MOTION OF IMMISCIBLE LIQUIDS IN A CRACKED HETEROGENEOUS POROUS MEDIUM WITH PRESSURE DEPENDENT PHASE DENSITIES

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(Communicated by R. S. Varma, F.N.I.)

(*Received 29 November 1967*)

Simultaneous motion of oil and water in a cracked heterogeneous porous medium is discussed for those situations in which the phases have pressure dependent densities. An analytical expression for phase saturation is obtained by the use of perturbation technique.

§ 1. INTRODUCTION

Bokserman *et al.* (1964) have recently given a formal description of the motion of immiscible liquids in a cracked homogeneous porous medium, i.e. medium traversed by a branched system of cracks of varying orientation. The author (*in press*) has obtained an analytical solution for a specific problem when pressure discontinuity between the phases is considered. In this paper we discuss the motion of oil and water in a cracked heterogeneous porous medium when the phases have pressure dependent densities. Water with constant velocity V is injected into a cracked porous seam saturated with oil. We have obtained an expression for phase saturation by using standard results of Jones (1946, 1949), Oroveanu (1963), Muskat (1949), Mattax and Kyte (1962), and Vezirov and Kocheshkov (1963).

§ 2. STATEMENT OF PROBLEM

Water with constant velocity V is injected into a seam saturated with oil and consisting of heterogeneous porous medium traversed by a branched system of cracks varying in orientation. It is assumed that the entire oil on the initial boundary of the seam, $x = 0$ (x is measured in the direction of displacement), is displaced through a small distance due to the impact of injecting water.

We are particularly interested in obtaining an analytical expression for phase saturation when the densities are pressure dependent. For definiteness, the laws of variation are taken in standard forms (§§ 5 and 6).

§ 3. Flow in a Cracked Porous Medium

The physics of flow in a cracked porous medium have been described in detail by Bokserman *et al.* (1964) (*see* Figs. 1 and 2).

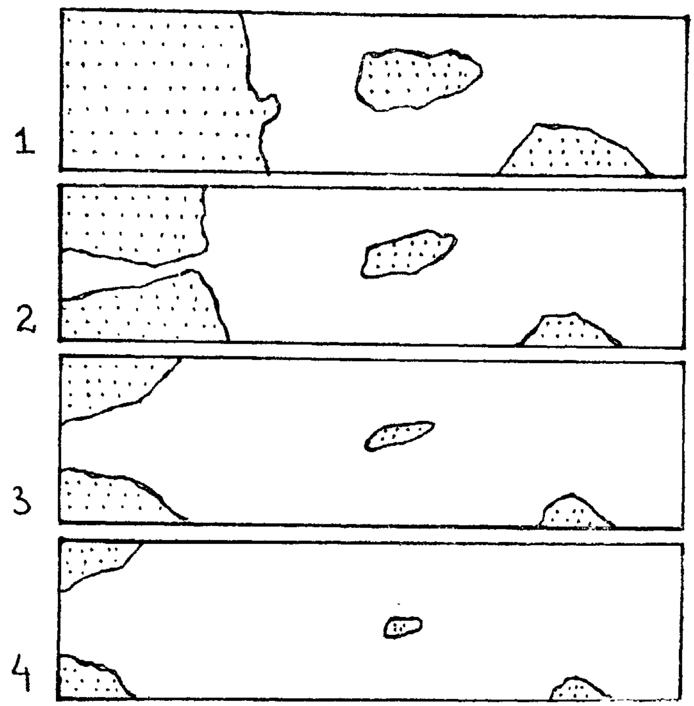

Fig. 1. Water impregnation of a cracked porous model at the successive moments 1, 2, 3, 4 when it moves from right to left (region shaded with dots represent oil).

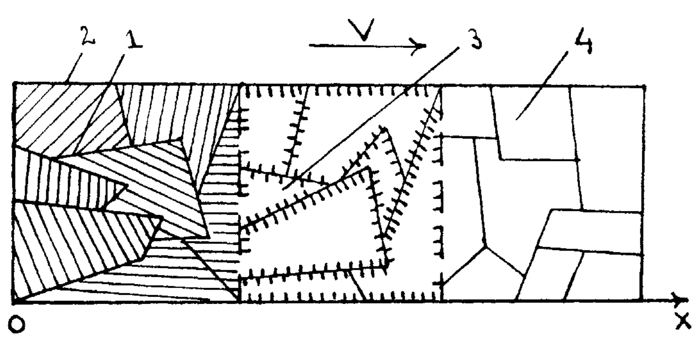

Fig. 2. Diagram showing the impregnation of a cracked one-dimensional porous seam (only a portion has been shown here) with water. The numbers in the diagram denote the following: (1) cracks, (2) completely impregnated blocks, (3) blocks being impregnated and (4) non-impregnated blocks.

In a cracked porous medium water penetrating the seam along the cracks is sucked into the blocks of rock under the action of capillary forces. The amount of water entering the blocks in an elementary volume of the seam (being a function of time) is designated by $\phi(t)$ and is called impregnation function.

Following Mattax and Kyte (1962) and Vezirov and Kocheshkov (1963), we take the expression for $\phi(t)$ as below:

$$\phi(t) = \frac{A}{2} m_B \rho_k \frac{s^2 \sigma \cos\theta \sqrt{k/m_B}}{\mu_o} \left(t \frac{s^2 \sigma \cos\theta \sqrt{k/m_B}}{\mu_o} \right)^{-\frac{1}{2}} \quad t \leqslant t_k \quad \ldots \quad (3.1)$$

where m_B denotes the porosity of the blocks; ρ_k, the saturation of the blocks with water at the moment t_k; s, the mean specific surface area of the blocks; σ, the surface tension; θ, the angle of wetting; μ_o, the kinematic viscosity of oil; A, a constant coefficient; and k, the permeability of the crack system which may be regarded constant in eqn. (3.1).

The following integral equation is obtained by Bokserman *et al.* (1964):

$$\left.\begin{aligned} &\int_0^{\xi(\tau = T)} \phi[T - \tau(\xi)]\, d\xi = \bar{q}(T) \\ &T = t \frac{\sigma \cos\theta\, s^2 \sqrt{k/m_B}}{\mu_o} \\ &\phi(T) = \frac{A}{2} m_B \rho_k \frac{\sigma \cos\theta\, s^2 \sqrt{k/m_B}}{\mu_o} T^{-\frac{1}{2}} \\ &\xi = \frac{x}{l} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (3.2)$$

where $\bar{q}(T)$ denotes the delivery of water per unit surface area perpendicular to the direction considered in the one-dimensional seam, l is the mean block size and $x = x(\tau)$ the seam length impregnated by water at the moment $\tau \leqslant t$.

In particular, if $\bar{q}$ = constant, the solution of eqn. (3.2) for the initial condition $\xi = 0$ at $T = 0$ is (Bokserman *et al.* 1964)

$$\tau = \bar{a}\xi^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

where

$$\bar{a} = \left(\frac{\pi A}{4\bar{q}} \frac{s^2 \sigma \cos\theta\, m_B \rho_k \sqrt{k/m_B}}{\mu_o} \right)^2$$

is a constant.

§ 4. Relative Permeability and Phase Saturation Relation

Following Jones (1946, 1949), the relationship between relative permeability and phase saturation may be taken as

$$\bar{K}_w = S_w^3, \quad \bar{K}_o = 1 - \alpha S_w \quad (\alpha = 1{\cdot}11) \quad \ldots \quad \ldots \quad (4.1)$$

where $\tilde{K}_w$ and $\tilde{K}_o$ denote the relative permeabilities of water and oil and S_w, the saturation of water (regarded as the wetting phase).

§ 5. Variation in Phase Densities

Regarding the liquids to be slightly compressible in the problem under investigation [as is frequently the case in problems of oil-water flow streams, encountered in petroleum technology (Muskat 1949, p. 634)], we may write, following Muskat (1949, p. 186), the densities of water (ρ_w) and oil (ρ_o) in the form

$$\rho_w = A_1[1+\alpha_1 p], \quad \rho_o = A_2[1+\alpha_2 p] \qquad \ldots \quad \ldots \quad (5.1)$$

where A_1, A_2, α_1 and α_2 are constants. The last two are small quantities.

§ 6. Heterogeneity in Porous Medium

The variations in porosity m and permeability k of the medium are taken in the following standard forms (Oroveanu 1963):

$$m = m(x) = \frac{1}{a-bx}, \quad k = k(x) = k_0(1+a_1 x) \qquad \ldots \quad \ldots \quad (6.1)$$

where a, b, k_0 and a_1 are constants. In order that the functions $m(x)$ and $k(x)$ remain finite, we assume further that $0 < x < \frac{a}{b}$.

§ 7. Fundamental Equations

Darcy's law gives the seepage velocity of water and oil as

$$v_w = -\frac{\tilde{K}_w}{\mu_w} k \frac{\partial p}{\partial x} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7.1)$$

$$v_o = -\frac{\tilde{K}_o}{\mu_o} k \frac{\partial p}{\partial x} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7.2)$$

where $k = k(x)$ is the permeability of the heterogeneous crack medium, $\tilde{K}_w$ and $\tilde{K}_o$ are the relative permeability of water and oil, which are functions of S_w and S_o (S_w and S_o are the saturations of water and oil) respectively, p denotes the pressure of both water and oil (capillary pressure is neglected), while μ_w and μ_o are the constant kinematic viscosities of the phases.

Following Ryzhik (as in Bokserman 1964), the equations of continuity for the flow of water and oil in the cracks with allowance for the capillary suction may be written as

$$m\frac{\partial}{\partial t}(\rho_w S_w)+\frac{\partial}{\partial x}(\rho_w v_w)+\phi[T-\tau(\xi)] = 0 \qquad \ldots \quad \ldots \quad (7.3)$$

$$m\frac{\partial}{\partial t}(\rho_o S_o)+\frac{\partial}{\partial x}(\rho_o v_o)-\phi[T-\tau(\xi)] = 0 \quad \ldots \quad \ldots \quad \ldots \quad (7.4)$$

where $m = m(x)$ is the porosity of medium and ρ_w and ρ_o, the densities of water and oil respectively, $\phi[T-\tau(\xi)]$ is the capillary suction function defined in § 3.

From the definition of phase saturation (Scheidegger 1960, p. 216), we have

$$S_w+S_o = 1 \qquad \ldots \quad (7.5)$$

§ 8. Equation of Motion and its Solution by Perturbation Method

Substituting the values of ρ_w and ρ_o from eqns. (5.2) and (5.3) in eqns. (7.3) and (7.4) respectively, we get

$$\left[mS_wA_1\alpha_1\frac{\partial p}{\partial t}+mA_1(1+\alpha_1 p)\frac{\partial S_w}{\partial t}+v_wA_1\alpha_1\frac{\partial p}{\partial x}+A_1(1+\alpha_1 p)\frac{\partial v_w}{\partial x}\right]+\phi[T-\tau(\xi)] = 0 \qquad \ldots \quad (8.1)$$

and

$$\left[mS_oA_2\alpha_2\frac{\partial p}{\partial t}+mA_2(1+\alpha_2 p)\frac{\partial S_o}{\partial t}+v_oA_2\alpha_2\frac{\partial p}{\partial x}+A_2(1+\alpha_2 p)\frac{\partial v_o}{\partial x}\right]-\phi[T-\tau(\xi)] = 0. \qquad \ldots \quad (8.2)$$

Putting the values of v_w and v_o from eqns. (7.1) and (7.2) in eqns. (8.1) and (8.2) we have

$$\left[m\frac{\partial S_w}{\partial t}-\frac{\partial}{\partial x}\left(\frac{\bar{K}_w}{\mu_w}k\frac{\partial p}{\partial x}\right)+\alpha_1\left\{mS_w\frac{\partial p}{\partial t}+mp\frac{\partial S_w}{\partial t}+\frac{\bar{K}_w}{\mu_w}k\left(\frac{\partial p}{\partial x}\right)^2-p\frac{\partial}{\partial x}\left(\frac{\bar{K}_w}{\mu_w}k\frac{\partial p}{\partial x}\right)\right\}\right]+\phi[T-\tau(\xi)] = 0 \qquad (8.3)$$

and

$$\left[m\frac{\partial S_o}{\partial t}-\frac{\partial}{\partial x}\left(\frac{\bar{K}_o}{\mu_o}k\frac{\partial p}{\partial x}\right)+\alpha_2\left\{mS_o\frac{\partial p}{\partial t}+mp\frac{\partial S_o}{\partial t}+\frac{\bar{K}_o}{\mu_o}k\left(\frac{\partial p}{\partial x}\right)^2-p\frac{\partial}{\partial x}\left(\frac{\bar{K}_o}{\mu_o}k\frac{\partial p}{\partial x}\right)\right\}\right]-\phi[T-\tau(\xi)] = 0. \qquad (8.4)$$

Elimination of time derivative for saturations from eqns. (8.3), (8.4) and (7.5) gives

$$\left[\alpha_1\left\{mS_w\frac{\partial p}{\partial t}+mp\frac{\partial S_w}{\partial t}+\frac{\bar{K}_w}{\mu_w}k\left(\frac{\partial p}{\partial x}\right)^2-p\frac{\partial}{\partial x}\left(\frac{\bar{K}_w}{\mu_w}k\frac{\partial p}{\partial x}\right)\right\}+\alpha_2\left\{mS_o\frac{\partial p}{\partial t}+mp\frac{\partial S_o}{\partial t}+\frac{\bar{K}_o}{\mu_o}k\left(\frac{\partial p}{\partial x}\right)^2-p\frac{\partial}{\partial x}\left(\frac{\bar{K}_o}{\mu_o}k\frac{\partial p}{\partial x}\right)\right\}-\frac{\partial}{\partial x}\left\{\left(\frac{\bar{K}_w}{\mu_w}+\frac{\bar{K}_o}{\mu_o}\right)k\frac{\partial p}{\partial x}\right\}\right] = 0. \qquad \ldots \quad (8.5)$$

The non-linear nature of some terms in (8.5) renders further analysis much cumbersome. This difficulty is, however, overcome by noting that α_1, α_2 and m are small [α_1 and α_2 by virtue of § 5 and m by virtue of Bokserman *et al.* (1964)] and hence the analytical solution is attempted by the perturbation method.

To start with, we will neglect terms involving α_1, α_2 and m so that eqns. (8.3), (8.4) and (8.5) become

$$\frac{\partial}{\partial x}\left(\frac{\bar{K}_w}{\mu_w}k\frac{\partial p}{\partial x}\right) = \phi[T-\tau(\xi)] \quad \ldots \quad (8.6)$$

$$\frac{\partial}{\partial x}\left(\frac{\bar{K}_o}{\mu_o}k\frac{\partial p}{\partial x}\right) = -\phi[T-\tau(\xi)] \quad \ldots \quad (8.7)$$

$$\frac{\partial}{\partial x}\left\{\left(\frac{\bar{K}_w}{\mu_w}+\frac{\bar{K}_o}{\mu_o}\right)k\frac{\partial p}{\partial x}\right\} = 0. \quad \ldots \quad (8.8)$$

Integrating (8.8) with respect to x we get

$$\left(\frac{\bar{K}_w}{\mu_w}+\frac{\bar{K}_o}{\mu_o}\right)k\frac{\partial p}{\partial x} = B(t) \quad \ldots \quad (8.9)$$

where $B(t)$ is an arbitrary function of time which is evaluated as follows. We note that

$$\text{at } x = 0, \quad \bar{K}_o = 0 \quad \text{for all time} \quad \ldots \quad (8.10)$$

(since the entire oil from the initial boundary of the formation, $x = 0$, has been displaced for all time by the impact of the striking water).

Further, since the striking water (at $x = 0$) maintains a uniform velocity V, we get from (7.1)

$$v_w(0, t) = -\left(\frac{\bar{K}_w}{\mu_w}k\frac{\partial p}{\partial x}\right)_{x=0} = V. \quad \ldots \quad (8.11)$$

Eqns. (8.9), (8.10) and (8.11) give

$$B(t) = V. \quad \ldots \quad (8.12)$$

Substituting this value of $B(t)$ in (8.9) we have

$$\frac{\partial p}{\partial x} = -\frac{V}{(\bar{K}_w/\mu_w+\bar{K}_o/\mu_o)k(x)}. \quad \ldots \quad (8.13)$$

Combining (8.6) and (8.13) we get

$$V\frac{\partial}{\partial x}\left(\frac{\frac{\bar{K}_w}{\mu_w}k}{\frac{\bar{K}_w}{\mu_w}k+\frac{\bar{K}_o}{\mu_o}k}\right) = -\phi[T-\tau(\xi)]. \quad \ldots \quad (8.14)$$

Since $\tilde{K}_w$ and $\tilde{K}_o$ are functions of S_w and S_o (i.e. $1-S_w$) we may write from (4.1) and (4.2)

$$\gamma(S_w) = \left(\frac{\tilde{K}_w/\mu_w}{k_w/\mu_w+\tilde{K}_o/\mu_o}\right) = \frac{PS_w^3}{PS_w^3+1-\alpha S_w}, \quad P = \mu_o/\mu_w . \qquad (8.15)$$

Differentiating (8.15) we get

$$\gamma'(S_w) = \frac{3PS_w^2-2P\alpha S_w^3}{(PS_w^3+1-\alpha S_w)^2} . \qquad (8.16)$$

From (8.14), (8.15) and (8.16) we get

$$V\left\{\frac{3PS_w^2-2P\alpha S_w^3}{(PS_w^3+1-\alpha S_w)^2}\right\}\frac{\partial S_w}{\partial x}+\phi[T-\tau(\xi)] = 0 \qquad (8.17)$$

where

$$\phi[T-\tau(\xi)] = D[T-\tau(\xi)]^{-\frac{1}{2}}, \quad D = \frac{A}{2}\, m_B \rho_k \frac{(\sigma \cos\theta + s^2 . \sqrt{k/m_B})}{\mu_o}$$

(D is a constant because the permeability k is to be regarded constant here according to the remark in § 3).

From (3.2) we have

$$\tau = \bar{a}\xi^2 = \frac{\bar{a}}{l^2}\, x^2 . \qquad (8.18)$$

Putting this value of $\tau(\xi)$ in (8.17) we get

$$V\left\{\frac{3PS_w^2-2P\alpha S_w^3}{(PS_w^3+1-\alpha S_w)^2}\right\}\frac{\partial S_w}{\partial x} = -\frac{D}{\sqrt{T-Rx^2}} \qquad (8.19)$$

where

$$R = \frac{\bar{a}}{l^2} .$$

Since, recently, Evgen'ev (1965) has shown that P (ratio of viscosities of oil and water) in most cases of practical interest is very large therefore we take $1/P$ as a small quantity. Keeping this in view, we get from (8.19) on integration

$$\left(\frac{1-\alpha S_w}{PS_w^3}\right) = \frac{D}{V\sqrt{R}}\sin^{-1}\left(\frac{x\sqrt{R}}{\sqrt{T}}\right)+E \qquad (8.20)$$

where E is an arbitrary constant which is evaluated by noting the following condition from (8.10) and (4.2):

$$S_w = \frac{1}{\alpha} \text{ at } x = 0. \qquad (8.21)$$

For this condition we find that $E = 0$ so that (8.20) becomes

$$\frac{D}{V\sqrt{R}}\sin^{-1}\left(\frac{x\sqrt{R}}{\sqrt{T}}\right) = \left(\frac{1-\alpha S_w}{PS_w^3}\right). \qquad (8.22)$$

Simplification of (8.22) yields

$$x = \sqrt{\frac{T}{R}} \sin \left\{ \frac{V\sqrt{R}}{D} \left(\frac{1-\alpha S_w}{PS_w^3} \right) \right\}. \qquad (8.23)$$

Partial differentiation of (8.22) gives

$$\frac{\partial S_w}{\partial T} = \frac{DPx}{2V\alpha T} \frac{S_w^3}{\sqrt{T-Rx^2}}. \qquad (8.24)$$

From eqns. (8.13), (4.1) and (4.2) we get

$$\frac{\partial p}{\partial x} = -\frac{V\mu_o}{(PS_w^3+1-\alpha S_w)k_0(1+a_1 x)}. \qquad (8.25)$$

Solving (8.23) for S_w and remembering the remark about $1/P$ we get

$$S_w = \left(\frac{VT^{\frac{1}{2}}}{DP} \right)^{\frac{1}{3}} \frac{1}{x^{\frac{1}{3}}}. \qquad (8.26)$$

Substituting this value in (8.25) we get

$$\frac{\partial p}{\partial x} = -\frac{\mu_o D}{k_0\sqrt{T}}(x - a_1 x^2 + \ldots). \qquad (8.27)$$

Integrating and using the condition that the pressure at initial boundary of the formation ($x = 0$) is considered as constant, p_E, for all time,

$$p = p_E - \frac{\mu_o D}{k_0\sqrt{T}} \left\{ \frac{x^2}{2} - a_1 \frac{x^3}{3} + \ldots \right\}. \qquad (8.28)$$

Differentiating with respect to T we get

$$\frac{\partial p}{\partial T} = \frac{\mu_o D}{2k_0 T^{\frac{3}{2}}} \left\{ \frac{x^2}{2} - a_1 \frac{x^3}{3} + \ldots \right\}. \qquad (8.29)$$

If the values of S_w, $\frac{\partial S_w}{\partial T}$, p, $\frac{\partial p}{\partial x}$, $\frac{\partial p}{\partial T}$ and $m(x)$ from (8.26), (8.24), (8.28), (8.27), (8.29) and (6.1) are substituted in terms involving m and α_1 in eqn. (8.3), viz.

$$\left[\epsilon m \frac{\partial S_w}{\partial T} + \alpha_1 \left\{ \epsilon m S_w \frac{\partial p}{\partial T} + \epsilon m p \frac{\partial S_w}{\partial T} + \frac{\tilde{K}_w}{\mu_w} k \left(\frac{\partial p}{\partial x} \right)^2 - p \frac{\partial}{\partial x} \left(\frac{\tilde{K}_w}{\mu_w} k \frac{\partial p}{\partial x} \right) \right\} \right] \qquad (8.30)$$

where

$$\epsilon = \sigma \cos\theta \sqrt{k/m_B} \, . \, s^2/\mu_o,$$

then (8.30) becomes some function of S_w, x and T, $\psi(S_w, x, T)$ say [t has been changed to T by (3.2)]. The method of perturbation consists in substituting $\psi(S_w, x, T)$ for terms involving α_1 and m in (8.3) and obtaining solution of the equation of motion so obtained.

Replacing terms involving α_1 and m in (8.3) by $\psi(S_w, x, T)$, the equation of motion becomes

$$V\left\{\frac{3PS_w^2-2P\alpha S_w^3}{(PS_w^3+1-\alpha S_w)^2}\right\}\frac{\partial S_w}{\partial x}+\phi[T-\tau(\xi)]+\psi(S_w, x, T)=0 \quad .. \quad (8.31)$$

where

$$-\frac{\partial}{\partial x}\left(\frac{\bar{K}_w}{\mu_w}k\frac{\partial p}{\partial x}\right)=V\frac{\partial}{\partial x}\left(\frac{\bar{K}_w/\mu_w}{\bar{K}_w/\mu_w+\bar{K}_o/\mu_o}\right)=V\left[\frac{3PS_w^2-2P\alpha S_w^3}{(PS_w^3+1-\alpha S_w)^2}\right]$$

and

$$\phi[T-\tau(\xi)]=\frac{D}{\sqrt{T-Rx^2}}$$

from (8.14), (8.15), (8.16), (8.17) and (8.18).

Eqn. (8.31) is an ordinary differential equation of the first order whose formal solution is easily obtained.

§ 9. Slight Heterogeneity in the Medium

Let us assume that the heterogeneity in porosity and permeability is small. Then it follows from eqns. (6.1) and (6.2) that b and α_1 are small quantities. Besides, we know that $1/P$ is also small.

Substituting the values of S_w, $\frac{\partial S_w}{\partial x}$, p, $\frac{\partial p}{\partial x}$, $\frac{\partial p}{\partial T}$ and $m(x)$ from (8.26), (8.24), (8.28), (8.27), (8.29) and (6.1) in (8.30) and considering the order of small quantities in the light of the above remarks, we have (retaining only the first order small quantities)

$$\psi(S_w, x, T)=\frac{1}{\sqrt{T-Rx^2}}\left[\{N_1\epsilon+M_1\alpha_1+L_1\alpha_1\}-(N_1\alpha\epsilon+\alpha\alpha_1)S_w\right.$$
$$+\frac{1}{S_w}\{\alpha^2bN_2\epsilon+M_2\alpha^2\alpha_1\}-\frac{1}{S_w^2}\{2\alpha bN_2\epsilon+2M_2\alpha\alpha_1\}$$
$$\left.+\frac{1}{S_w^3}\{bN_2\epsilon+M_2\alpha_1\}\right] \quad .. \quad .. \quad .. \quad .. \quad .. \quad (9.1)$$

where

$$N_1=\frac{1}{2\alpha a\sqrt{T}}, \quad N_2=\frac{V}{2\alpha a^2PD}=N_1\frac{V\sqrt{T}}{PDa}$$

$$M_1=\epsilon\frac{1}{2\alpha a\sqrt{T}}p_E, \quad M_2=M_1\frac{b}{a}\frac{V\sqrt{T}}{PD}$$

$$L_1=Va_1p_E.$$

Thus our problem reduces to obtaining the solution of (8.31) with $\psi(S_w, x, T)$ replaced by (9.1), so that the equation of motion for saturation becomes

$$V\left\{\frac{3PS_w^2-2P\alpha S_w^3}{(PS_w^3+1-\alpha S_w)^2}\right\}\frac{\partial S_w}{\partial x}=-\frac{1}{\sqrt{T-Rx^2}}\Big[\{N_1\epsilon+M_1\alpha_1+L_1\alpha_1+D\}$$
$$-(N_1\alpha\epsilon+\alpha\alpha_1)S_w+(\alpha^2bN_2\epsilon+M_2\alpha^2\alpha_1)\frac{1}{S_w}-(2\alpha bN_2\epsilon+2M_2\alpha\alpha_1)\frac{1}{S_w^2}$$
$$+(bN_2\epsilon+M_2\alpha_1)\frac{1}{S_w^3}\Big]. \quad \ldots \quad (9.2)$$

Simplification and rearrangement gives

$$-\frac{V}{Z^2P}\left[\delta_1\frac{1}{S_w^2}+\delta_2\frac{1}{S_w^3}+\delta_3\frac{1}{S_w^4}+\delta_4\frac{1}{S_w^5}+\delta_5\frac{1}{S_w^6}+\delta_6\frac{1}{S_w^7}\right]\frac{\partial S_w}{\partial x}=\frac{1}{\sqrt{T-Rx^2}} \quad (9.3)$$

where

$\delta_1=-2\alpha^2(N_1\epsilon+\alpha_1)$, $\delta_2=N_1\alpha\epsilon+3\alpha\alpha_1-2\alpha\alpha_1M_1-2\alpha\alpha_1L_1-2\alpha D$,

$\delta_3=3(N_1\epsilon+M_1\alpha_1+L_1\alpha_1+D)+2\alpha^3(bN_2\epsilon+M_2\alpha_1)$,

$\delta_4=-7\alpha^2(bN_2\epsilon+M_2\alpha_1)$, $\delta_5=8\alpha(bN_2\epsilon+M_2\alpha_1)$,

$\delta_6=-3(bN_2\epsilon+M_2\alpha_1)$, $Z=(N_1\epsilon+M_1\alpha_1+L_1\alpha_1+D)$.

Eqn. (9.3) is an ordinary differential equation of the first order which on integration gives

$$\frac{V}{Z^2P}\left[\delta_1\left(\frac{1}{S_w}\right)+\left(\frac{\delta_2}{2}\right)\frac{1}{S_w^2}+\left(\frac{\delta_3}{3}\right)\frac{1}{S_w^3}+\left(\frac{\delta_4}{4}\right)\frac{1}{S_w^4}+\left(\frac{\delta_5}{5}\right)\frac{1}{S_w^5}+\left(\frac{\delta_6}{6}\right)\frac{1}{S_w^6}\right]$$
$$+G=\frac{1}{\sqrt{R}}\sin^{-1}\left(\frac{x\sqrt{R}}{\sqrt{T}}\right) \quad \ldots \quad (9.4)$$

where G is the constant of integration. For the initial condition

$$S_w=\frac{1}{\alpha} \text{ at } x=0$$

the value of G from (9.4) is

$$G=-\frac{V}{Z^2P}\left[\delta_1\alpha+\left(\frac{\delta_2}{2}\right)\alpha^2+\left(\frac{\delta_3}{3}\right)\alpha^3+\left(\frac{\delta_4}{4}\right)\alpha^4+\left(\frac{\delta_5}{5}\right)\alpha^5+\left(\frac{\delta_6}{6}\right)\alpha^6\right]. \quad \ldots \quad (9.5)$$

Putting this value of G in (9.4) we get the solution of the equation of motion for saturation as

$$x=\sqrt{\frac{T}{R}}\sin\left[\frac{V\sqrt{R}}{Z^2P}\left\{\delta_1\left(\frac{1}{S_w}\right)+\left(\frac{\delta_2}{2}\right)\frac{1}{S_w^2}+\left(\frac{\delta_3}{3}\right)\frac{1}{S_w^3}+\left(\frac{\delta_4}{4}\right)\frac{1}{S_w^4}\right.\right.$$
$$\left.\left.+\left(\frac{\delta_5}{5}\right)\frac{1}{S_w^5}+\left(\frac{\delta_6}{6}\right)\frac{1}{S_w^6}+G\alpha\right\}\right] \quad \ldots \quad (9.6)$$

where

$$G_\alpha = -\left\{\delta_1\alpha+\left(\frac{\delta_2}{2}\right)\alpha^2+\left(\frac{\delta_3}{3}\right)\alpha^3+\left(\frac{\delta_4}{4}\right)\alpha^4+\left(\frac{\delta_5}{5}\right)\alpha^5+\left(\frac{\delta_6}{6}\right)\alpha^6\right\}.$$

Eqn. (9.6) gives the desired result.

§ 10. Discussion

The following deductions can be made from our discussion:

(i) *Cracked Heterogeneous Porous Medium without Variation in Phase Density*

For deducing this case we put $b \neq 0$, $a_1 \neq 0$ and $\alpha_1 = 0$ in eqn. (9.6). This gives

$$x = \sqrt{\frac{T}{R}} \sin\left[\frac{V\sqrt{R}}{Z'^2P}\left\{\delta_1'\left(\frac{1}{S_w}\right)+\left(\frac{\delta_2'}{2}\right)\frac{1}{S_w^2}+\left(\frac{\delta_3'}{3}\right)\frac{1}{S_w^3}+\left(\frac{\delta_4'}{4}\right)\frac{1}{S_w^4}\right.\right.$$
$$\left.\left.+\left(\frac{\delta_5'}{5}\right)\frac{1}{S_w^5}+\left(\frac{\delta_6'}{6}\right)\frac{1}{S_w^6}+G_\alpha'\right\}\right] \quad .. \ (10.1)$$

where

$$\delta_1' = -2\alpha^2N_1\epsilon, \quad \delta_2' = N_1\alpha\epsilon-2\alpha D$$
$$\delta_3' = 3(N_1\epsilon+D)+2\alpha^3bN_2\epsilon, \quad \delta_4' = -7\alpha^2bN_2\epsilon$$
$$\delta_5' = 8\alpha bN_2\epsilon, \quad \delta_6' = -3bN_2\epsilon, \quad Z' = N_1\epsilon+D$$

and

$$G_\alpha' = -\left\{\delta_1'(\alpha)+\left(\frac{\delta_2'}{2}\right)\alpha^2+\left(\frac{\delta_3'}{3}\right)\alpha^3+\left(\frac{\delta_4'}{4}\right)\alpha^4+\left(\frac{\delta_5'}{5}\right)\alpha^5+\left(\frac{\delta_6'}{6}\right)\alpha^6\right\}.$$

(ii) *Cracked Homogeneous Porous Medium with Variation in Phase Density*

For deducing this case we put $b = 0$, $a_1 = 0$ and $\alpha_1 \neq 0$ in eqn. (9.6). This gives

$$x = \sqrt{\frac{T}{R}} \sin\left[\frac{V\sqrt{R}}{Z^2P}\left\{\delta_1\left(\frac{1}{S_w}\right)+\left(\frac{\delta_2}{2}\right)\frac{1}{S_w^2}+\left(\frac{\delta_3''}{3}\right)\frac{1}{S_w^3}+\left(\frac{\delta_4''}{4}\right)\frac{1}{S_w^4}\right.\right.$$
$$\left.\left.+\left(\frac{\delta_5''}{5}\right)\frac{1}{S_w^5}+\left(\frac{\delta_6''}{6}\right)\frac{1}{S_w^6}+G_\alpha''\right\}\right] \quad .. \ (10.2)$$

where

$$\delta_3'' = 3(N_1\epsilon+M_1\alpha_1+L_1\alpha_1+D)+2\alpha^3M_2\alpha_1,$$
$$\delta_4'' = -7\alpha^2\alpha_1M_2, \quad \delta_5'' = 8\alpha^2\alpha_1M_2,$$
$$\delta_6'' = -3M_2\alpha_1,$$
$$G_\alpha'' = -\left\{\delta_1\alpha+\left(\frac{\delta_2}{2}\right)\alpha^2+\left(\frac{\delta_3''}{3}\right)\alpha^3+\left(\frac{\delta_4''}{4}\right)\alpha^4+\left(\frac{\delta_5''}{5}\right)\alpha^5+\left(\frac{\delta_6''}{6}\right)\alpha^6\right\},$$

and δ_1, δ_2 and Z have the same value as in (9.3).

(iii) *Cracked Homogeneous Porous Medium without Variation in Phase Density*

For deducing this case we put $b = 0$, $a_1 = 0$ and $\alpha_1 = 0$ in eqn. (9.6). This gives

$$x = \sqrt{\frac{T}{R}} \sin\left[\frac{V\sqrt{R}}{Z'^2P}\left\{\delta_1'\left(\frac{1}{S_w}\right)+\left(\frac{\delta_2'}{2}\right)\frac{1}{S_w^2}+\left(\frac{\delta_3'''}{3}\right)\frac{1}{S_w^3}+G_\alpha'''\right\}\right] \quad .. \quad (10.3)$$

where

$$\delta_3''' = 3(N_1\epsilon+D)$$

$$G_\alpha''' = -\left\{\delta_1'\alpha+\left(\frac{\delta_2'}{2}\right)\alpha^2+\left(\frac{\delta_3'''}{3}\right)\alpha^3\right\}$$

and Z', δ_1' and δ_2' have the same value as in (10.1).

We will now show that the deduction for homogeneous porous medium without variation in phase density from our discussion is equivalent to the result arrived at by Bokserman *et al.* (1964) for the motion of immiscible liquids in a cracked homogeneous porous medium without considering the variation of phase density.

Bokserman *et al.* (1964) have obtained the following relation:

$$\gamma(S_w) = -\frac{1}{V}\int \phi[T-\tau(\xi)]\,d\xi+F \quad .. \quad .. \quad .. \quad (10.4)$$

where $\phi[T-\tau(\xi)]$ and $\gamma(S_w)$ are the same as in (8.17) and (8.15) respectively and F is the constant of integration. To evaluate F, we note that the condition of our problem, viz.

$$S_w = \frac{1}{\alpha} \text{ at } x = 0,$$

is equivalent to (as is evident from eqn. 8.15)

$$\gamma(S_w) = 1 \text{ at } x = 0.$$

Substituting the value of $\phi[T-\tau(\xi)]$ from (8.17) in (10.4) and evaluating the integral for the condition just stated we get

$$\gamma(S_w) = 1-\frac{D}{V\sqrt{R}}\sin^{-1}\left(\frac{x\sqrt{R}}{\sqrt{T}}\right) \quad .. \quad .. \quad (10.5)$$

where

$$R = \frac{\bar{a}}{l^2}.$$

For deduction of these results from our analysis we put $b = 0$, $a_1 = 0$ and $\alpha_1 = 0$ in eqn. (8.31) and neglect $\epsilon m \frac{\partial S_w}{\partial T}$ on account of the smallness (following Bokserman *et al.* 1964). This gives the equation of motion for saturation as

$$V\gamma'(S_w)\frac{\partial S_w}{\partial x}+\phi[T-\tau(\xi)] = 0 \quad .. \quad .. \quad .. \quad (10.6)$$

where $\gamma'(S_w)$ has been written for

$$\frac{3PS_w^2 - 2P\alpha S_w^3}{(PS_w^3+1-\alpha S_w)^2}$$

by virtue of (8.16).

Putting the value of $\phi[T-\tau(\xi)]$ from (8.17) and (8.18) we have

$$\gamma'(S_w)\,dS_w = -\frac{D}{V\sqrt{T-Rx^2}}\,dx, \quad R = \frac{\bar{a}}{l^2}. \qquad \ldots \qquad \ldots (10.7)$$

Integrating and taking $r(S_w) = 1$ at $x = 0$, we get

$$\gamma(S_w) = 1 - \frac{D}{V\sqrt{R}} \sin^{-1}\left(\frac{X\sqrt{R}}{\sqrt{T}}\right). \qquad \ldots \qquad \ldots (10.8)$$

Our contention follows from the equivalence of eqns. (10.5) and (10.8).

§ 11. Conclusion

An expression for the phase saturation in a cracked porous medium of specific heterogeneity has been obtained when the phases have pressure dependent densities. Deductions for heterogeneous medium without variation in phase density and homogeneous medium with or without phase density variation have been made.

Acknowledgement

The author gratefully acknowledges the valuable guidance of Prof. G. Paria of Shri Govindram Seksaria Technological Institute, Indore, in the preparation of this paper.

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