beta,\ \ldots,\ 1-a_n-\beta,\ 1-\beta+b+c-\nu-u\\ 1-b_1-\beta,\ \ldots,\ 1-b_m-\beta\end{matrix}\right)du. \quad .. \quad (4.3)$$

(ii) In (4.2) and (4.3), taking $\beta = 1$, we get a known pair [Bajpai (*in press*)].

(iii) In (4.2) and (4.3), putting $b+c=2\nu$ and $\beta=1$, we obtain the transform pair due to Wimp (1964-65).

(iv) With $b+c = 2\nu$, $\sigma = 0$, $u = iz$ and in view of $g(x) = g(-x)$ from (4.2) and (4.3), we have the transform pair:

$$g(w) = \frac{1}{\beta}\int_0^\infty G^{m,\, n+2}_{p+2,\, q}\left(t^{\frac{1}{\beta}}\middle|\begin{matrix}1-\nu-iw,\ 1-\nu+iw,\ a_1,\ \ldots,\ a_p\\ b_1,\ \ldots,\ b_q\end{matrix}\right)f(t)\,dt, \quad .. \quad (4.4)$$

$$f(w) = \frac{ie^{\pi i(\beta-\nu-1)}}{\pi}\int_0^\infty ze^{\pi z}\Lambda\left(w^{\frac{1}{\beta}}e^{\pi i}\middle|\, 1-\beta+\nu+iz,\ 1-\beta+\nu-iz\right)g(z)\,dz$$
$$-\frac{ie^{\pi i(\beta-\nu-1)}}{\pi}\int_0^\infty ze^{-\pi z}\Lambda\left(w^{\frac{1}{\beta}}e^{\pi i}\middle|\, 1-\beta+\nu-iz,\ 1-\beta+\nu+iz\right)g(z)\,dz, \quad .. \quad (4.5)$$

where

$$\Lambda(\tau\,|\,r_1,\ r_2) = G^{q-m,\, p-n+1}_{p+2,\, q}\left(\tau\middle|\begin{matrix}1-a_{n+1}-\beta,\ \ldots,\ 1-a_p-\beta,\\ 1-b_{m+1}-\beta,\ \ldots,\ 1-b_q-\beta,\end{matrix}\right.$$
$$\left.\begin{matrix}r_1,\ 1-a_1-\beta,\ \ldots,\ 1-a_n-\beta,\ r_2\\ 1-b_1-\beta,\ \ldots,\ 1-b_m-\beta\end{matrix}\right). \quad .. \quad (4.6)$$

(v) In (4.4) and (4.5), putting $n = p = 0$, $m = q = 1$, $b_1 = \frac{1}{2}-\nu-k$, using eqns. (9) and (8) given by Erdelyi *et al.* (1953, pp. 209 and 216 resp.) and

$$G^{1,\, 0}_{1,\, 2}\left(w\middle|\begin{matrix}\alpha\\ \gamma,\ \delta\end{matrix}\right) = \frac{e^{\frac{w}{2}}w^{\frac{\delta+\gamma-1}{2}}}{\Gamma(1+\gamma-\delta)\Gamma(\alpha-\gamma)} M_{\alpha-\frac{\gamma+\delta+1}{2},\ \frac{\gamma-\delta}{2}}(w) \quad .. \quad (4.7)$$

$$M_{k,\,-m}(xe^{-\pi i}) = e^{(m-\frac{1}{2})\pi i}M_{-k,\,-m}(x) \quad .. \quad .. \quad .. \quad .. \quad (4.8)$$

and (1.7) of Slater (1960), i.e.

$$W_{k,m}(x)=\frac{\pi}{\sin 2m\pi}\left\{\frac{M_{k,-m}(x)}{\Gamma(\frac{1}{2}+m-k)\Gamma(1-2m)}-\frac{M_{k,m}(x)}{\Gamma(\frac{1}{2}-m-k)\Gamma(1+2m)}\right\}$$

$$=W_{k,-m}(x), \quad \ldots \quad (4.9)$$

also replacing β by $1/\beta$, we arrive at the transform pair:

$$g(w)=\beta\Gamma(\tfrac{1}{2}-k+iw)\Gamma(\tfrac{1}{2}-k-iw)\int_0^\infty e^{\frac{1}{2t^\beta}}\, t^{\frac{\beta}{2}(1-2\nu)}\, W_{k,iw}(t^{-\beta})f(t)\,dt \quad \ldots \quad (4.10)$$

$$f(w)=\frac{e^{-\frac{1}{2w^\beta}}\, w^{\frac{1}{2}\beta(1+2\nu)-1}}{\pi^2}\int_0^\infty z\sinh(2\pi z)W_{k,iz}(w^{-\beta})\,g(z)\,dz. \quad \ldots \quad (4.11)$$

(vi) In (4.10) replace t by $1/t$, in (4.11) replace w by $1/w$ and then replacing $e^{\frac{1}{2}t^\beta}\, t^{\frac{1}{2}\beta(2\nu-1)-2} f(t^{-1})$ by $f(t)$, we have the transform pair:

$$g(w)=\beta\Gamma(\tfrac{1}{2}-k+iw)\Gamma(\tfrac{1}{2}-k-iw)\int_0^\infty W_{k,iw}(t^\beta)\,f(t)\,dt, \quad \ldots \quad (4.12)$$

$$f(w)=\frac{w^{-\beta-1}}{\pi^2}\int_0^\infty z\sinh(2\pi z)W_{k,iz}(w^\beta)\,g(z)\,dz. \quad \ldots \quad (4.13)$$

(vii) Putting $k=0$, using

$$W_{0,m}(x)=\sqrt{\frac{x}{\pi}}K_m\left(\frac{x}{2}\right) \quad \ldots \quad (4.14)$$

and replacing $\frac{t^{\frac{1}{2}\beta}}{\sqrt{\pi}}f(t)$ by $f(t)$ in (4.12) and (4.13), we have the transform pair:

$$g(w)=\beta\Gamma(\tfrac{1}{2}+iw)\Gamma(\tfrac{1}{2}-iw)\int_0^\infty K_{iw}\left(\frac{t^\beta}{2}\right)f(t)\,dt, \quad \ldots \quad (4.15)$$

$$f(w)=\frac{1}{w\pi^3}\int_0^\infty z\sinh(2\pi z)K_{iz}\left(\frac{w^\beta}{2}\right)g(z)\,dz. \quad \ldots \quad (4.16)$$

(viii) In (4.15) and (4.16), setting $\beta=1$, replacing $g(t)$ by $\frac{\pi^3}{t\sinh 2\pi t}g(t)$, $K_{it}\left(\frac{w}{2}\right)$ by $K_{it}(w)$, $f(x)$ by $x^{-1}f(x)$, then using $\sin\pi z=\frac{\pi}{\Gamma(z)\Gamma(1-z)}$, we obtain the Kontorovich-Lebedev transform pair (Erdelyi *et al.* 1954, p. 173):

$$g(w)=\frac{2}{\pi^2}w\sinh\pi w\int_0^\infty t^{-1}K_{iw}(t)f(t)\,dt, \quad \ldots \quad (4.17)$$

$$f(w)=\int_0^\infty K_{iz}(w)\,g(z)\,dz. \quad \ldots \quad (4.18)$$

(ix) In (4.4) and (4.5), taking $n = p = 0$, $m = 1$, $q = 2$, $b_1 = \frac{1}{2}-\nu-\frac{m+n}{2}$, $b_2 = \frac{1}{2}-\nu+\frac{m-n}{2}$; using the relation

$$G_{2,2}^{1,n}\left(x\middle|\begin{matrix}a_1, a_2\\ b_1, b_2\end{matrix}\right) \equiv \frac{\prod_{j=1}^{n} \Gamma(1+b_1-a_j)x^{b_1}}{\Gamma(1+b_1-b_2)\prod_{j=n+1}^{2}\Gamma(a_j-b_1)}\, {}_2F_1\left[\begin{matrix}1+b_1-a_1, 1+b_1-a_2; (-1)^{n+1}x\\ 1+b_1-b_2\end{matrix}\right] \quad .. (4.19)$$

and replacing β by $1/\beta$, we get the transform pair:

$$g(w) = \frac{\beta\Gamma\left(\frac{1}{2}+iw-\frac{m+n}{2}\right)\Gamma\left(\frac{1}{2}-iw-\frac{m+n}{2}\right)}{\Gamma(1-m)} \times \int_0^\infty t^{\frac{1}{2}\beta(1-2\nu-m-n)}\, {}_2F_1\left[\begin{matrix}\frac{1}{2}+iw-\frac{m+n}{2}, \frac{1}{2}-iw-\frac{m+n}{2}; -t^\beta\\ 1-m;\end{matrix}\right] f(t)\,dt, \quad (4.20)$$

$$f(w) = \frac{w^{\frac{\beta}{2}(1+2\nu-m+n)-1}\, e^{\frac{\pi i}{2}(n-m)}}{\pi\Gamma(1-m)} \times \int_0^\infty \left\{\frac{e^{\pi z}\Gamma\left(\frac{1}{2}-iz-\frac{m-n}{2}\right)}{\Gamma\left(\frac{1}{2}-iz+\frac{m-n}{2}\right)} - \frac{e^{-\pi z}\Gamma\left(\frac{1}{2}+iz-\frac{m-n}{2}\right)}{\Gamma\left(\frac{1}{2}+iz+\frac{m-n}{2}\right)}\right\}$$

$$\times\, {}_2F_1\left[\begin{matrix}\frac{1}{2}-iz-\frac{m-n}{2}, \frac{1}{2}+iz-\frac{m-n}{2}; -w^\beta\\ 1-m;\end{matrix}\right] g(z)\,dz. \quad .. \quad .. (4.21)$$

(x) Now using the relation*

$$P_k^{m,n}(z) \equiv \frac{2^n(z+1)^{-\frac{n}{2}}}{\Gamma(1-m)(z-1)^{\frac{m}{2}}}\, {}_2F_1\left[\begin{matrix}-k-\frac{m+n}{2}, 1+k-\frac{m+n}{2}; \frac{1-z}{2}\\ 1-m;\end{matrix}\right] \quad .. (4.22)$$

(with z replaced by $(2t^\beta+1)$ and k replaced by $(iw-\frac{1}{2})$ in (4.20)) and the relation (Kuipers and Meulenbeld 1957, p. 440)

$$P_k^{m,n}(z) = \frac{1}{\Gamma(1-m)}\frac{(z+1)^{\frac{n}{2}}}{(z-1)^{\frac{m}{2}}}\, {}_2F_1\left[\begin{matrix}1+k-\frac{m-n}{2}, -k-\frac{m-n}{2}; \frac{1-z}{2}\\ 1-m;\end{matrix}\right] \quad .. (4.23)$$

* (4.22) follows from (4.23) in view of the transformation

$$F(a, b; c; x) = (1-x)^{c-a-b}F(c-a, c-b; c; x)$$

(with z replaced by $(2w^{\beta}+1)$ and k replaced by $(iz-\frac{1}{2})$ in (4.21)), we get the transform pair:

$$g(w) = 2^{\frac{m-n}{2}}\beta\Gamma\left(\tfrac{1}{2}+iw-\frac{m+n}{2}\right)\Gamma\left(\tfrac{1}{2}-iw-\frac{m+n}{2}\right)\int_0^{\infty} t^{\frac{\beta}{2}(1-2\nu-n)}(1+t^{\beta})^{\frac{n}{2}}$$

$$\times P^{m,n}_{iw-\frac{1}{2}}(2t^{\beta}+1)f(t)\,dt, \quad \ldots \quad (4.24)$$

$$f(w) = \frac{2^{\frac{m-n}{2}} e^{\frac{\pi i}{2}(n-m)} w^{\frac{\beta}{2}(1+2\nu+n)-1}(w^{\beta}+1)^{-\frac{n}{2}}}{\pi}$$

$$\times\int_0^{\infty}\left\{\frac{e^{\pi z}\Gamma\left(\frac{1}{2}-iz-\frac{m-n}{2}\right)}{\Gamma\left(\frac{1}{2}-iz+\frac{m-n}{2}\right)}-\frac{e^{-\pi z}\Gamma\left(\frac{1}{2}+iz-\frac{m-n}{2}\right)}{\Gamma\left(\frac{1}{2}+iz+\frac{m-n}{2}\right)}\right\}$$

$$\times P^{m,n}_{iz-\frac{1}{2}}(2w^{\beta}+1)z\,g(z)\,dz. \quad \ldots \quad (4.25)$$

(xi) With $m = n = k$, (4.24) and (4.25) reduce to the transform pair:

$$g(w) = \beta\Gamma(\tfrac{1}{2}-k+iw)\Gamma(\tfrac{1}{2}-k-iw)\int_0^{\infty} t^{\frac{\beta}{2}(1-2\nu-k)}(1+t^{\beta})^{\frac{k}{2}}P^{k}_{iw-\frac{1}{2}}(2t^{\beta}+1)f(t)\,dt, \quad \ldots \quad (4.26)$$

$$f(w) = \frac{2w^{\frac{1}{2}\beta(2\nu+k+1)-1}(w^{\beta}+1)^{-\frac{k}{2}}}{\pi}\int_0^{\infty} z\sinh \pi z\, P^{k}_{iz-\frac{1}{2}}(2w^{\beta}+1)\,g(z)\,dz. \quad \ldots \quad (4.27)$$

(xii) Now in (4.26) and (4.27) replacing $f(x)$ by $x^{-\frac{1}{2}\beta(1-2\nu-k)}(1+x^{\beta})^{-\frac{k}{2}}f(x)$ and $g(x)$ by $\dfrac{\pi\, g(x)}{2x\sinh \pi x}$, we get the transform pair:

$$g(w) = \frac{2\beta w\sinh \pi w}{\pi}\Gamma(\tfrac{1}{2}-k+iw)\Gamma(\tfrac{1}{2}-k-iw)\int_0^{\infty} P^{k}_{iw-\frac{1}{2}}(2t^{\beta}+1)f(t)\,dt, \quad (4.28)$$

$$f(w) = w^{\beta-1}\int_0^{\infty} P^{k}_{iz-\frac{1}{2}}(2w^{\beta}+1)\,g(z)\,dz. \quad \ldots \quad (4.29)$$

(xiii) Setting $\beta = 1$ in (4.28) and (4.29); in (4.28) replacing $2t+1$ by t; in (4.29) replacing $2w+1$ by w and then replacing $f\left(\frac{x-1}{2}\right)$ by $f(x)$, we obtain the generalized Mehler transform pair (Rosenthal 1961):

$$g(w) = \frac{w\sinh \pi w}{\pi}\Gamma(\tfrac{1}{2}-k+iw)\Gamma(\tfrac{1}{2}-k-iw)\int_1^{\infty} P^{k}_{iw-\frac{1}{2}}(t)f(t)\,dt, \quad \ldots \quad (4.30)$$

$$f(w) = \int_0^{\infty} P^{k}_{iz-\frac{1}{2}}(w)\,g(z)\,dz. \quad \ldots \quad (4.31)$$

Acknowledgement

The author wishes to express his sincere thanks to Dr. R. K. Saxena of G. S. Technological Institute, Indore, for his kind help and guidance during the preparation of this paper.

References

Bajpai, S. D. (*in press*). A class of integral transforms. *La Ricerca*.

Braaksma, B. L. J. (1963). Asymptotic expansions and analytic continuations for a class of Barnes integrals. *Compositio Math.*, **15**, 239–341.

Erdelyi, A. *et al.* (1953). Higher Transcendental Functions. Vol. I. McGraw-Hill Book Co. Inc., New York.

——— (1954). Tables of Integral Transforms. Vol. II. McGraw-Hill Book Co. Inc., New York.

Fox, C. (1961). The *G*- and *H*-functions as symmetrical Fourier kernels. *Trans. Am. math. Soc.*, **98**, 395–429.

Gupta, K. C., and Jain, U. C. (*unpublished data*). The *H*-function—II.

Kuipers, L., and Meulenbeld, B. (1957). On a generalization of Legendre's associated differential equation—I. *Proc. K. Ned. Akad. Wet.*, **60**, 436–443.

Rainville, E. D. (1960). Special Functions. MacMillan, New York.

Rosenthal, P. L. (1961). On a Generalization of Mehler's Inversion Formula. Dissertation, Oregon State University.

Slater, L. T. (1960). Confluent Hypergeometric Functions. Cambridge University Press.

Titchmarsh, E. C. (1962). Introduction to the Theory of Fourier Integrals. Oxford.

Wimp, J. (1964-65). A class of integral transforms. *Proc. Edinb. math. Soc.*, **14**, 33–40.