A MODEL OF TURBULENCE WITH THE POSSIBILITY OF ENERGY OF TURBULENCE VANISHING WITH DISTANCE FROM A FIXED ORIGIN

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A model of turbulence which is symmetric about a fixed line in space with the characteristic of turbulence at any point dependent on its distance from the central point of entrance face of the tunnel has been envisaged. For this model the turbulence energy shows the possibility of vanishing with distance measured from the central point of entrance face of the tunnel as origin. The study of this model is of interest, as all other models so far studied fail to furnish us with this possibility, in absence of which the scheme of turbulence is termed 'conservative' and not suitable in too extensive a region (cf. Bass 1954).

1. Introduction

The simplest model of turbulence is of homogeneous and isotropic nature. In order of simplicity, next comes the homogeneous and axisymmetric turbulence as studied by Batchelor (1946) and Chandrasekhar (1950). Bass (1954) considered, in course of a detail analysis of all types of turbulence so far studied, another scheme of axisymmetric turbulence which he termed as cylindrically symmetric turbulence. The characteristics of turbulence in this type of turbulence have all the symmetry properties of ordinary axisymmetry (in the sense of Batchelor) and in addition they are dependent on the distances of points under consideration from a fixed origin measured parallel to the direction of symmetry. In this work Bass pointed out that the mathematical models of turbulence so far constructed including his model of cylindrically symmetric turbulence show no possibility of energy of turbulence vanishing with distance from its source of initiation.

The present author in an earlier attempt (Ghosh 1968) presented a model of turbulence which is symmetric about a fixed axis in space, the axis being considered as the line-source of turbulence energy (obviously this line is along the direction of mean flow) and turbulence characteristics depending on the distances of points under consideration from this fixed line. In that model, too, the turbulence energy did not indicate the possibility of vanishing with distances from this fixed axis.

In the current work the model presented is almost of the same nature as discussed by the author which is referred to above, but here one more parametric dependence has been introduced. This parameter is the distance of the point under consideration from the entrance face of the tunnel measured along the axis of symmetry.

The interesting point in this model is that here comes the possibility of turbulence energy vanishing with distance from the central point of the entrance face of the tunnel. In that sense this scheme of turbulence is 'not conservative' and suitable in too extensive a region.

2. Kinematics

Consider a turbulent medium extending to infinity in all directions. Let OX be the fixed line of symmetry represented by the unit vector $\vec{\lambda}$. P and Q be two arbitrary space points in close neighbourhood inside the turbulent medium. Let us associate two unit vectors i and j at P and Q respectively. PN and QN' are drawn perpendicular from P and Q on the λ-axis. Further let us assume that the point O on the λ-axis is the central point of entrance face of the tunnel while λ-axis directs the direction of mean flow. Lengths ON and NP as shown in Fig. 1 are denoted by l and p respectively. The directions NP and $N'Q$ (which are not necessarily in the same plane) are given by unit vectors $\vec{\psi}$ and $\vec{\psi}'$ respectively. We have further denoted $\vec{NP} = \vec{\mu}$ $(= p\vec{\psi})$ and $\vec{N'Q} = \vec{\mu}'$ $(= p'\vec{\psi}')$ where the magnitude of $N'Q$ is taken as p'.

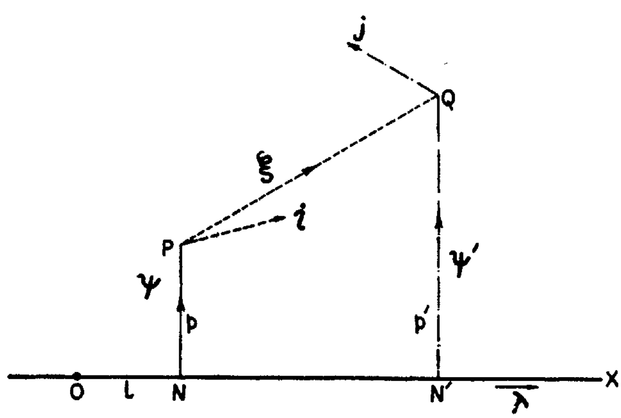

Fig. 1. Diagram showing the positions of points with their corresponding components of fluctuating velocities inside the turbulent fluid.

Next consider u_i and u_j' as the i and jth components of fluctuating parts of fluid velocities at P and Q respectively at a certain instant of time. For the type of turbulence which is symmetric about the fixed line OX (*see*

Fig. 1) and whose characteristics at a point P (say) depend on the distances PN ($= p$) and ON ($= l$), the second order velocity correlations F_{ij} are given by

$$\overline{u_i u_j'} \equiv F_{ij} = A\xi_i\xi_j + B\delta_{ij} + C\lambda_i\lambda_j + D\lambda_i\xi_j + E\lambda_j\xi_i + F\lambda_i\psi_j + G\lambda_j\psi_i + H\xi_i\psi_j + I\xi_j\psi_i + J\psi_i\psi_j \quad \ldots \quad (1)$$

where A, B, C, D, E, F, G, H, I, J are functions of

$$\left(\vec{\xi} \cdot \vec{\xi}\right) = r^2, \quad \left(\vec{\xi} \cdot \vec{\lambda}\right) = s, \quad \left(\vec{\xi} \cdot \vec{\psi}\right) = q, \ p \text{ and } l$$

in the case of stationary state of the scheme of turbulence envisaged here. It is worth while to mention that the derivation of the form (1) for F_{ij} is done following the well-known procedure developed by Robertson (1940).

Equations of Continuity

The equations of continuity for an incompressible fluid applied at P and Q give us, after usual process of averaging,

$$\frac{\partial}{\partial x_i} F_{ij} = 0 \quad \text{and} \quad \frac{\partial}{\partial x'} F_{ij} = 0,$$

where the coordinates of the points P and Q are respectively denoted by (x_i) and (x_i') with origin O and the x-axis along λ-axis. The two equations of continuity stated above are reducible to the forms

$$\left(\lambda_i \frac{\partial}{\partial l} + \psi_i \frac{\partial}{\partial p}\right) F_{ij} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

and

$$\frac{\partial}{\partial \xi_j} F_{ij} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where the operator $\frac{\partial}{\partial \xi_j}$ means $\frac{\xi_j}{r}\frac{\partial}{\partial r} + \lambda_j \frac{\partial}{\partial s} + \psi_j \frac{\partial}{\partial q}$.

Substituting the expression for F_{ij} given by (1) in eqns. (2) and (3), we obtain

$$\xi_j\left[s\frac{\partial A}{\partial l} + \frac{\partial D}{\partial l} + q\frac{\partial A}{\partial p} + \frac{\partial I}{\partial p}\right] + \lambda_j\left[\frac{\partial B}{\partial l} + \frac{\partial C}{\partial l} + s\frac{\partial E}{\partial l} + q\frac{\partial E}{\partial p} + \frac{\partial G}{\partial p}\right]$$
$$+\psi_j\left[\frac{\partial F}{\partial l} + s\frac{\partial H}{\partial l} + \frac{\partial B}{\partial p} + q\frac{\partial H}{\partial p} + \frac{\partial J}{\partial p}\right] = 0$$

and

$$\xi_i\left[4A + r\frac{\partial A}{\partial r} + s\frac{\partial A}{\partial s} + q\frac{\partial A}{\partial q} + \frac{1}{r}\frac{\partial B}{\partial r} + \frac{s}{r}\frac{\partial E}{\partial r} + \frac{\partial E}{\partial s} + \frac{q}{r}\frac{\partial H}{\partial r} + \frac{\partial H}{\partial q}\right]$$
$$+\lambda_i\left[\frac{\partial B}{\partial s} + \frac{s}{r}\frac{\partial C}{\partial r} + \frac{\partial C}{\partial s} + 3D + r\frac{\partial D}{\partial r} + s\frac{\partial D}{\partial s} + q\frac{\partial D}{\partial q} + E + \frac{q}{r}\frac{\partial F}{\partial r} + \frac{\partial F}{\partial q}\right]$$
$$+\psi_i\left[\frac{\partial B}{\partial q} + \frac{s}{r}\frac{\partial G}{\partial r} + \frac{\partial G}{\partial s} + H + 3I + r\frac{\partial I}{\partial r} + s\frac{\partial I}{\partial s} + q\frac{\partial I}{\partial q} + \frac{q}{r}\frac{\partial J}{\partial r} + \frac{\partial J}{\partial q}\right] = 0.$$

Thus for arbitrariness of ξ_j, λ_j, ψ_j and ξ_i, λ_i, ψ_i, we get

$$\left.\begin{aligned} & s\frac{\partial A}{\partial l}+\frac{\partial D}{\partial l}+q\frac{\partial A}{\partial p}+\frac{\partial I}{\partial p}=0 \\ & \frac{\partial B}{\partial l}+\frac{\partial C}{\partial l}+s\frac{\partial E}{\partial l}+q\frac{\partial E}{\partial p}+\frac{\partial G}{\partial p}=0 \\ & \frac{\partial F}{\partial l}+s\frac{\partial H}{\partial l}+\frac{\partial B}{\partial p}+q\frac{\partial H}{\partial p}+\frac{\partial J}{\partial p}=0 \end{aligned}\right\} \quad \ldots \quad (4)$$

and

$$\left.\begin{aligned} & 4A+r\frac{\partial A}{\partial r}+s\frac{\partial A}{\partial s}+q\frac{\partial A}{\partial q}+\frac{1}{r}\frac{\partial B}{\partial r}+\frac{s}{r}\frac{\partial E}{\partial r}+\frac{\partial E}{\partial s}+\frac{q}{r}\frac{\partial H}{\partial r}+\frac{\partial H}{\partial q}=0 \\ & \frac{\partial B}{\partial s}+\frac{s}{r}\frac{\partial C}{\partial r}+\frac{\partial C}{\partial s}+3D+r\frac{\partial D}{\partial r}+s\frac{\partial D}{\partial s}+q\frac{\partial D}{\partial q}+E+\frac{q}{r}\frac{\partial F}{\partial r}+\frac{\partial F}{\partial q}=0 \\ & \frac{\partial B}{\partial q}+\frac{s}{r}\frac{\partial G}{\partial r}+\frac{\partial G}{\partial s}+H+3I+r\frac{\partial I}{\partial r}+s\frac{\partial I}{\partial s}+q\frac{\partial I}{\partial q}+\frac{q}{r}\frac{\partial J}{\partial r}+\frac{\partial J}{\partial q}=0 \end{aligned}\right\} \quad \ldots \quad (5)$$

From the geometry of the configuration given in Fig. 1, we write

$$\vec{\mu}' = -s\,\vec{\lambda}+\vec{\mu}+\vec{\xi}$$

which is equivalent to

$$\vec{\psi}' = -\frac{s}{p'}\vec{\lambda}+\frac{p}{p'}\vec{\psi}+\frac{\vec{\xi}}{p'} \quad \ldots \quad (6)$$

The consideration of u_i at P and u_j' at Q and taking the ensemble average of the product, viz. $\overline{u_i u_j'}$, is the same as to obtain $\overline{u_j' u_i}$ which one gets from the consideration of u_j' at Q as first point and u_i at P as second point and taking the ensemble average of the product. Although the result looks trivial, yet consideration of P as first point and Q as second point in the previous case and consideration of Q as first point and P as second point in the subsequent case with components of fluctuating parts of velocities at the two points same in both cases supplement additional relations. These relations are said to be the outcome of the symmetry of the geometric configuration. Accordingly $\vec{\xi}$ of the first case is to be replaced by $-\vec{\xi}$, $\vec{\mu}$ of this case to be changed to $\vec{\mu}'$ with suffixes i and j interchanged to give us the same picture that we realize in the second case. Consequently

$$\begin{aligned} & A\xi_i\xi_j+B\delta_{ij}+C\lambda_i\lambda_j+D\lambda_i\xi_j+E\xi_i\lambda_j+F\lambda_i\psi_j+G\lambda_j\psi_i+H\xi_i\psi_j+I\xi_j\psi_i+J\psi_i\psi_j \\ & = A'(-\xi_j)(-\xi_i)+B'\delta_{ji}+C'\lambda_j\lambda_i+D'\lambda_j(-\xi_i)+E'(-\xi_j)\lambda_i \\ & \quad +F'\lambda_j\psi_i'+G'\lambda_i\psi_j'+H'(-\xi_j)\psi_i'+I'(-\xi_i)\psi_j'+J'\psi_j'\psi_i' \end{aligned} \quad \ldots \quad (7)$$

where $A = A(r^2, s, q\,;\ l, p)$

and $A' = A(r^2, -s, q'\,;\ l+s, p')$,

q' being $\left(= -\vec{\xi}\,.\,\vec{\psi}'\right)$. Such functional relationships are also true for pairs of defining scalars (B, B'), (C, C'), (D, D'), (E, E'), etc.

Substituting the geometric relation (6) in eqn. (7), we easily obtain the following relations by comparing coefficients of corresponding elements on both sides of simplified form of eqn. (7):

$$\left.\begin{aligned}
A &= A' - \frac{H'}{p'} - \frac{I'}{p'} + \frac{J'}{p'^2} \\
B &= B', \\
C &= C' - \frac{sF'}{p'} - \frac{sG'}{p'} + \frac{s^2}{p'^2} J' \\
D &= -E' + \frac{G'}{p'} + \frac{sH'}{p'} - \frac{s}{p'^2} J' \\
E &= -D' + \frac{F'}{p'} + \frac{sI'}{p'} - \frac{s}{p'^2} J' \\
F &= \frac{pG'}{p'} - \frac{sp}{p'^2} J' \\
G &= \frac{pF'}{p'} - \frac{ps}{p'^2} J' \\
H &= -\frac{p}{p'} I' + \frac{p}{p'^2} J' \\
I &= -\frac{p}{p'} H' + \frac{p}{p'^2} J' \\
J &= \frac{p^2}{p'^2} J'
\end{aligned}\right\} \qquad \ldots \quad \ldots \quad (8)$$

3. Limited Expansions of defining Scalars in Powers of r, s and q

In § 2, we have derived all the kinematical relations that are possible in the stationary state of the scheme of turbulence discussed above. For further simplification of these kinematical relations, we take help of limited expansions of defining scalars in powers of r, s and q correct up to second order in r (i.e in r, s and q, as s and q are of the same order as r). This simplification of the whole analysis will give us somewhat a true picture of the phenomena as we know that in the case of measurements in wind-tunnels the correlations of fluctuating velocity components at widely separated points asymptotically vanish. Accordingly the points P and Q are to be chosen in sufficient close neighbourhood of each other to give us non-zero correlation values and in that context results correct up to second order in r can give us a true picture of the phenomena. Hence for the expression of F_{ij} to be correct up to second order in r, we set

$$\left.\begin{aligned}
A &= A_0, \quad B = B_0+B_1s+B_2q+B_{11}r^2+B_{22}s^2+B_{33}q^2+2B_{23}sq \\
C &= C_0+C_1s+C_2q+C_{11}r^2+C_{22}s^2+C_{33}q^2+2C_{23}sq \\
D &= D_0+D_1s+D_2q \\
E &= E_0+E_1s+E_2q \\
F &= F_0+F_1s+F_2q+F_{11}r^2+F_{22}s^2+F_{33}q^2+2F_{23}sq \\
G &= G_0+G_1s+G_2q+G_{11}r^2+G_{22}s^2+G_{33}q^2+2G_{23}sq \\
H &= H_0+H_1s+H_2q \\
I &= I_0+I_1s+I_2q \\
J &= J_0+J_1s+J_2q+J_{11}r^2+J_{22}s^2+J_{33}q^2+2J_{23}sq
\end{aligned}\right\} \quad \ldots \quad (9)$$

where A_0, B_0, B_1, . . . , J_{23} are functions of l and p.

Now using the expansions (9) in eqns. (4) and (5) we obtain

$$\left.\begin{aligned}
\frac{\partial D_0}{\partial l}+\frac{\partial I_0}{\partial p} &= 0 \\
\frac{\partial A_0}{\partial l}+\frac{\partial D_1}{\partial l}+\frac{\partial I_1}{\partial p} &= 0 \\
\frac{\partial D_2}{\partial l}+\frac{\partial A_0}{\partial p}+\frac{\partial I_2}{\partial p} &= 0
\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.1')$$

$$\left.\begin{aligned}
\frac{\partial B_0}{\partial l}+\frac{\partial C_0}{\partial l}+\frac{\partial G_0}{\partial p} &= 0 \\
\frac{\partial B_1}{\partial l}+\frac{\partial C_1}{\partial l}+\frac{\partial E_0}{\partial l}+\frac{\partial G_1}{\partial p} &= 0 \\
\frac{\partial B_2}{\partial l}+\frac{\partial C_2}{\partial l}+\frac{\partial E_0}{\partial p}+\frac{\partial G_2}{\partial p} &= 0 \\
\frac{\partial B_{11}}{\partial l}+\frac{\partial C_{11}}{\partial l}+\frac{\partial G_{11}}{\partial p} &= 0 \\
\frac{\partial B_{22}}{\partial l}+\frac{\partial C_{22}}{\partial l}+\frac{\partial E_1}{\partial l}+\frac{\partial G_{22}}{\partial p} &= 0 \\
\frac{\partial B_{33}}{\partial l}+\frac{\partial C_{33}}{\partial l}+\frac{\partial E_2}{\partial p}+\frac{\partial G_{33}}{\partial p} &= 0 \\
2\frac{\partial B_{23}}{\partial l}+2\frac{\partial C_{23}}{\partial l}+\frac{\partial E_2}{\partial l}+\frac{\partial E_1}{\partial p}+2\frac{\partial G_{23}}{\partial p} &= 0
\end{aligned}\right\} \quad \ldots \quad \ldots \quad (4.2')$$

$$\left.\begin{aligned}
\frac{\partial F_0}{\partial l}+\frac{\partial B_0}{\partial p}+\frac{\partial J_0}{\partial p} &= 0\\
\frac{\partial F_1}{\partial l}+\frac{\partial H_0}{\partial l}+\frac{\partial B_1}{\partial p}+\frac{\partial J_1}{\partial p} &= 0\\
\frac{\partial F_2}{\partial l}+\frac{\partial H_0}{\partial p}+\frac{\partial B_2}{\partial p}+\frac{\partial J_2}{\partial p} &= 0\\
\frac{\partial F_{11}}{\partial l}+\frac{\partial B_{11}}{\partial p}+\frac{\partial J_{11}}{\partial p} &= 0\\
\frac{\partial F_{22}}{\partial l}+\frac{\partial H_1}{\partial l}+\frac{\partial B_{22}}{\partial p}+\frac{\partial J_{22}}{\partial p} &= 0\\
\frac{\partial F_{33}}{\partial l}+\frac{\partial H_2}{\partial p}+\frac{\partial B_{33}}{\partial p}+\frac{\partial J_{33}}{\partial p} &= 0\\
2\frac{\partial F_{23}}{\partial l}+\frac{\partial H_2}{\partial l}+\frac{\partial H_1}{\partial p}+2\frac{\partial B_{23}}{\partial p}+2\frac{\partial J_{23}}{\partial p} &= 0
\end{aligned}\right\} \quad \ldots \quad (4.3')$$

and

$$4A_0+2B_{11}+E_1+H_2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (5.1')$$

$$\left.\begin{aligned}
B_1+C_1+3D_0+E_0+F_2 &= 0\\
2B_{22}+2C_{11}+2C_{22}+4D_1+E_1+2F_{23} &= 0\\
2B_{23}+2C_{23}+4D_2+E_2+2F_{11}+2F_{33} &= 0
\end{aligned}\right\} \quad \ldots \quad \ldots \quad (5.2')$$

$$\left.\begin{aligned}
B_2+G_1+H_0+3I_0+J_2 &= 0\\
2B_{23}+2G_{11}+H_1+4I_1+2J_{23} &= 0\\
2B_{33}+H_2+4I_2+2J_{11}+2J_{33} &= 0
\end{aligned}\right\}. \quad \ldots \quad \ldots \quad (5.3')$$

The geometric relation (6) is in vector form. Now we reduce this in scalar form and expanding the resultant binomially and keeping terms correct up to second order in r we find

$$\left.\begin{aligned}
p' &\simeq p+q+\frac{1}{2p}(r^2-s^2-q^2)\\
q' &\simeq -q-\frac{1}{p}(r^2-s^2-q^2)
\end{aligned}\right\}. \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

Introducing the limited expansions (9) and geometric relation (10) in the set of equations given in (8), we derive after Taylor expansion round about $r = 0$ the following relations:

$$p(H_0+I_0) = J_0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8.1')$$

$$B_1 = \frac{1}{2}\frac{\partial B_0}{\partial l}; \quad B_2 = \frac{1}{2}\frac{\partial B_0}{\partial p} \quad \ldots \quad \ldots \quad \ldots \quad (8.2')$$

$$\left.\begin{aligned} C_1 &= \frac{1}{2}\frac{\partial C_0}{\partial l} - \frac{1}{2p}(F_0+G_0) \\ C_2 &= \frac{1}{2}\frac{\partial C_0}{\partial p} \\ F_1+G_1 &- \frac{1}{2}\frac{\partial}{\partial l}(F_0+G_0)+\frac{J_0}{p} = 0 \\ F_2+G_2 &- \frac{1}{2}\frac{\partial}{\partial p}(F_0+G_0)+\frac{1}{p}(F_0+G_0) = 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (8.3')$$

$$\left.\begin{aligned} D_0+E_0 &= \frac{G_0}{p} \\ D_1 &= -\frac{\partial E_0}{\partial l}+E_1+\frac{1}{p}\frac{\partial G_0}{\partial l} - \frac{1}{p}G_1+\frac{H_0}{p} - \frac{J_0}{p^2} \\ D_2 &= -\frac{\partial E_0}{\partial p}+E_2-\frac{1}{p^2}G_0+\frac{1}{p}\frac{\partial G_0}{\partial p} - \frac{G_2}{p} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (8.4')$$

$$\left.\begin{aligned} E_0+D_0 &= \frac{F_0}{p} \\ E_1 &= -\frac{\partial D_0}{\partial l}+D_1+\frac{1}{p}\frac{\partial F_0}{\partial l} - \frac{1}{p}F_1+\frac{I_0}{p} - \frac{J_0}{p^2} \\ E_2 &= -\frac{\partial D_0}{\partial p}+D_2-\frac{1}{p^2}F_0+\frac{1}{p}\frac{\partial F_0}{\partial p} - \frac{F_2}{p} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (8.5')$$

$$\left.\begin{aligned} F_0 &= G_0 \\ F_1 &= \frac{\partial G_0}{\partial l} - G_1 - \frac{J_0}{p} \\ F_2 &= -\frac{G_0}{p}+\frac{\partial G_0}{\partial p} - G_2 \\ F_{11} &= -\frac{1}{2p^2}G_0+\frac{1}{2p}\frac{\partial G_0}{\partial p} - \frac{1}{p}G_2+G_{11} \\ F_{22} &= \frac{1}{2p^2}G_0-\frac{1}{2p}\frac{\partial G_0}{\partial p}+\frac{1}{2}\frac{\partial^2 G_0}{\partial l^2} - \frac{\partial G_1}{\partial l}+\frac{G_2}{p} \\ &\quad +G_{22}-\frac{1}{2}\frac{\partial J_0}{\partial l}+\frac{1}{p}J_1 \\ F_{33} &= \frac{3}{2p^2}G_0-\frac{3}{2p}\frac{\partial G_0}{\partial p}+\frac{1}{2}\frac{\partial^2 G_0}{\partial p^2}+\frac{G_2}{p} - \frac{\partial G_2}{\partial p}+G_{33} \\ 2F_{23} &= -\frac{1}{p}\frac{\partial G_0}{\partial l}+\frac{\partial^2 G_0}{\partial l \partial p}+\frac{G_1}{p} - \frac{\partial G_1}{\partial p} - \frac{\partial G_2}{\partial l}+2G_{23} \\ &\quad +\frac{2}{p^2}J_0-\frac{1}{p}\frac{\partial J_0}{\partial p}+\frac{1}{p}J_2 \,. \end{aligned}\right\} \quad \ldots \quad (8.6')$$

$$\left.\begin{aligned}
G_0 &= F_0 \\
G_1 &= \frac{\partial F_0}{\partial l} - F_1 - \frac{1}{p} J_0 \\
G_2 &= -\frac{F_0}{p} + \frac{\partial F_0}{\partial p} - F_2 \\
G_{11} &= -\frac{1}{2p^2} F_0 + \frac{1}{2p} \cdot \frac{\partial F_0}{\partial p} - \frac{1}{p} F_2 + F_{11} \\
G_{22} &= \frac{1}{2p^2} F_0 - \frac{1}{2p} \cdot \frac{\partial F_0}{\partial p} + \frac{1}{2} \frac{\partial^2 F_0}{\partial l^2} - \frac{\partial F_1}{\partial l} + \frac{1}{p} F_2 + F_{22} - \frac{1}{p} \cdot \frac{\partial J_0}{\partial l} + \frac{1}{p} J_1 \\
G_{33} &= \frac{3}{2p^2} F_0 - \frac{3}{2p} \cdot \frac{\partial F_0}{\partial p} + \frac{1}{2} \frac{\partial^2 F_0}{\partial p^2} + \frac{F_2}{p} - \frac{\partial F_2}{\partial p} + F_{33} \\
2G_{23} &= -\frac{1}{p} \frac{\partial F_0}{\partial l} + \frac{\partial^2 F_0}{\partial l \partial p} + \frac{F_1}{p} - \frac{\partial F_1}{\partial p} - \frac{\partial F_2}{\partial l} + 2F_{23} + \frac{2}{p^2} J_0 - \frac{1}{p} \frac{\partial J_0}{\partial p} + \frac{1}{p} J_2
\end{aligned}\right\} \quad (8.7')$$

$$\left.\begin{aligned}
H_0 &= -I_0 + \frac{J_0}{p} \\
H_1 &= -\frac{\partial I_0}{\partial l} + I_1 + \frac{1}{p} \frac{\partial J_0}{\partial l} - \frac{J_1}{p} \\
H_2 &= \frac{1}{p} I_0 - \frac{\partial I_0}{\partial p} + I_2 - \frac{2}{p^2} J_0 + \frac{1}{p} \frac{\partial J_0}{\partial p} - \frac{J_2}{p}
\end{aligned}\right\} \quad \ldots \quad \ldots \quad (8.8')$$

$$\left.\begin{aligned}
I_0 &= -H_0 + \frac{J_0}{p} \\
I_1 &= -\frac{\partial H_0}{\partial l} + H_1 + \frac{1}{p} \frac{\partial J_0}{\partial l} - \frac{J_1}{p} \\
I_2 &= \frac{1}{p} H_0 - \frac{\partial H_0}{\partial p} + H_2 - \frac{2}{p^2} J_0 + \frac{1}{p} \frac{\partial J_0}{\partial p} - \frac{J_2}{p^2}
\end{aligned}\right\} \quad \ldots \quad (8.9')$$

$$\left.\begin{aligned}
J_1 &= \frac{1}{2} \frac{\partial J_0}{\partial l} \\
J_2 &= -\frac{J_0}{p} + \frac{1}{2} \frac{\partial J_0}{\partial p}
\end{aligned}\right\}. \quad \ldots \quad \ldots \quad \ldots \quad (8.10')$$

4. Final Results obtained from Continuity and Symmetry Conditions

All the relations that have been obtained by virtue of continuity and symmetry conditions and subsequent limited expansions of the defining scalars of the second order velocity correlations can be combined and put in the following forms:

$$\left.\begin{aligned}
D_0 &= E_0 - \frac{F_0}{p} = -\int^{l} \frac{\partial I_0}{\partial p}\, dl + f_1(p) \\
F_0 &= G_0
\end{aligned}\right\} \quad \ldots \quad \ldots \quad (11.1)$$

$$\left.\begin{aligned} E_1 &= -\frac{\partial D_0}{\partial l}+D_1+\frac{1}{p}\,\frac{\partial F_0}{\partial l}-\frac{1}{p}F_1+\frac{I_0}{p}-\frac{J_0}{p^2} \\ &= \frac{\partial E_0}{\partial l}+D_1-\frac{1}{p}\,\frac{\partial G_0}{\partial l}+\frac{1}{p}G_1-\frac{H_0}{p}+\frac{J_0}{p^2} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (11.2)$$

$$\left.\begin{aligned} A_0 &= -D_1-\int^l \frac{\partial I_1}{\partial p}\,dl+f_2(p) \\ &= -\frac{B_{11}}{2}-\frac{E_1}{4}-\frac{H_2}{4} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (11.3)$$

$$\left.\begin{aligned} D_2 &= -\int^l \frac{\partial}{\partial p}(A_0+I_2)\,dl+f_3(p) = E_2+\frac{\partial D_0}{\partial p}+\frac{1}{p^2}F_0-\frac{1}{p}\,\frac{\partial F_0}{\partial p}+\frac{F_2}{p} \\ &= E_2-\frac{\partial E_0}{\partial p}-\frac{1}{p^2}G_0+\frac{1}{p}\,\frac{\partial G_0}{\partial p}-\frac{G_2}{p} \end{aligned}\right\} \quad (11.4)$$

$$B_0+C_0 = -\int^l \frac{\partial G_0}{\partial p}\,dl+f_4(p) \quad \ldots \quad \ldots \quad \ldots \quad (11.5)$$

$$\left.\begin{aligned} B_1+C_1+E_0 &= -3D_0-F_2 = -\int^l \frac{\partial G_1}{\partial p}\,dl+f_5(p) \\ C_1 &= \frac{1}{2}\,\frac{\partial C_0}{\partial l}-\frac{1}{2p}(F_0+G_0) \\ C_2 &= \frac{1}{2}\,\frac{\partial C_0}{\partial p} \end{aligned}\right\} \quad \ldots \quad (11.6)$$

$$\frac{1}{2}\,\frac{\partial B_0}{\partial p} = B_2 = -G_1-H_0-3I_0-J_2 = -C_2-\int^l \frac{\partial}{\partial p}(E_0+G_2)\,dl+f_6(p) \quad (11.7)$$

$$\left.\begin{aligned} C_{11} &= -B_{22}-C_{22}-2D_1-\tfrac{1}{2}E_1-F_{23} \\ &= -B_{11}-\int^l \frac{\partial G_0}{\partial p}\,dl+f_7(p) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (11.8)$$

$$B_{22}+C_{22}+E_1 = -\int^l \frac{\partial G_{22}}{\partial p}\,dl+f_8(p) \quad \ldots \quad \ldots \quad (11.9)$$

$$\left.\begin{aligned} 2B_{33} &= -H_2-4I_2-2J_{11}-2J_{33} \\ &= -2C_{33}-2\int^l \frac{\partial}{\partial p}(E_2+G_{33})\,dl-2f_9(p) \end{aligned}\right\} \quad \ldots \quad (11.10)$$

$$\left.\begin{aligned} 2(B_{23}+C_{23})+E_2 &= -4D_1-2F_{11}-2F_{33} \\ &= -\int^l \frac{\partial}{\partial p}(E_1+2G_{23})\,dl+f_{10}(p) \end{aligned}\right\} \quad \ldots \quad (11.11)$$

$$J_0 = p(H_0+I_0) = -B_0-\int^p \frac{\partial F_0}{\partial l}\,dp+\psi_1(l) \quad \ldots \quad \ldots \quad (11.12)$$

$$B_1 = \frac{1}{2}\frac{\partial B_0}{\partial l} = -J_1 - \int^p \frac{\partial}{\partial l}(F_1+H_0)\,dp + \psi_2(l) \qquad \text{.. (11.13)}$$

$$H_0+B_2+J_2 = -\int^p \frac{\partial F_2}{\partial l}\,dp + \psi_3(l) \quad \text{..} \quad \text{..} \quad \text{.. (11.14)}$$

$$B_{22} = -J_{22} - \int^p \frac{\partial}{\partial l}(F_{22}+H_1)\,dp + \psi_4(l) \quad \text{..} \quad \text{.. (11.15)}$$

$$\left.\begin{aligned} -4I_2+2\left[B_{11}+\int^p \frac{\partial F_{11}}{\partial l}\,dp+\psi_3(l)\right] \\ = -H_2-2\int^p \frac{\partial F_{33}}{\partial l}\,dp+2\psi_5(l) \end{aligned}\right\} \quad \text{..} \quad \text{.. (11.16)}$$

$$\left.\begin{aligned} 2B_{23}+2J_{23}+H_1 &= -2G_{11}-4I_1 \\ &= -\int^p \frac{\partial}{\partial l}(2F_{23}+H_2)\,dp+\psi_6(l) \end{aligned}\right\} \quad \text{.. (11.17)}$$

$$F_1+G_1 = \frac{\partial G_0}{\partial l} - \frac{J_0}{p} \quad \text{..} \quad \text{..} \quad \text{.. (11.18)}$$

$$F_2+G_2 = -\frac{G_0}{p} + \frac{\partial G_0}{\partial p} \quad \text{..} \quad \text{..} \quad \text{.. (11.19)}$$

$$H_1-I_1 = -\frac{\partial I_0}{\partial l} = \frac{\partial H_0}{\partial l} \quad \text{..} \quad \text{..} \quad \text{.. (11.20)}$$

$$\left.\begin{aligned} F_{11}-G_{11} &= -\frac{1}{2p^2}G_0+\frac{1}{2p}\frac{\partial G_0}{\partial p}-\frac{1}{p}G_2 \\ &= \frac{1}{2p^2}F_0-\frac{1}{2p}\frac{\partial F_0}{\partial p}+\frac{1}{p}F_2 \end{aligned}\right\} \quad \text{..} \quad \text{.. (11.21)}$$

$$\left.\begin{aligned} F_{22}-G_{22} &= \frac{1}{2p^2}G_0-\frac{1}{2p}\frac{\partial G_0}{\partial p}+\frac{1}{2}\frac{\partial^2 G_0}{\partial l^2}-\frac{\partial G_1}{\partial l}+\frac{G_2}{p} \\ &= -\frac{1}{2p^2}F_0+\frac{1}{2p}\frac{\partial F_0}{\partial p}-\frac{1}{2}\frac{\partial^2 F_0}{\partial l^2}+\frac{\partial F_1}{\partial l}-\frac{F_2}{p} \end{aligned}\right\} \quad \text{.. (11.22)}$$

$$\left.\begin{aligned} F_{33}-G_{33} &= \frac{3}{2p^2}G_0-\frac{3}{2p}\frac{\partial G_0}{\partial p}+\frac{1}{2}\frac{\partial^2 G_0}{\partial p^2}+\frac{G_2}{p}-\frac{\partial G_2}{\partial p} \\ &= -\frac{3}{2p^2}F_0+\frac{3}{2p}\frac{\partial F_0}{\partial p}-\frac{1}{2}\frac{\partial^2 F_0}{\partial p^2}-\frac{F_2}{p}+\frac{\partial F_2}{\partial p} \end{aligned}\right\} \quad \text{.. (11.23)}$$

$$\left.\begin{aligned} 2(F_{23}-G_{23}) &= -\frac{1}{p}\frac{\partial G_0}{\partial l}+\frac{\partial^2 G_0}{\partial l\,\partial p}+\frac{G_1}{p}-\frac{\partial G_1}{\partial p}-\frac{\partial G_2}{\partial p}-\frac{J_2}{p} \\ &= \frac{1}{p}\frac{\partial F_0}{\partial l}-\frac{\partial^2 F_0}{\partial l\,\partial p}-\frac{F_1}{p}+\frac{\partial F_1}{\partial p}+\frac{\partial F_2}{\partial p}-\frac{J_2}{p} \end{aligned}\right\} \quad \text{.. (11.24)}$$

$$\left.\begin{aligned} H_2-I_2 &= \frac{1}{p}I_0-\frac{\partial I_0}{\partial p}+\frac{3J_0}{p^2}-\frac{J_2}{p} \\ &= -\frac{1}{p}H_0+\frac{\partial H_0}{\partial p}+\frac{3J_0}{p^2}-\frac{J_2}{p} \end{aligned}\right\}. \quad \text{..} \quad \text{.. (11.25)}$$

It is to be noted that while solving the eqns. (4.1′) and (4.2′) we have introduced ten arbitrary functions of p, namely $f_1(p)$, $f_2(p)$, $f_3(p)$, $f_4(p)$, $f_5(p)$, $f_6(p)$, $f_7(p)$, $f_8(p)$, $f_9(p)$, $f_{10}(p)$, $f_{11}(p)$, $f_{12}(p)$, and while solving the eqns. (4.3′) we have introduced six arbitrary functions of l, namely $\psi_1(l)$, $\psi_2(l)$, $\psi_3(l)$, $\psi_4(l)$, $\psi_5(l)$, $\psi_6(l)$ respectively.

5. Conclusions

From eqns. (11.1) to (11.25) we may draw the following conclusions:

(i) The tensor F_{ij} is not symmetrical in its indices i, j even up to second order in r, s, q, as the pairs of coefficients (D, E), (F, G) and (H, I) are not equal even up to the order of r, s and q as given in (9).

(ii) The energy parts associated with three component directions $\vec{\lambda}$, $\vec{\psi}$, $\vec{\chi}$ $(\vec{\chi}$ being azimuthal direction$)$ at a point within the turbulent medium are respectively given by

$$[\lambda_i\lambda_j F_{ij}]_{r=0} = B_0+C_0 = -\int^l \frac{\partial G_0}{\partial p}\,dl+f_4(p)$$

(cf. eqn. (11.5))

$$[\psi_i\psi_j F_{ij}]_{r=0} = B_0+J_0 = -\int^p \frac{\partial F_0}{\partial l}\,dp+\psi_1(l)$$

(cf. eqn. (11.12))

$$[\chi_i\chi_j F_{ij}]_{r=0} = B_0 = 2\int^p B_2\,dp + \text{a function of } l \text{ alone}$$

(cf. eqn. (11.7))

(iii) The energy parts associated with three component directions at a point in the turbulent medium so envisaged have the possibility of vanishing with distance from the origin which is here the central point of the entrance face of the tunnel. Thus the scheme of turbulence proposed in this work is 'not conservative' and is suitable in too extensive a region.

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