ON THE UNSTEADY FLOW OF CONDUCTING LIQUID BETWEEN TWO PARALLEL PLATES

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(Communicated by B. Sen, F.N.I.)

(*Received 19 November 1967*)

The unsteady flow of viscous conducting liquid between two parallel plates, when one of them is set in motion parallel to the other by a periodic force, has been discussed by using the methods of Laplace transforms.

1. Introduction

The problem of flow of conducting liquids—both steady and unsteady—between parallel plates traversed by a magnetic field is considered to be an important problem in magneto-hydrodynamics and, chiefly because of this, it has engaged the attention of several researchers, namely Yen and Chang (1961), Pavlov and Tarasov (1960), Kapur and Jain (1962), Sukla (1963), Vijay Kumar (1966), Rathy (1963), Jagdeesan (1964). All these problems may be described as attempts to accommodate magnetic fields in purely hydrodynamic problems and hence to study the effects of such fields on the hydrodynamic motions. An important problem in hydrodynamics, vide Chaudhury (1962), is the flow of viscous liquid between parallel plates, when one of the plates is set in motion by a given force and accordingly a magneto-hydrodynamic analogue of this may be thought of by having a magnetic field in the direction perpendicular to the parallel plates. The solution of the problem has been achieved by Laplace transform techniques for a periodic force.

2. Problem, Fundamental Equations and Boundary Conditions

Let us take the x-axis along the plates in the direction of the flow and y-axis perpendicular to the plates. There is an original magnetic field of strength B_0 perpendicular to the plates. Let the plate at $y = 0$ be at rest and that at $y = h$ be set in motion at time $t = 0$ by a force σX along the x-axis, where σ is the mass per unit area of the plate at $y = h$. Our object is to investigate the motion when the prescribed force is periodic in character.

The governing equations of the problem are constituted by (i) the equation of motion of the plate and (ii) the equation of motion of the liquid in the presence of a magnetic field. Then for the motion of the plate we have

$$\frac{dU}{dt} = X+\frac{\rho\nu}{\sigma}\left(\frac{\partial u}{\partial y}\right)_{y=h} \quad \ldots \quad (1)$$

and for the unsteady motion of the liquid between the parallel plates, we have

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma'}{\rho}B_0^2 u, \quad 0<y<h,\ t>0 \quad \ldots \quad (2)$$

where u is the velocity of the liquid; U, velocity of the plate; ρ, density of the liquid; ν, kinematic coefficient of viscosity; and σ', conductivity of the liquid. The derivation of eqn. (2) is given in Ali Bulent Cambel (1963) with full explanations of the terms involved and need not be repeated here, taking into account, of course, the fact that the pressure gradient is zero as done by Rossow (1957), Ong and Nicholls (1959). The boundary conditions are

$$\left.\begin{array}{ll} u = 0 & \text{when } y = 0 \\ u = U & \text{when } y = h \end{array} \quad t \geqslant 0 \right\}. \quad \ldots \quad (3)$$

3. Solution of the Problem

Let us write $X = \alpha \sin \omega t$ where α and ω are real constants. Then in this case (1) becomes

$$\frac{dU}{dt} = \alpha \sin \omega t+\frac{\rho\nu}{\sigma}\left(\frac{\partial u}{\partial y}\right)_{y=h}. \quad \ldots \quad (4)$$

Let us introduce Laplace transform $\bar{f}(p)$ of $f(t)$ given by

$$\bar{f}(p) = \int_0^\infty e^{-pt} f(t)\, dt.$$

The Laplace transform of (4) is given by

$$p\bar{U} = \frac{\alpha\omega}{p^2+\omega^2}+\frac{\rho\nu}{\sigma}\left(\frac{d\bar{u}}{dy}\right)_{y=h} \quad \ldots \quad (5)$$

and the Laplace transform of (2) is given by

$$p\bar{u} = \nu\frac{d^2\bar{u}}{dy^2}-\frac{\sigma'}{\rho}B_0^2\bar{u}.$$

Therefore,

$$\frac{d^2\bar{u}}{dy^2}-(A+Bp)\bar{u} = 0 \quad \ldots \quad (6)$$

where

$$A = \frac{\sigma'}{\rho}\frac{B_0^2}{\nu}, \quad B = \frac{1}{\nu}.$$

The boundary conditions (3) become, when transformed,

$$\left.\begin{array}{ll} \bar{u} = 0 & \text{when } y = 0 \\ \bar{u} = \bar{U} & \text{when } y = h \end{array}\right\}. \quad \ldots \quad (7)$$

The solution of (6) is

$$\bar{u} = c_1 \cosh\left(y\sqrt{A+Bp}\right)+c_2 \sinh\left(y\sqrt{A+Bp}\right)$$

where c_1 and c_2 are constants.

By (7) $c_1 = 0$, $c_2 = \dfrac{\bar{U}}{\sinh(h\sqrt{A+Bp})}$

$$\therefore \quad \bar{u} = \frac{\bar{U}\sinh(y\sqrt{A+Bp})}{\sinh(h\sqrt{A+Bp})} \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

$$\therefore \quad \left(\frac{d\bar{u}}{dy}\right)_{y=h} = \bar{U}\sqrt{A+Bp}\coth(h\sqrt{A+Bp}). \quad \ldots \quad \ldots \quad (9)$$

Then from (5) and (9) we get on simplification,

$$\bar{U} = \frac{\alpha\omega}{p^2+\omega^2}\,\frac{\sinh(h\sqrt{A+Bp})}{\left\{p\sinh(h\sqrt{A+Bp})-\frac{\rho\nu}{\sigma}\sqrt{A+Bp}\cosh(h\sqrt{A+Bp})\right\}}. \quad (10)$$

Substituting the value of $\bar{U}$ in (8) we find

$$\bar{u} = \frac{\alpha\omega}{p^2+\omega^2}\,\frac{\sinh(y\sqrt{A+Bp})}{\left\{p\sinh(h\sqrt{A+Bp})-\frac{\rho\nu}{\sigma}\sqrt{A+Bp}\cosh(h\sqrt{A+Bp})\right\}}. \quad (11)$$

Therefore, the velocity $u(y, t)$ of the liquid is obtained from (11) by applying the inversion theorem, vide Carslaw and Jaeger (1962), and is given by

$$u(y, t) = \frac{\alpha\omega}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}\sinh(y\sqrt{A+B\lambda})\,d\lambda}{(\lambda^2+\omega^2)\left\{\lambda\sinh(h\sqrt{A+B\lambda})-\frac{\rho\nu}{\sigma}\sqrt{A+B\lambda}\cosh(h\sqrt{A+B\lambda})\right\}}. \quad \ldots \quad (12)$$

The integrand of (12) is a single-valued function of λ with simple poles at

$$\lambda = \pm i\omega \text{ and at } \lambda = -\frac{\alpha_s^2}{h^2B}-\frac{A}{B} \quad (s = 1, 2, \ldots)$$

where α_s's are the real positive roots of the equation,

$$\left(\frac{\alpha_s^2}{h^2B}+\frac{A}{B}\right)\sin\alpha_s+\frac{\rho\nu}{\sigma}\alpha_s\cos\alpha_s = 0.$$

We next choose the contour of integration as the part of the straight line $\lambda = \gamma$ and a part of the circle $|\lambda| = R$, no pole being on the circle. Then the integral over the part of the circle tends to zero as R tends to infinity and the residues of the integrand at the pole $\lambda = -\dfrac{\alpha_s^2}{h^2B}-\dfrac{A}{B}$ are

$$-\frac{2\rho\sigma\nu h^2B\alpha_s^2 e^{-\left(\frac{\alpha_s^2}{h^2B}+\frac{A}{B}\right)t}\sin\left(y\cdot\frac{\alpha_s}{h}\right)}{\left\{\left(\frac{\alpha_s^2}{h^2B}+\frac{A}{B}\right)^2+\omega^2\right\}\left\{\sigma^2(\alpha_s^2+Ah^2)^2+(\alpha_s^2+Ah^2)\alpha_s\rho\sigma\nu hB+\alpha_s^2\rho^2\nu^2\sigma h^3B^2-2\alpha_s^2\rho\sigma\nu h^2B\right\}\sin\alpha_s}. \quad \ldots \quad (13)$$

The residue at $\lambda = i\omega$ is

$$\frac{e^{i\omega t}\sinh\left(y\sqrt{A+Bi\omega}\right)}{2i\omega\left\{i\omega\sinh\left(h\sqrt{A+Bi\omega}\right)-\frac{\rho\nu}{\sigma}\sqrt{A+Bi\omega}\cosh\left(h\sqrt{A+Bi\omega}\right)\right\}} \quad \ldots \quad (14)$$

and at $\lambda = -i\omega$ is

$$\frac{e^{-i\omega t}\sinh\left(y\sqrt{A+Bi\omega}\right)}{2i\omega\left\{i\omega\sinh\left(h\sqrt{A-Bi\omega}\right)+\frac{\rho\nu}{\sigma}\sqrt{A-Bi\omega}\cosh\left(h\sqrt{A-Bi\omega}\right)\right\}} . \quad (15)$$

Therefore, from (12) we have

$$u(y,t) = \frac{\alpha}{2i}\,\frac{e^{i\omega t}\sinh\left(y\sqrt{A+Bi\omega}\right)}{i\omega\sinh\left(h\sqrt{A+Bi\omega}\right)-\frac{\rho\nu}{\sigma}\sqrt{A+Bi\omega}\cosh\left(h\sqrt{A+Bi\omega}\right)}$$

$$+\frac{\alpha}{2i}\,\frac{e^{-i\omega t}\sinh\left(y\sqrt{A-Bi\omega}\right)}{i\omega\sinh\left(h\sqrt{A-Bi\omega}\right)+\frac{\rho\nu}{\sigma}\sqrt{A-Bi\omega}\cosh\left(h\sqrt{A-Bi\omega}\right)}$$

$$-2\alpha\omega\rho\sigma\nu h^2B\sum_{s=1}^{\infty}\frac{\alpha_s^2 e^{-\left(\frac{\alpha_s^2}{h^2B}+\frac{A}{B}\right)t}\sin\left(y\cdot\frac{\alpha_s}{h}\right)}{\left\{\left(\frac{\alpha_s^2}{h^2B}+\frac{A}{B}\right)^2+\omega^2\right\}\left\{\sigma^2\left(\alpha_s^2+Ah^2\right)^2+\left(\alpha_s^2+Ah^2\right)\alpha_s\rho\sigma\nu hB+\right.}$$

$$\left.+\alpha_s^2\rho^2\nu^2\sigma h^3B^2-2\alpha_s^2\rho\sigma\nu h^2B\right\}\sin\alpha_s \quad \ldots \quad (16)$$

$$= A\alpha\sin(\omega t+\phi)-2\alpha\omega\rho\sigma\nu h^2B\sum_{s=1}^{\infty}\frac{\alpha_s^2 e^{-\left(\frac{\alpha_s^2}{h^2B}+\frac{B}{A}\right)t}\sin\left(y\cdot\frac{\alpha_s}{h}\right)}{\left\{\left(\frac{\alpha_s^2}{h^2B}+\frac{A}{B}\right)^2+\omega^2\right\}\times}$$

$$\times\left\{\sigma^2\left(\alpha_s^2+Ah^2\right)^2+\left(\alpha_s^2+Ah^2\right)\alpha_s\rho\sigma\nu hB+\alpha_s^2\rho^2\nu^2\sigma h^3B^2-2\alpha_s^2\rho\sigma\nu h^2B\right\}\sin\alpha_s \quad \ldots \quad (17)$$

where

$$A = \sqrt{L^2+M^2}, \quad \phi = \tan^{-1}\frac{M}{L}$$

$$L = \frac{C\beta_1-D\alpha_1}{C^2+D^2}, \quad M = -\frac{C\alpha_1+D\beta_1}{C^2+D^2}$$

$$C = \omega\alpha_2-\frac{\rho\nu}{\sigma}k_2\gamma\sin\frac{\theta_1}{2}-\frac{\rho\nu}{\sigma}k_2\delta\cos\frac{\theta_1}{2}$$

$$D = \omega\beta_2+\frac{\rho\nu}{\sigma}k_2\gamma\cos\frac{\theta_1}{2}-\frac{\rho\nu}{\sigma}k_2\delta\sin\frac{\theta_1}{2}$$

$$\alpha_1 = \cos\left(yk_2\sin\frac{\theta_1}{2}\right)\sinh\left(yk_2\cos\frac{\theta_1}{2}\right)$$

$$\alpha_2 = \cos\left(hk_2\sin\frac{\theta_1}{2}\right)\sinh\left(hk_2\cos\frac{\theta_1}{2}\right)$$

$$\beta_1 = \sin\left(yk_2 \sin\frac{\theta_1}{2}\right)\cosh\left(yk_2\cos\frac{\theta_1}{2}\right)$$

$$\beta_2 = \sin\left(hk_2 \sin\frac{\theta_1}{2}\right)\cosh\left(hk_2\cos\frac{\theta_1}{2}\right)$$

$$\gamma = \cos\left(hk_2 \sin\frac{\theta_1}{2}\right)\cosh\left(hk_2\cos\frac{\theta_1}{2}\right)$$

$$\delta = \sin\left(hk_2 \sin\frac{\theta_1}{2}\right)\sinh\left(hk_2\cos\frac{\theta_1}{2}\right)$$

$$k_2 = w^{\frac{1}{2}}B^{\frac{1}{2}}(1+k_1^2)^{\frac{1}{4}}$$

$$k_1 = \frac{A}{B}$$

$$\theta_1 = \tan^{-1}\left(\frac{1}{k_1}\right). \qquad \ldots \quad (18)$$

Also for the velocity of the plate at any time t, we have from (10) with the help of inversion theorem,

$$U = \frac{\alpha\omega}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}\sinh\left(h\sqrt{A+B\lambda}\right)d\lambda}{(\lambda^2+\omega^2)\left\{\lambda\sinh\left(h\sqrt{A+B\lambda}\right)-\frac{\rho\nu}{\sigma}\sqrt{A+B\lambda}\cosh\left(h\sqrt{A+B\lambda}\right)\right\}}. \qquad \ldots \quad (19)$$

The integrand of (19) is also a single-valued function of λ and has the same poles as those of the integrand of (12). Therefore, using the same contour as in the previous case and calculating residues we get finally,

$$U = \frac{\alpha}{2i}\frac{e^{i\omega t}\sinh\left(h\sqrt{A+Bi\omega}\right)}{i\omega\sinh\left(h\sqrt{A+Bi\omega}\right)-\frac{\rho\nu}{\sigma}\sqrt{A+Bi\omega}\cosh\left(h\sqrt{A+Bi\omega}\right)}$$

$$+\frac{\alpha}{2i}\frac{e^{i\omega t}\sinh\left(h\sqrt{A-Bi\omega}\right)}{i\omega\sinh\left(h\sqrt{A-Bi\omega}\right)+\frac{\rho\nu}{\sigma}\sqrt{A-Bi\omega}\cosh\left(h\sqrt{A-Bi\omega}\right)}$$

$$-2\alpha\omega\rho\sigma\nu h^2B\sum_{s=1}^{\infty}\frac{\alpha_s^2 e^{-\left(\frac{\alpha_s^2}{h^2B}+\frac{A}{B}\right)t}}{\left\{\left(\frac{\alpha_s^2}{h^2B}+\frac{B}{A}\right)^2+\omega^2\right\}}$$

$$\times\left\{\sigma^2(\alpha_s^2+Ah^2)^2+(\alpha_s^2Ah^2)\alpha_s\rho\sigma\nu hB+\alpha_s^2\rho^2\nu^2\sigma h^3B-2\alpha_s^2\rho\sigma\nu h^2B\right\} \qquad \ldots \quad (20)$$

$$= A'\alpha\sin(\omega t+\phi'')-2\alpha\omega\rho\sigma\nu h^2B\sum_{s=1}^{\infty}\frac{\alpha_s^2 e^{-\left(\frac{\alpha_s^2}{h^2B}+\frac{A}{B}\right)t}}{\left\{\left(\frac{\alpha_s^2}{h^2B}+\frac{A}{B}\right)^2+\omega^2\right\}}$$

$$\times\left\{\sigma^2(\alpha_s^2+Ah^2)^2+(\alpha_s^2+Ah^2)\alpha_s\rho\sigma\nu hB+\alpha_s^2\rho^2\nu^2\sigma h^3B^2-2\alpha_s^2\rho\sigma\nu h^2B\right\} \qquad (21)$$

where

$$A' = \sqrt{L'^2+M'^2}, \quad \phi' = \tan^{-1}\frac{M'}{L'}$$

$$L' = \frac{C\beta_2-D\alpha_2}{C^2+D^2}, \quad M' = -\frac{C\alpha_2+D\beta_2}{C^2D^2}$$

and C, D, γ, δ, α_1, α_2, β_1, β_2, k_1, k_2, θ_1 are the same as in (18).
The expressions for u and U bring out their characteristics, viz. each of them is partly periodic and partly transient with respect to time. The case for a transient force can easily be derived from the previous result by replacing $i\omega$ by $-n$ $(n > 0)$.

Acknowledgements

The author wishes to express his sincere gratitude to Dr. D. K. Sinha, Department of Mathematics, Jadavpur University, for his kind help and constant guidance in the preparation of this paper. He is also thankful to the referee for his valuable comments.

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