CONCENTRIC ELLIPTIC INCLUSION IN A CIRCULAR REGION

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The problem of an elliptic inclusion concentrically situated in a circular region is considered in this paper. Previous to this, the inclusions were supposed to be present in an infinite or semi-infinite medium, which considerably simplified the solution. The old methods, e.g. the point-force method used by Eshelby (1957, 1959), Jaswon and Bhargava (1961) and others and the energy methods used by Bhargava and Radhakrishnan (1963), cannot be used, simply because the point-force method needs an infinite medium and the energy methods require a considerable guessing of the equilibrium interface which is not possible in the present problem. However, the theory of the Hilbert problem can be applied. This has been done to provide explicit solution to the problem stated as above. The problem has been generalized for the case when the Poisson's ratios of the two materials are different but shear moduli are the same.

The Problem

Consider a circular ring of outer radius R with concentric elliptic hole. Let a and b be the semi-major and minor axes of the ellipse respectively. This medium has been named as matrix. Suppose an elastic solid called 'inclusion' of dimensions slightly larger than those of the hole but remaining within the limits of proportional elasticity is embedded in the matrix. Because of misfit in size, the stresses would develop both in the matrix and in the inclusion. The evaluation of the elastic field everywhere is the main theme of this paper.

This model can be seen from the other point of view. An elliptic region concentrically situated in a circular plate undergoes spontaneous deformation, which in the absence of matrix would be of a prescribed shape and size. Because of the presence of matrix, stresses would develop both in the inclusion and matrix. It can be seen easily that these two models are mathematically identical and the solution of one can be interpreted for the other.

Such problems were first studied by Frenkel (1946), Mott and Nabarro (1940) in connection with their theory of precipitation hardening of alloys. A systematic investigation was undertaken by Eshelby introducing the point-force concept; and Jaswon and Bhargava and others relating it to two-dimensional problems.

The method for the solution of the present problem is based upon Hilbert's theorem which states that a sectionally holomorphic function can be represented in the form of the sum of a holomorphic function and a Cauchy integral, evaluated along the line of discontinuity, provided the region we are considering is finite (Muskhelishvilli 1953).

Let the outer boundary of the matrix be denoted by L_0 and the inner boundary by L. The centre of L_0 is taken as the origin and is denoted by O. By using the complex variable method of determining the elastic field, the problem is reduced to the determination of two sectionally holomorphic functions $\phi(z)$ and $\psi(z)$ with the line of discontinuity L.

For the purpose of this problem, it is enough to state as follows: Let the region defined as inclusion be named as positive region and that of the matrix as negative region. The holomorphic functions in inclusion and matrix are distinguished by writing + or − superscripts. If $F(z)$ is analytic within the inclusion and matrix and if the discontinuity of $F(z)$ be $f(t)$ along the interface L, i.e. if

$$F^{+}(t)-F^{-}(t) = f(t) \text{ on } L,$$

then

$$F(z) = F^{*}(z)+\frac{1}{2\pi i}\int_{L}\frac{f(t)\,dt}{t-z} \text{ everywhere}$$

where t denotes the boundary point on L and $F^{*}(z)$ is the function analytic in the whole region.

$F^{*}(z)$ is evaluated by the boundary conditions. Necessary results will be given at the appropriate places.

The inclusion in the absence of the matrix undergoes a displacement $(\epsilon_1 x+\gamma_1 y,\ \epsilon_2 y+\gamma_2 x)$ with respect to the origin at O. Let (u^{+}, v^{+}) be the displacement components of the inclusion and (u^{-}, v^{-}) be those of the matrix.

At the equilibrium interface, if the displacement components be denoted by

$$\left(u_b^{+}, v_b^{+}\right) \text{ and } \left(u_b^{-}, v_b^{-}\right),$$

then

$$u_b^{+}-u_b^{-} = -(\epsilon_1 x+\gamma_1 y) = g_1(t),$$

$$v_b^{+}-v_b^{-} = -(\epsilon_2 y+\gamma_2 x) = g_2(t).$$

It may be remarked that $\left(u_b^{+}, v_b^{+}\right)$, the displacement at the boundary of the inclusion is measured from its natural state: the state it would have in the absence of matrix; while $\left(u_b^{-}, v_b^{-}\right)$ is the displacement at the boundary of matrix measured from its natural state, which is the state when the inner hole is elliptic.

We shall write $z = x+iy$. For a point on L, we shall denote z by t.

Let $\phi(z)$ and $\psi(z)$ be the functions sectionally holomorphic having the line of discontinuity L. We write

$$\phi(z) = \phi_0(z)+\phi_*(z)$$

and

$$\psi(z) = \psi_0(z)+\psi_*(z)$$

where $\phi_0(z)$ and $\psi_0(z)$ are the functions analytic in the whole region and $\phi_*(z)$ and $\psi_*(z)$ are defined as follows:

$$\phi_*(z) = \frac{\mu}{\pi i(\kappa+1)}\int_L \frac{g(t)\,dt}{t-z}$$

and

$$\psi_*(z) = \frac{\mu}{\pi i(\kappa+1)}\int_L \frac{h(t)\,dt}{t-z}$$

where

$$g(t) = g_1(t)+i\,g_2(t)$$

$$h(t) = -\overline{g(t)}-\bar{t}\,g'(t)$$

and $\kappa = 3-4\sigma$ in the plane strain case and $\kappa = (3-\sigma)/(1+\sigma)$ for the plane stress case, σ being the Poisson's ratio.

Taking the origin at the centre of L_0, it can be seen that

$$g(t) = -\{\epsilon_1+\epsilon_2-i(\gamma_1-\gamma_2)\}\frac{t}{2}-\{\epsilon_1-\epsilon_2+i(\gamma_1+\gamma_2)\}\frac{\bar{t}}{2},$$

$$h(t) = (\epsilon_1+\epsilon_2)\bar{t}+\{\epsilon_1-\epsilon_2-i(\gamma_1+\gamma_2)\}\frac{t}{2}+\{\epsilon_1-\epsilon_2+i(\gamma_1+\gamma_2)\}\frac{\bar{t}}{2}\frac{d\bar{t}}{dt}.$$

So that

$$\phi_*(z) = -\frac{\mu}{(\kappa+1)}\{\epsilon_1+\epsilon_2-i(\gamma_1-\gamma_2)\}z-\frac{\mu}{(\kappa+1)}\frac{(a-b)}{(a+b)}$$
$$\times\{\epsilon_1-\epsilon_2+i(\gamma_1+\gamma_2)\}z \text{ for inclusion,}$$
$$= \frac{2\mu}{(\kappa+1)}\frac{ab}{a^2-b^2}\{\epsilon_1-\epsilon_2+i(\gamma_1+\gamma_2)\}(z-\sqrt{z^2-c^2}) \text{ for matrix,}$$

and

$$\psi_*(z) = \frac{\mu}{(\kappa+1)}\Bigg[2(\epsilon_1+\epsilon_2)\frac{(a-b)}{(a+b)}+\{\epsilon_1-\epsilon_2+i(\gamma_1+\gamma_2)\}\frac{(a-b)^2}{(a+b)^2}$$
$$+\{\epsilon_1-\epsilon_2)-i(\gamma_1+\gamma_2)\}\Bigg]z \text{ for inclusion,}$$
$$= -\frac{4\mu}{(\kappa+1)}\frac{ab}{a^2-b^2}(\epsilon_1+\epsilon_2)\{z-\sqrt{z^2-c^2}\}-\frac{2\mu}{(\kappa+1)}\frac{ab(a^2+b^2)}{(a^2-b^2)^2}$$
$$\times\{\epsilon_1-\epsilon_2+i(\gamma_1+\gamma_2)\}\{2(z-\sqrt{z^2-c^2})-c^2/\sqrt{z^2-c^2}\} \text{ for matrix}$$

where $c^2 = a^2-b^2$.

The boundary condition satisfied on the outer boundary L_0 is as follows:

$$\phi_0(t)+t\overline{\phi_0'(t)}+\overline{\psi_0(t)} = f_0(t)$$

where

$$f_0(t) = f(t) - \phi_*(t) - t\overline{\phi_*'(t)} - \overline{\psi_*(t)};$$

bar denotes the conjugate complex, and $f(t)$ accounts for the forces on the boundary. In the present case $f(t) = 0$.

The equation of L_0 is $|z| = R$.

$\phi_0(z)$ and $\psi_0(z)$ are functions analytic in the region $|z| \leqslant R$. Using integrodifferential approach (Sokolnikoff 1956) after some calculations these analytic functions can be found out. The expressions of sectionally holomorphic functions $\phi(z)$ and $\psi(z)$ are as follows:

$$\phi(z) = \frac{\mu}{(\kappa+1)}\frac{2ab}{(a^2-b^2)}\Bigg[\{\epsilon_1-\epsilon_2-i(\gamma_1+\gamma_2)\}\left\{\frac{R^2}{(R^4-c^2z^2)^{\frac{1}{2}}}-1\right\}z$$

$$-2(\epsilon_1+\epsilon_2)\left\{(R^4-c^2z^2)^{\frac{1}{2}}-R^2+\frac{c^2z^2}{4R^2}\right\}\times\frac{1}{z}-\frac{2(a^2+b^2)}{(a^2-b^2)}$$

$$\times\{\epsilon_1-\epsilon_2-i(\gamma_1+\gamma_2)\}\left\{(R^4-c^2z^2)^{\frac{1}{2}}-R^2+\frac{c^2z^2}{4R^2}\right\}\times\frac{1}{z}$$

$$-(a^2+b^2)\{\epsilon_1-\epsilon_2-i(\gamma_1+\gamma_2)\}\left\{\frac{R^2}{(R^4-c^2z^2)^{\frac{1}{2}}}-\frac{1}{2}\right\}\frac{z}{R^2}\Bigg]$$

$$-\frac{\mu}{(\kappa+1)}\{\epsilon_1+\epsilon_2-i(\gamma_1-\gamma_2)\}z-\frac{\mu}{(\kappa+1)}\frac{(a-b)}{(a+b)}$$

$$\times\{\epsilon_1-\epsilon_2+i(\gamma_1+\gamma_2)\}z \quad \text{for inclusion,}$$

$$=\phi_0(z)+\frac{\mu}{(\kappa+1)}\frac{2ab}{(a^2-b^2)}\{\epsilon_1-\epsilon_2+i(\gamma_1+\gamma_2)\}(z-\sqrt{z^2-c^2}) \quad \text{for matrix,}$$

where $\phi_0(z)$ is the expression in the square brackets of $\phi(z)$ for inclusion given above;

and

$$\psi(z) = \frac{\mu}{(\kappa+1)}\frac{2ab}{(a^2-b^2)}\Bigg[\{\epsilon_1-\epsilon_2-i(\gamma_1+\gamma_2)\}\Bigg\{(R^4-c^2z^2)^{\frac{1}{2}}$$

$$-\frac{R^8}{(R^4-c^2z^2)^{\frac{3}{2}}}\Bigg\}\times\frac{1}{z}-2(\epsilon_1+\epsilon_2)\left\{\frac{R^6}{(R^4-c^2z^2)^{\frac{1}{2}}}-R^4-\frac{c^2z^2}{2}\right\}\times\frac{1}{z^3}$$

$$-\frac{2(a^2+b^2)}{(a^2-b^2)}\{\epsilon_1-\epsilon_2-i(\gamma_1+\gamma_2)\}\left\{\frac{R^6}{(R^4-c^2z^2)^{\frac{1}{2}}}-R^4-\frac{c^2z^2}{2}\right\}\times\frac{1}{z^3}$$

$$+(a^2+b^2)\{\epsilon_1-\epsilon_2-i(\gamma_1+\gamma_2)\}\left\{\frac{R^6}{(R^4-c^2z^2)^{\frac{3}{2}}}-1\right\}\times\frac{1}{z}\Bigg]$$

$$+\frac{2\mu}{(\kappa+1)}(\epsilon_1+\epsilon_2)\frac{(a-b)}{(a+b)}z+\frac{\mu}{(\kappa+1)}\frac{(a-b)^2}{(a+b)^2}\{\epsilon_1-\epsilon_2+i(\gamma_1+\gamma_2)\}z$$

$$+\frac{\mu}{(\kappa+1)}\{\epsilon_1-\epsilon_2-i(\gamma_1+\gamma_2)\}z \quad \text{for inclusion,}$$

$$\psi(z) = \psi_0(z) - \frac{\mu}{(\kappa+1)}\,\frac{4ab}{(a^2-b^2)}(\epsilon_1+\epsilon_2)(z-\sqrt{z^2-c^2})$$

$$-\frac{\mu}{(\kappa+1)}\,\frac{2ab(a^2+b^2)}{(a^2-b^2)^2}\{\epsilon_1-\epsilon_2+i(\gamma_1+\gamma_2)\}\left\{2(z-\sqrt{z^2-c^2})-\frac{c^2}{\sqrt{z^2-c^2}}\right\}$$

for matrix,

where $\psi_0(z)$ is the expression in the square brackets of $\psi(z)$ for inclusion given above.

Stresses can be determined by using the formulae:

$$\tau_{rr}+\tau_{\theta\theta} = 2[\phi'(z)+\overline{\phi'(z)}]$$

$$\tau_{\theta\theta}-\tau_{rr}+2i\tau_{\gamma\theta} = 2e^{2i\theta}[\bar{z}\,\phi''(z)+\psi'(z)].$$

It can be verified that the normal stress and shearing stress on the boundary of the circle $|z| = R$ vanish as they should. The continuity of normal stress and shearing stress at the equilibrium interface can be checked by using the transformation (Timoshenko and Goodier 1951).

$$z = c\cosh\zeta, \quad \zeta = \xi+i\eta.$$

So that the boundary of the ellipse $x^2/a^2+y^2/b^2 = 1$ is defined by $\xi = \xi_0$ where ξ_0 and c are defined by the equations

$$c\cosh\xi_0 = a$$

and

$$c\sinh\xi_0 = b.$$

If we denote the hoop stress at the equilibrium boundary by $\rho_{\eta\eta}$, then the difference in the hoop stress outside and hoop stress inside is given by

$$\{\rho_{\eta\eta}\}_{\text{matrix}} - \{\rho_{\eta\eta}\}_{\text{inclusion}} = \frac{8\mu}{(\kappa+1)(a^2-b^2)(\cosh 2\xi_0-\cos 2\eta)}$$

$$\times\{\epsilon_1 a^2(1-\cos 2\eta)+\epsilon_2 b^2(1+\cos 2\eta)-ab(\gamma_1+\gamma_2)\sin 2\eta\}.$$

In the above analysis the elastic constants for the inclusion and the matrix were assumed to be the same. However, if shear moduli for both the regions be the same and Poisson's ratios be different, then the problem may be solved as follows:

The boundary conditions in this case shall be

$$\phi(t)+t\,\overline{\phi'(t)}+\overline{\psi(t)} = f(t) \text{ on } L_0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$\phi^+(t)+t\,\overline{\phi'^+(t)}+\overline{\psi^+(t)} = \phi^-(t)+t\,\overline{\phi^-(t)}+\overline{\psi^-(t)} \text{ on } L \quad \ldots \quad \ldots \quad (2)$$

$$\kappa\phi^+(t)-t\,\overline{\phi'^+(t)}-\overline{\psi^+(t)} = \kappa'\phi^-(t)-t\,\overline{\phi'^-(t)}-\overline{\psi^-(t)}+2\mu g(t) \text{ on } L \quad \ldots \quad (3)$$

where $+$ and $-$ regions are the same as before and σ and σ' are Poisson's ratios of the inclusion and matrix respectively, κ and κ' are the quantities already explained for the inclusion and matrices respectively. Adding (2) and

(3) we get

$$\phi^{+}(t)-K\,\phi^{-}(t) = \frac{2\mu\, g(t)}{\kappa+1} \quad \text{on } L \qquad \ldots \quad \ldots \quad \ldots \qquad (4)$$

where

$$K = \frac{\kappa'+1}{\kappa+1}.$$

Also from (2) we get

$$\psi^{+}(t)-\psi^{-}(t) = -[\overline{\phi^{+}(t)}-\overline{\phi^{-}(t)}]-\bar{t}[\phi'^{+}(t)-\phi'^{-}(t)] \text{ on } L.$$

Equation (4) can be reduced to the equation of the form

$$F^{+}(t)-F^{-}(t) = \frac{2\mu\, g(t)}{(\kappa+1)}$$

when $\phi(z)$ is defined as follows:

$$\phi^{+}(z) = F^{+}(z)$$

and

$$K\,\phi^{-}(z) = F^{-}(z).$$

Having found ϕ^{+} and ϕ^{-}, we can determine ψ^{+} and ψ^{-} and the problem is solved.

Some of the results previously obtained may be derived from this problem: for example if $R \to \infty$ the results for elliptic inclusion in an infinite medium may be obtained. The results of circular inclusion in an infinite medium may be obtained by taking the limits as $c \to 0$ and $R \to \infty$. Also by taking the limit as $c \to 0$, we obtain the results of concentric circular inclusion.

Discussion

Numerical work for the calculation of maximum shearing stress for the case when $\epsilon_1 = \epsilon_2 = \epsilon$ has been done on IBM 7044 computer. The radius of the circle is taken as 4 and the semi-major axis of the ellipse has been taken to be equal to 1, 2, 3 respectively. For each value of a, the ratio a/b is taken as 2, 10. The ratio of the maximum shearing stress to $\mu\epsilon/(\kappa+1)$ has been denoted by τ. The graphs (Figs. 1–6) for these cases have been drawn. It is interesting to note that for slender ellipses the lines of maximum shearing stress emanating from the boundary of the ellipse resemble to some extent to those of thin rectangular inclusion in an infinite medium (Bhargava and Kapoor 1966).

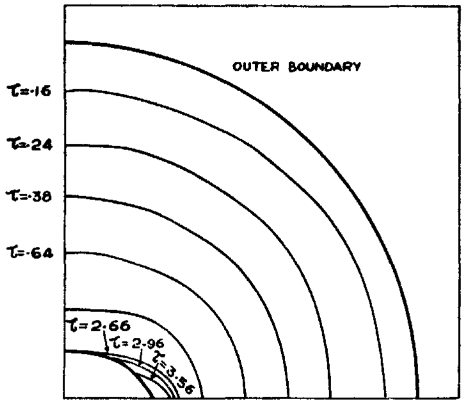

FIG. 1. Lines of maximum shearing stress for an ellipse $a = 1$, $b = 0{\cdot}5$ and $\epsilon_1 = \epsilon_2 = \epsilon$.

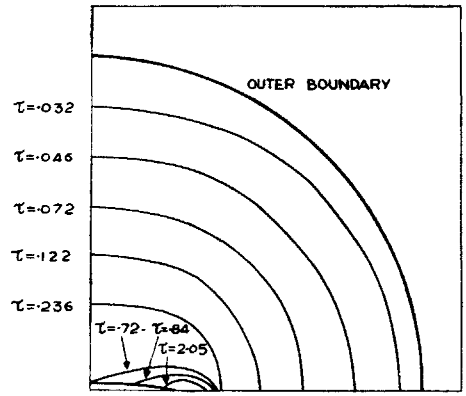

FIG. 2. Lines of maximum shearing stress for an ellipse $a = 1$, $b = 0{\cdot}1$ and $\epsilon_1 = \epsilon_2 = \epsilon$.

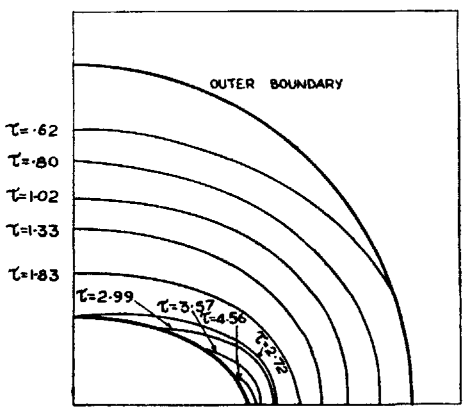

FIG. 3. Lines of maximum shearing stress for an ellipse $a = 2$, $b = 1{\cdot}0$ and $\epsilon_1 = \epsilon_2 = \epsilon$.

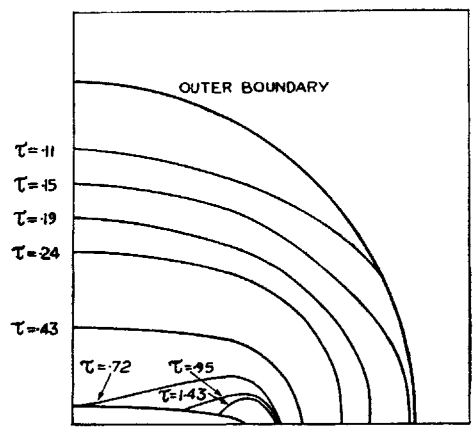

FIG. 4. Lines of maximum shearing stress for an ellipse $a = 2$, $b = 0{\cdot}2$ and $\epsilon_1 = \epsilon_2 = \epsilon$.

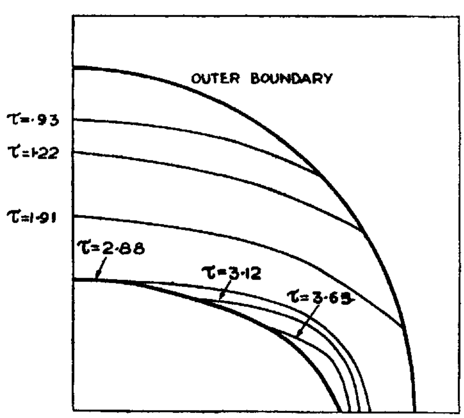

FIG. 5. Lines of maximum shearing stress for an ellipse $a = 3$, $b = 1{\cdot}5$ and $\epsilon_1 = \epsilon_2 = \epsilon$.

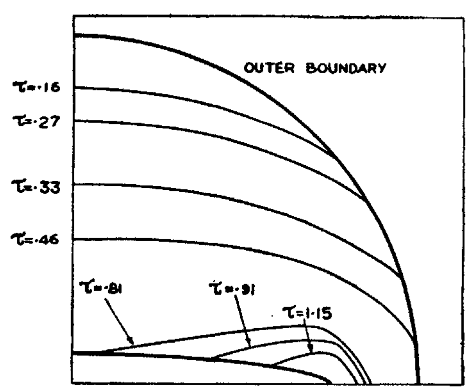

FIG. 6. Lines of maximum shearing stress for an ellipse $a = 3$, $b = 0{\cdot}3$ and $\epsilon_1 = \epsilon_2 = \epsilon$.

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