AXISYMMETRIC PROBLEMS IN THE THEORY OF ELASTICITY FOR A SEMI-INFINITE TWO-DIMENSIONAL MEDIUM CONTAINING A GRIFFITH CRACK

by K. N. Srivastava *and* M. Kumar, *M.A. College of Technology, Bhopal (M.P.)*

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The elastic equilibrium of a semi-infinite two-dimensional medium containing a Griffith crack situated parallel to the free boundary is investigated. Throughout it is assumed that the equations of the classical theory of elasticity hold. Two problems are considered. In the first problem it is assumed that the free edge is stress-free while in the second the free edge is assumed to be rigidly clamped. In the case of an axisymmetric loading the problems are first reduced to a system of simultaneous dual integral equations involving trigonometric kernels and then to simultaneous Fredholm integral equations of the second kind. Finally, analytical expressions for stress intensity factor, shape of the crack and the crack energy are derived.

1. Introduction

In recent years interest in fracture mechanics has focused attention to the problem of determining stress in the vicinity of cracks. Discussions of brittle fracture phenomena due to cracks [Irwin (1958) and Barenblatt (1962)] have been based on the calculations of stress in the neighbourhood of cracks. The theory of cracks in two-dimensional medium was first developed by Griffith (1920). Sneddon and Elliot (1946) solved the problem of finding the distribution of stress in the neighbourhood of crack by considering the corresponding mixed boundary value problem for two-dimensional infinite medium. They used Fourier transforms technique with the help of which they derived expressions for stress components and components of displacement vector. Following Sneddon and Elliot we shall be using the method of Fourier transform in our study of the problem under discussion.

In the formulation of problems we shall follow the method suggested in the work of Kuz'min and Ufland (1965). Two problems in which different boundary conditions are satisfied on the surface of free boundary are discussed. Each problem is first reduced to a system of simultaneous dual integral equations involving trigonometric kernels and then to simultaneous Fredholm integral equations of the second kind. These integral equations are best solved by numerical methods. The numerical calculations are being carried out which will be announced elsewhere. However, in the case when $h \gg 1$, it is possible to solve them iteratively.

To illustrate the application of the results derived in the following sections, expressions for stress intensity factor, shape of the crack and the energy required to open the crack, in the case when the shearing stress on the surface of crack is zero and the normal stress is constant, are derived.

2. Formulation of the Problem

We shall consider displacement field in a perfectly elastic medium. With regard to its mechanical properties the medium is assumed to be isotropic and homogeneous. In the problem that we shall consider, it is assumed that there is symmetry about y-axis. For a symmetric deformation the displacement vector U may be taken to have the components $(u, v, 0)$ and the components of stress tensor will be represented by σ_x, σ_y and τ_{xy}.

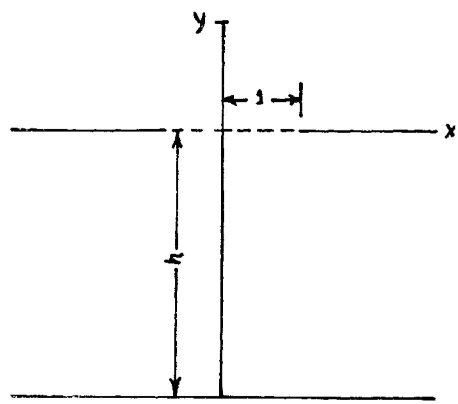

Fig. 1. Griffith crack in a semi-infinite medium situated parallel to free boundary.

Let us assume that the body is divided in two domains: (1) The layer $-h < y < 0$ where h is measured in terms of the distance of the ends of the crack from origin which is taken to be the unit of measurement and (2) the half space $0 < y < \infty$. Let the given stresses and components of displacement be denoted by indices 1 and 2 in the two layers. We shall consider two problems.

Problem 1

The free boundary is assumed stress-free and stresses on the surface of crack are prescribed. The boundary conditions can be written as

$$\sigma_y\big|_{y=-h} = 0, \qquad \tau_{xy}\big|_{y=-h} = 0 \qquad \text{for all } x \quad \ldots \quad (2.1)$$

$$\left.\begin{aligned} \sigma_y\big|_{y=0^-} &= \sigma_1(x), & \tau_{xy}\big|_{y=0^-} &= \tau_1(x) \\ \sigma_y\big|_{y=0^+} &= \sigma_2(x), & \tau_{xy}\big|_{y=0^+} &= \tau_2(x) \end{aligned}\right\} \quad \text{for } -1 < x < 1 \quad \ldots \quad (2.2)$$

Problem 2

When the free boundary is rigidly clamped and the stresses are prescribed on the surface of the crack. In this case we have

$$u\,|_{y=-h}=0, \quad v\,|_{y=-h}=0 \quad \text{for all } x \qquad \text{.. .. (2.3)}$$

together with the condition (2.2).

In addition, for $y = 0$, to pass through the region not occupied by the crack, the values of displacements and stresses should be continuous. This requires the following additional conditions

$$\left.\begin{array}{ll} u\,|_{y=0^+}=u\,|_{y=0^-}, & \sigma_y\,|_{y=0^+}=\sigma y\,|_{y=0^-} \\ v\,|_{y=0^+}=v\,|_{y=0^-}, & \tau_{xy}\,|_{y=0^+}=\tau_{xy}\,|_{y=0^-} \end{array}\right\} \text{ for } 1<x<\infty \qquad \text{.. (2.4)}$$

3. Equations of Equilibrium for the Elastic Field

The basic equations of equilibrium for an elastic medium in two dimensions in the absence of body forces are

$$(2-2\eta)\frac{\partial^2 u}{\partial x^2}+(1-2\eta)\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\,\partial y}=0 \qquad \text{.. .. (3.1)}$$

$$(1-2\eta)\frac{\partial^2 v}{\partial x^2}+2(1-\eta)\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 u}{\partial x\,\partial y}=0 \qquad \text{.. .. (3.2)}$$

$$\tau_{xy}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \qquad \text{.. .. .. (3.3)}$$

$$\sigma_y=\frac{2\mu}{1-2\eta}\left\{(1-\eta)\frac{\partial v}{\partial y}+\eta\frac{\partial u}{\partial x}\right\} \qquad \text{.. .. .. (3.4)}$$

where $\mu = E/2(1+\eta)$; η is Poisson's ratio and E Young's modulus of the elastic material.

For the solution of these partial differential equations (3.1) and (3.2) we introduce Fourier sine and cosine transforms of $u(x, y)$ and $v(x, y)$. We define

$$\bar{u}(\xi, y)=\mathfrak{F}_s[u(x, y), x\to\xi]=\sqrt{\frac{2}{\pi}}\int_0^\infty u(x, y)\sin\xi x\,dx \qquad \text{.. (3.5)}$$

$$\bar{v}(\xi, y)=\mathfrak{F}_c[v(x, y), x\to\xi]=\sqrt{\frac{2}{\pi}}\int_0^\infty v(x, y)\cos\xi x\,dx. \qquad \text{.. (3.6)}$$

Multiply (3.1) by $\sin\xi x$ and $\cos\xi x$ and integrate with respect to x from 0 to infinity to get

$$\left[(1-2\eta)\frac{d^2}{dy^2}-(2-2\eta)\xi^2\right]\bar{u}-\xi\frac{d\bar{v}}{dy}=0 \qquad \text{.. .. (3.7)}$$

$$\left[2(1-\eta)\frac{d^2}{dy^2}-(1-2\eta)\xi^2\right]\bar{v}+\xi\frac{d\bar{u}}{dy}=0 \qquad \text{.. .. (3.8)}$$

From (3.3) and (3.4) we get

$$\bar{\tau}_{xy} = \mathfrak{F}_s[\tau_{xy}(x, y), x \to \xi] = \mu\left[\frac{d\bar{u}}{dy} - \xi\bar{v}\right] \quad .. \quad .. \quad .. \quad (3.9)$$

$$\bar{\sigma}_y = \mathfrak{F}_c[\sigma_y(x, y), x \to \xi] = \frac{2\mu}{1-2\eta}\left[(1-\eta)\frac{d\bar{v}}{dy} + \eta\,\xi\bar{u}\right]. \quad .. \quad (3.10)$$

4. Solutions for Semi-infinite Plane

In this case when the medium is assumed free from disturbances at infinity, we are interested in the solutions which tend to zero as $y \to \infty$. Hence the solutions of (3.7) and (3.8) are

$$\bar{u} = [A(\xi) + B(\xi)\,\xi y]e^{-\xi y}, \quad \bar{v} = [A_1(\xi) + B_1(\xi)\,\xi y]e^{-\xi y} \quad .. \quad (4.1)$$

where

$$B(\xi) = B_1(\xi), \quad A_1(\xi) = A(\xi) + (3-4\eta)B(\xi).$$

Hence the components of displacement vector and stress tensors are

$$u(x, y) = \mathfrak{F}_s[\{A(\xi) + B(\xi)\,\xi y\}e^{-\xi y}, \; \xi \to x] \quad .. \quad .. \quad .. \quad (4.2)$$

$$v(x, y) = \mathfrak{F}_c[\{A_1(\xi) + B(\xi)\,\xi y\}e^{-\xi y}, \; \xi \to x] \quad .. \quad .. \quad .. \quad (4.3)$$

$$\sigma_y(x, y) = -2\mu\,\mathfrak{F}_c[\{A(\xi) + 2(1-\eta)B(\xi) + B(\xi)\,\xi y\}e^{-\xi y}\,.\,\xi, \; \xi \to x] \quad (4.4)$$

$$\tau_{xy}(x, y) = -2\mu\,\mathfrak{F}_s[\{A(\xi) + (1-2\eta)B(\xi) + B(\xi)\,\xi y\}e^{-\xi y}\,.\,\xi, \; \xi \to x]. \quad (4.5)$$

5. Solutions for the Layer $-h < y < 0$

The appropriate solutions in this case are

$$\bar{u} = [\{C + D\xi(y+h)\}\cosh\xi(y+h) + \{E + F\xi(y+h)\}\sinh\xi(y+h)]/\sinh\xi h \quad (5.1)$$

$$\bar{v} = [\{C_1 + D_1\xi(y+h)\}\cosh\xi(y+h) + \{E_1 + F_1\xi(y+h)\}\sinh\xi(y+h)]/\sinh\xi h \quad (5.2)$$

where $C, D, E \ldots$ etc. are functions of ξ and

$$\left.\begin{array}{ll} D = -F_1, & C + E_1 + (3-4\eta)D_1 = 0 \\ D_1 = -F. & C_1 + E - (3-4\eta)D = 0 \end{array}\right\}. \quad .. \quad .. \quad (5.3)$$

Problem 1

When the free boundary is stress-free we have to satisfy the conditions $\sigma_y = 0$ and $\tau_{xy} = 0$ for $y = -h$. From (3.9) and (3.10) we have

$$\bar{\tau}_{xy} = \frac{\mu\xi}{\sinh\xi h}[\{C + D\xi(y+h) + F - E_1 - F_1\xi(y+h)\}\sinh\xi(y+h)$$
$$+ \{E + F\xi(y+h) + D - C_1 - D_1(y+h)\xi\}\cosh\xi(y+h)] \quad .. \quad .. \quad (5.4)$$

$$\bar{\sigma}_y = \frac{\mu\xi}{\sinh\xi h}[\{(1-\eta)(C_1 + D_1\xi(y+h) + F_1) + \eta(E + F(y+h)\xi)\}\sinh\xi(y+h)$$
$$+ \{(1-\eta)(E_1 + F_1\xi(y+h) + D_1) + \eta(C + D\xi(y+h))\}\cosh\xi(y+h)]. \quad (5.5)$$

These equations imply that

$$E+D-C_1 = 0 \qquad \ldots \qquad (5.6)$$

$$(1-\eta)(E_1+D_1)+\eta C = 0. \qquad \ldots \qquad (5.7)$$

From (5.3), (5.6) and (5.7) we have

$$\left.\begin{array}{ll} E = (1-2\eta)D, & C_1 = 2(1-\eta)D \\ E_1 = -(1-2\eta)D_1, & C = -2(1-\eta)D_1 \end{array}\right\}. \qquad \ldots \qquad (5.8)$$

Hence the expressions for the components of displacement vector and stress tensors are

$$u(x, y) = \mathcal{F}_s[(\{D\xi(y+h)-2(1-\eta)D_1\} \cosh (y+h)\xi+\{(1-2\eta)D-D_1\xi(y+h)\} \times \sinh (y+h)\xi)/\sinh \xi h\,;\ \xi \to x] \qquad \ldots \qquad (5.9)$$

$$v(x, y) = \mathcal{F}_c[(\{2(1-\eta)D+D_1\xi(y+h)\} \cosh (y+h)\xi-\{(1-2\eta)D_1+D\xi(y+h)\} \times \sinh (y+h)\xi)/\sinh \xi h\,;\ \xi \to x] \qquad \ldots \qquad (5.10)$$

$$\sigma_y(x, y) = 2\mu \mathcal{F}_c[(\{D+D_1\xi(y+h)\} \sinh (y+h)\xi-D\xi(y+h) \cosh (y+h)\xi)\xi/\sinh \xi h\,;\ \xi \to x] \qquad \ldots \qquad (5.11)$$

$$\tau_{xy}(x, y) = -2\mu\mathcal{F}_s[(\{D_1-D\xi(y+h)\} \sinh (y+h)\xi+D_1\xi(y+h) \cosh (y+h)\xi)\xi/\sinh \xi h\,;\ \xi \to x] \qquad \ldots \qquad (5.12)$$

Problem 2

In this case we have to satisfy the conditions $u = 0$, $v = 0$ for $y = -h$. These conditions imply that

$$C = C_1 = 0, \quad E = (3-4\eta)D, \quad E_1 = -(3-4\eta)D_1. \qquad \ldots \qquad (5.13)$$

Hence we have

$$u(x, y) = \mathcal{F}_s[(D\xi(y+h) \cosh (y+h)\xi+\{(3-4\eta)D-D_1\xi(y+h)\} \sinh (y+h)\xi)/\sinh \xi h\,;\ \xi \to x] \qquad \ldots \qquad (5.14)$$

$$v(x, y) = \mathcal{F}_c[(D_1\xi(y+h) \cosh (y+h)\xi-\{(3-4\eta)D_1+D\xi(y+h)\} \sinh (y+h)\xi)/\sinh \xi h\,;\ \xi \to x] \qquad \ldots \qquad (5.15)$$

$$\sigma_y(x, y) = -2\mu\mathcal{F}_c[(\{(1-2\eta)D-D_1(y+h)\xi\} \sinh (y+h)\xi + \{D(y+h)\xi+2(1-\eta)D_1\} \cosh (y+h)\xi)\xi/\sinh \xi h\,;\ \xi \to x] \qquad \ldots \qquad (5.16)$$

$$\tau_{xy}(x, y) = -2\mu\mathcal{F}_s[(\{D(y+h)\xi-D_1(1-2\eta)\} \sinh (y+h)\xi + \{D_1(y+h)\xi-2(1-\eta)D\} \cosh (y+h)\xi)\xi/\sinh \xi h\,;\ \xi \to x] \qquad (5.17)$$

6. Reduction of the Problems to a System of Simultaneous Triple Integral Equations

Problem 1

We still have to satisfy the boundary conditions for $y = 0$. For the sake of simplicity we assume that $\tau_1(x) = \tau_2(x) = \tau(x)$ and $\sigma_1(x) = \sigma_2(x) = -p(x)$.

Hence for $0 < x < 1$, we have

$$\int_0^\infty [D_1\,\xi h + D - D\,\xi h \coth \xi h]\xi \cos \xi x\, d\xi$$

$$= -\int_0^\infty [A + 2(1-\eta)B]\xi \cos \xi x\, d\xi = -\frac{\pi P(x)}{8(1-\eta)} \quad \ldots \quad (6.1)$$

$$\int_0^\infty [D_1 - D\,\xi h + D_1\,\xi h \coth \xi h]\xi \sin \xi x\, d\xi$$

$$= \int_0^\infty [A + (1-2\eta)B]\xi \sin \xi x\, d\xi = -\frac{\pi T(x)}{8(1-\eta)} \quad \ldots \quad (6.2)$$

where

$$P(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{2(1-\eta)}{\mu}\, p(x), \quad T(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{2(1-\eta)}{\mu}\, \tau(x)$$

and for $1 < x < \infty$, we have

$$\int_0^\infty [D_1\,\xi h + D - D\,\xi h \coth \xi h + A + 2(1-\eta)B]\xi \cos \xi x\, d\xi = 0 \quad \ldots \quad \ldots \quad \ldots \quad (6.3)$$

$$\int_0^\infty [D_1 - D\,\xi h + D_1\,\xi h \coth \xi h - A - (1-2\eta)B]\xi \sin \xi x\, d\xi = 0 \quad \ldots \quad \ldots \quad \ldots \quad (6.4)$$

$$\int_0^\infty [\{D\,\xi h - 2D_1(1-\eta)\} \coth \xi h + (1-2\eta)B - D_1\,\xi h - A] \sin \xi x\, d\xi = 0 \quad \ldots \quad (6.5)$$

$$\int_0^\infty [\{2(1-\eta) + D_1\,\xi h\} \coth \xi h - D\,\xi h - D_1(1-2\eta) - A - (3-4\eta)B] \cos \xi x\, d\xi = 0 \quad (6.6)$$

From (6.1) to (6.4) we have

$$X = D(1 - z \coth z) + D_1 z = -[A + 2(1-\eta)B] \quad \ldots \quad \ldots \quad (6.7)$$

$$Y = -zD + D_1\{1 + z \coth z\} = [A + (1-2\eta)B]\,. \quad \ldots \quad \ldots \quad (6.8)$$

Let us write

$$M = D\{z \coth z + (1-2\eta)\} - D_1\{z + 2(1-\eta) \coth z\} - A \quad \ldots \quad \ldots \quad (6.9)$$

$$N = D\{2(1-\eta) \coth z - z\} + D_1\{z \coth z - 1 + 2\eta\} - A - (3-4\eta)B \quad (6.10)$$

where $z = \xi h$.

From these equations we have

$$4(1-\eta)X = N + I(z)N + J(z)M \quad \ldots \quad \ldots \quad \ldots \quad (6.11)$$

$$-4(1-\eta)Y = M + K(z)M + L(z)N \quad \ldots \quad \ldots \quad \ldots \quad (6.12)$$

where

$$I(z) = -(1 + 2z + 2z^2)e^{-2z}, \quad J(z) = L(z) = -2z^2 e^{-2z}$$

$$K(z) = -(1 - 2z + 2z^2)e^{-2z}.$$

Eqns. (6.1), (6.2), (6.5) and (6.6) together with (6.9), (6.10) to (6.12) lead to a system of simultaneous dual integral equations

$$\int_0^\infty M(\xi)\sin\xi x\,d\xi = 0,\quad 1<x<\infty \qquad (6.13)$$

$$\int_0^\infty N(\xi)\cos\xi x\,d\xi = 0,\quad 1<x<\infty \qquad (6.14)$$

$$\int_0^\infty [N(\xi)+I(z)N(\xi)+M(\xi)J(z)]\xi\cos\xi x\,d\xi = -\frac{\pi}{2}P(x),\quad 0<x<1 \qquad (6.15)$$

$$\int_0^\infty [M(\xi)+K(z)M(\xi)+L(z)N(\xi)]\xi\sin\xi x\,d\xi = \frac{\pi}{2}T(x).\quad 0<x<1. \qquad (6.16)$$

Problem 2

The conditions (2.2) and (2.4) and exactly similar procedure, as for problem 1, leads to the relations

$$X = D\{z\coth z+1-2\eta\}+D_1\{2(1-\eta)\coth z-z\} = A+2(1-\eta)B$$
$$Y = -D\{2(1-\eta)\coth z+z\}+D_1\{z\coth z-1+2\eta\} = A+(1-2\eta)B$$
$$M = D\{z\coth z+3-4\eta\}-D_1 z-A$$
$$N = -Dz+D_1\{z\coth z-3+4\eta\}-A-(3-4\eta)B.$$

Hence

$$4(1-\eta)X = -[N+NI(z)+MJ(z)]$$
$$4(1-\eta)Y = -[M+MK(z)+NL(z)]$$

where

$$\left.\begin{array}{ll} I(z) = 2(b+z+z^2)e^{-2z}/(3-4\eta), & b=\tfrac{1}{2}\{5-12\eta+8\eta^2\} \\ K(z) = 2(b-z+z^2)e^{-2z}/(3-4\eta), & c = 2(1-\eta)(1-2\eta) \\ J(z) = L(z) = 2(z^2+c)/(3-4\eta), & \end{array}\right\}. \qquad (6.17)$$

Once again we are led to the system of simultaneous dual integral equations (6.13) to (6.16) with the functions I, J, K and L having the values given in (6.17).

7. Some Useful Results

We list here, for ready reference, some results that we shall use in future sections. All these results are well known and can be found in Gradshteyn and Ryzhik (1965). Here $H(t)$ is Heaviside's function.

$$\int_0^\infty J_0(\xi t)\cos\xi x\,d\xi = (t^2-x^2)^{-\frac{1}{2}}H(t-x) \qquad (7.1)$$

$$\int_0^\infty J_0(\xi t)\sin\xi x\,d\xi = (x^2-t^2)^{-\frac{1}{2}}H(x-t) \qquad (7.2)$$

$$\int_0^\infty J_1(\xi t)\sin\xi x\,d\xi = xt^{-1}(t^2-x^2)^{-\frac{1}{2}}H(t-x) \qquad (7.3)$$

$$\int_0^t (t^2-x^2)^{-\frac{1}{2}} \cos \xi x \, d\xi = \frac{\pi}{2} J_0(\xi t) \quad \ldots \quad (7.4)$$

$$\int_0^t (t^2-x^2)^{-\frac{1}{2}} x \sin \xi x \, d\xi = \frac{\pi}{2} t J_1(\xi t). \quad \ldots \quad (7.5)$$

The solution of the integral equation

$$f(x) = \int_0^x \frac{g(t)}{(x^2-t^2)^{\frac{1}{2}}} \, dt \quad \ldots \quad (7.6)$$

is

$$g(t) = \frac{2}{\pi} \frac{d}{dt} \int_0^t \frac{x f(x)}{(t^2-x^2)^{\frac{1}{2}}} \, dx. \quad \ldots \quad (7.7)$$

8. Solution of the Simultaneous Dual Integral Equations

We have to find solutions of the integral equations

$$\int_0^\infty M(\xi) \sin \xi x \, d\xi = 0, \quad 1 < x < \infty \quad \ldots \quad (8.1)$$

$$\int_0^\infty N(\xi) \cos \xi x \, d\xi = 0, \quad 1 < x < \infty \quad \ldots \quad (8.2)$$

$$\left.\begin{aligned} &\int_0^\infty [N(\xi)+I(z)N(\xi)+J(z)M(\xi)]\xi \cos \xi x \, d\xi = -\frac{\pi}{2} P(x) \quad (8.3) \\ &\int_0^\infty [M(\xi)+K(z)M(\xi)+L(z)N(\xi)]\xi \sin \xi x \, d\xi = \frac{\pi}{2} T(x) \quad (8.4) \end{aligned}\right\} \quad 0 < x < 1, \; z = \xi h.$$

where I, J, K, L, P and T are known functions and M and N are the unknown functions to be determined. We shall presently show that these equations can be reduced to simultaneous Fredholm integral equations of the second kind which are best solved by numerical methods. However, in the case when $h \gg 1$ and the integrals

$$\int_0^\infty u^n I(u) \, du, \quad n = 0, 1, 2, \ldots$$

and similar integrals for J, K and L are convergent iterative solution can be derived.

Let the trail solutions be

$$M(\xi) = \int_0^1 m(t) J_1(\xi t) \, dt = -\frac{m(1)}{\xi} J_0(\xi) + \frac{1}{\xi} \int_0^1 m'(t) J_0(\xi t) \, dt, \quad m(0) = 0 \quad (8.5)$$

$$N(\xi) = \int_0^1 n(t) J_0(\xi t) \, dt. \quad \ldots \quad (8.6)$$

With the help of (7.1) and (7.2) it is easy to see that (8.1) and (8.2) are satisfied for these choice of $M(\xi)$ and $N(\xi)$. We can write (8.3) and (8.4) as

$$\frac{d}{dx}\int_0^\infty N(\xi)\sin\xi x\,d\xi = \psi(x) = -\frac{\pi}{2}P(x)-\int_0^\infty [I(z)N(\xi)+J(z)M(\xi)]\xi\cos\xi x\,d\xi \quad .. \quad (8.7)$$

$$\int_0^\infty M(\xi)\sin\xi x\,.\,\xi\,d\xi = \phi(x) = \frac{\pi}{2}T(x)-\int_0^\infty [K(z)M(\xi)+L(z)N(\xi)]\xi\sin\xi x\,d\xi. \quad .. \quad (8.8)$$

Substituting the values of $M(\xi)$ and $N(\xi)$ from (8.5) and (8.6), changing the order of integration and the use of (7.2) gives

$$\frac{d}{dx}\int_0^x \frac{n(t)}{(x^2-t^2)^{\frac{1}{2}}}\,dt = \psi(x) \quad .. \quad .. \quad .. \quad (8.9)$$

$$\int_0^x \frac{m'(t)}{(x^2-t)^{\frac{1}{2}}}\,dt = \phi(x). \quad .. \quad .. \quad .. \quad (8.10)$$

The solutions of these integral equations are

$$n(t) = \frac{2t}{\pi}\int_0^t \frac{\psi(x)}{(t^2-x^2)^{\frac{1}{2}}}\,dx \quad .. \quad .. \quad .. \quad .. \quad (8.11)$$

$$m'(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{x\,\phi(x)}{(t^2-x^2)^{\frac{1}{2}}}\,dx. \quad .. \quad .. \quad .. \quad (8.12)$$

Integrating (8.12) we have

$$m(t) = \frac{2}{\pi}\int_0^t \frac{x\,\phi(x)}{(t^2-x^2)^{\frac{1}{2}}}\,dx. \quad .. \quad .. \quad .. \quad .. \quad (8.13)$$

Since

$$\frac{2t}{\pi}\int_0^t \frac{\psi(x)}{(t^2-x^2)^{\frac{1}{2}}}\,dx = -t\int_0^t \frac{P(x)}{(t^2-x^2)^{\frac{1}{2}}}\,dx - \frac{2t}{\pi}\Bigg[\int_0^1 n(u)\int_0^\infty \xi\, I(z)J_0(\xi u)$$
$$\times\int_0^t \frac{\cos\xi x}{(t^2-u^2)^{\frac{1}{2}}}\,dx\,d\xi\,du + \int_0^1 m(u)\int_0^\infty \xi\, J(z)J_1(\xi u)\int_0^t \frac{\cos\xi x}{(t^2-x^2)^{\frac{1}{2}}}\,dx\,d\xi\,du\Bigg]$$

$$\frac{2}{\pi}\int_0^t \frac{x\,\phi(x)}{(t^2-x^2)^{\frac{1}{2}}}\,dx = \int_0^t \frac{x\,T(x)}{(t^2-x^2)^{\frac{1}{2}}}\,dx + \frac{2}{\pi}\Bigg[\int_0^1 m(u)\int_0^\infty \xi\, K(z)J_1(\xi u)\int_0^t \frac{x\sin\xi x}{(t^2-x^2)^{\frac{1}{2}}}\,dx\,d\xi\,du$$
$$+\int_0^1 n(u)\int_0^\infty \xi\, L(z)J_0(\xi u)\int_0^t \frac{x\sin\xi x}{(t^2-x^2)^{\frac{1}{2}}}\,dx\,d\xi\,du\Bigg].$$

By using (7.4) and (7.5) it is seen that (8.11) and (8.13) are the integral equations

$$n(t)+\int_0^1 [n(u)\Gamma(u,t)+m(u)\delta(u,t)]\,du = -t\int_0^t \frac{P(x)}{(t^2-x^2)^{\frac{1}{2}}}\,dx \quad .. \quad (8.14)$$

$$m(t)+\int_0^1 [m(u)\Gamma_1(u,t)+n(u)\delta_1(u,t)]\,du = \int_0^t \frac{x\,\Gamma(x)}{(t^2-x^2)^{\frac{1}{2}}}\,dx \quad .. \quad (8.15)$$

where

$$\Gamma(u,t) = \frac{t}{h^2}\int_0^\infty z\,I(z)J_0\left(\frac{t}{h}z\right)J_0\left(\frac{u}{h}z\right)dz$$

$$\delta(u,t) = \frac{t}{h^2}\int_0^\infty z\,J(z)J_0\left(\frac{t}{h}z\right)J_1\left(\frac{u}{h}z\right)dz$$

$$\Gamma_1(u,t) = \frac{t}{h^2}\int_0^\infty z\,K(z)J_1\left(\frac{t}{h}z\right)J_1\left(\frac{u}{h}z\right)dz$$

$$\delta_1(u,t) = \frac{t}{h^2}\int_0^\infty z\,L(z)J_1\left(\frac{t}{h}z\right)J_0\left(\frac{u}{h}z\right)dz$$

for the determination of the unknown functions $m(t)$ and $n(t)$.

When $h \gg 1$, one can write the expansions of Bessel functions in power series to get

$$\Gamma(u,t) = \frac{I_0 t}{h^2}+\frac{I_1 t}{h^4}(u^2+t^2)+O(h^{-6}) \qquad (8.16)$$

$$\delta(u,t) = \frac{J_0 ut}{h^3}+\frac{J_1 ut}{h^5}(u^2+2t^2)+O(h^{-7}) \qquad (8.17)$$

$$\Gamma_1(u,t) = \frac{K_0 ut^2}{h^4}+\frac{K_1 ut^2}{h^6}(u^2+t^2)+O(h^{-8}) \qquad (8.18)$$

$$\delta_1(u,t) = \frac{L_0 t^2}{h^3}+\frac{L_1 t^2}{h^5}(t^2+2u^2)+O(h^{-7}) \qquad (8.19)$$

where

$$I_0 = \int_0^\infty z\,I(z)\,dz, \quad J_0 = \frac{1}{2}\int_0^\infty z^2 J(z)\,dz, \quad K_0 = \frac{1}{4}\int_0^\infty z^3 K(z)\,dz, \quad L_0 = \frac{1}{2}\int_0^\infty z^2 L(z)\,dz$$

$$I_1 = -\frac{1}{4}\int_0^\infty z^3 I(z)\,dz, \quad J_1 = -\frac{1}{16}\int_0^\infty z^4 J(z)\,dz, \quad K_1 = -\frac{1}{32}\int_0^\infty z^5 K(z)\,dz$$

$$L_1 = -\frac{1}{16}\int_0^\infty z^4 L(z)\,dz.$$

The values of these constants for the problems under discussion are:

Problem 1

$$I_0 = -\frac{3}{2}, \quad J_0 = L_0 = -\frac{3}{4}, \quad K_0 = -\frac{21}{32}$$

$$I_1 = \frac{45}{32}, \quad J_1 = L_1 = \frac{45}{64}, \quad K_1 = -\frac{15}{16}$$

Problem 2

$$I_0 = -\frac{2b+1}{4(3-4\eta)}, \quad J_0 = L_0 = \frac{c+3}{4(3-4\eta)}, \quad K_0 = \frac{3(b+3)}{16(3-4\eta)}$$

$$I_1 = -\frac{3b+9}{16(3-4\eta)}, \quad J_1 = L_1 = -\frac{6c+45}{64(3-4\eta)}, \quad K_1 = -\frac{15(2b+15)}{512(3-4\eta)}.$$

Suppose that the solutions of (8.14) and (8.15) can be written as

$$n(t) = n_0(t)+\frac{n_1(t)}{h}+\frac{n_2(t)}{h^2}+\ldots+\frac{n_5(t)}{h^5}+O(h^{-6}) \quad \ldots \quad (8.20)$$

$$m(t) = m_0(t)+\frac{m_1(t)}{h}+\frac{m_2(t)}{h^2}+\ldots+\frac{m_5(t)}{h^5}+O(h^{-6}). \quad \ldots \quad (8.21)$$

Substituting the above values of $n(t)$ and $m(t)$ we get

$$n_0(t) = -t\int_0^t \frac{Px}{(t^2-x^2)^{\frac{1}{2}}}\,dx, \quad n_1(t) = 0$$

$$n_2(t) = -I_0 t\int_0^1 n_0(u)\,du$$

$$n_3(t) = -J_0 t\int_0^1 u m_0(u)\,du$$

$$n_4(t) = -I_0 t\int_0^1 n_2(u)\,du - J_0 t\int_0^1 u m_1(u)\,du - I_1 t\int_0^1 (u^2+t^2) n_0(u)\,du$$

$$n_5(t) = -I_0 t\int_0^1 n_3(u)\,du - J_0 t\int_0^1 u m_2(u)\,du - J_1 t\int_0^1 u(u^2+2t^2) m_0(u)\,du$$

$$m_0(t) = \int_0^t \frac{x\,T(x)}{(t^2-x^2)^{\frac{1}{2}}}\,dx$$

$$m_1(t) = m_2(t) = 0$$

$$m_3(t) = -L_0 t^2\int_0^1 n_0(u)\,du$$

$$m_4(t) = -K_0 t^2\int_0^1 u m_0(u)\,du$$

$$m_5(t) = -L_0 t^2\int_0^1 n_2(u)\,du - L_1 t^2\int_0^1 (t^2+2u^2) n_0(u)\,du.$$

We are, however, interested in the case when the shearing stress on the surface of crack is zero and the normal stress is constant. This implies that $T(x) = 0$ and $P(x) = P$. Hence

$$n_0(t) = -\frac{P\pi t}{2}, \quad n_1(t) = n_3(t) = n_5(t) = 0$$

$$n_2(t) = \frac{P\pi I_0 t}{4}, \quad n_4(t) = -\frac{P\pi I_0^2 t}{8}+\frac{I_1 P\pi t}{4}(t^2+\tfrac{1}{2})$$

$$m_0(t) = m_1(t) = m_2(t) = m_4(t) = 0, \quad m_3(t) = \frac{L_0 P\pi t^2}{4}$$

$$m_5(t) = -\frac{L_0 I_0 P\pi t^2}{8}+\frac{L_1 P\pi t^2}{4}(1+t^2)$$

9. The Stress Intensity Factors

Expressions for stress intensity factors are of great importance for workers in fracture mechanics. We shall now derive analytical expressions for them. They are given by the relations

$$\chi = \lim_{x \to 1^+} (x-1)^{\frac{1}{2}}[\sigma_y(x, 0)]_{x>1} \quad \ldots \quad (9.1)$$

$$\zeta = \lim_{x \to 1^+} (x-1)^{\frac{1}{2}}[\tau_{xy}(x, 0)]_{x>1}. \quad \ldots \quad (9.2)$$

It is easy to show that

$$\frac{(2\pi)^{\frac{1}{2}}}{\mu}(1-\eta)\sigma_y(x, 0) = \int_0^\infty [N(\xi)+N(\xi)I(\xi h)+M(\xi)J(\xi h)]\xi \cos \xi x \, d\xi \quad (9.3)$$

$$\frac{(2\pi)^{\frac{1}{2}}}{\mu}(1-\eta)\tau_{xy}(x, 0) = \int_0^\infty [M(\xi)+M(\xi)K(\xi h)+N(\xi)L(\xi h)]\xi \sin \xi x \, d\xi. \quad (9.4)$$

Substituting the values of $N(\xi)$, $M(\xi)$ we get

$$\frac{(2\pi)^{\frac{1}{2}}}{\mu}(1-\eta)\sigma_y(x, 0) = \frac{d}{dx}\int_0^1 n(t)\int_0^\infty \sin \xi x \, J_0(\xi t)\, d\xi \, dt$$
$$+\frac{1}{h^2}\int_0^1 n(t)\int_0^\infty z\, I(z) \cos \frac{zx}{h} J_0\left(\frac{z}{h}t\right) dz\, dt+\frac{1}{h^2}\int_0^1 m(t)\int_0^\infty z\, J(z) \cos \frac{z}{h} x\, J_0\left(\frac{z}{h}t\right) dz\, dt \quad \ldots \quad (9.5)$$

$$\frac{(2\pi)^{\frac{1}{2}}}{\mu}(1-\eta)\tau_{xy}(x, 0) = \int_0^\infty \sin \xi x\left\{-m(1)J_0(\xi)+\int_0^1 m'(t)J_0(\xi t)\, dt\right\} d\xi$$
$$+\frac{1}{h^2}\int_0^1 m(t)\int_0^\infty z\, K(z) \sin \frac{z}{h} x\, J_1\left(\frac{z}{h}t\right) dz\, dt+\frac{1}{h^2}\int_0^1 n(t)\int_0^\infty z\, L(z) \sin \frac{z}{h} x\, J_0\left(\frac{z}{ht}\right) dz\, dt. \quad \ldots \quad (9.6)$$

Substituting the value of $m(t)$ and $n(t)$ and evaluating the integrals with the help of results given in section 7, we get

$$[\sigma_y(x, 0)]_{x>1} = p\left[\left\{\frac{x}{(x^2-1)^{\frac{1}{2}}}-1\right\}\left\{1-\frac{I_0}{2h^2}+\frac{I_0^2}{4h^4}\right\}+\frac{I_0}{2h^2}-\frac{I_0^2}{4h^4}+\frac{I_1}{h^4}x^4\right.$$
$$\left.-\frac{4I_1}{3h^4}\left\{\frac{x(3x^2-1)}{(x^2-1)^{\frac{1}{2}}}-3x^2-\tfrac{1}{2}\right\}\right] \quad \ldots \quad (9.7)$$

$$[\tau_{xy}(x, 0)]_{x>1} = \frac{p}{2}\left[\left(\frac{2x^2-1}{(x^2-1)^{\frac{1}{2}}}-x\right)\left(\frac{L_0}{h^3}-\frac{L_0 I_0}{2h^5}\right)+\frac{L_0}{h^3}-\frac{2L_1 I_0}{3h^5}+\frac{2L_1}{3h^5}(2x^2+\tfrac{3}{2})\right.$$
$$\left.+\frac{2L_1}{3h^5}\left\{\frac{4x^4+x^2-2}{(x^2-1)^{\frac{1}{2}}}-x(4x^2+1)-2\right\}+O(h^{-7})\right. \quad \ldots \quad (9.8)$$

Hence the stress intensity factors are given by the following expressions:

$$\chi = \frac{p}{\sqrt{2}}\left\{1-\frac{I_0}{2h^2}+\frac{I_0^2}{4h^4}-\frac{8I_1}{3h^4}+O(h^{-6})\right\} \quad \ldots \quad (9\ 9)$$

$$\zeta = \frac{p}{2\sqrt{2}}\left\{\frac{L_0}{h^3}-\frac{L_0 I_0}{h^5}+\frac{2L_1}{h^5}+O(h^{-})^7\right\}. \quad \ldots \quad (9.10)$$

10. Shape of the Crack

We shall consider the case when the free edge is stress-free. The other case can be dealt in the same way. It is easy to show that the components of displacement vector can be expressed as

$$(2\pi)^{\frac{1}{2}}(1-\eta)u(x,0) = \int_0^\infty (M(\xi)[1-\eta+\{(1-\eta)(1-2\mu)+\mu^2\}e^{-2\mu}]$$
$$+N(\xi)[\tfrac{1}{2}-\eta+\{(\tfrac{1}{2}-\eta)(1-2\mu)+\mu^2\}e^{-2\mu}])\sin \xi x\, d\xi \quad \ldots \quad (10.1)$$

$$(2\pi)^{\frac{1}{2}}(1-\eta)v(x,0) = \int_0^\infty (M(\xi)[\tfrac{1}{2}-\eta+\{\mu^2-(\tfrac{1}{2}-\eta)(1-2\mu)\}e^{-2\mu}$$
$$+N(\xi)[1-\eta+\{(1-\eta)(1+2\mu)+\mu^2\}e^{-2\mu}])\cos \xi x\, d\xi, \quad \mu=\xi h \quad \ldots \quad (10.2)$$

Substituting the values of $M(\xi)$ and $N(\xi)$ and after some calculations we get

$$[u(x,0)]_{0<x<1} = \frac{\pi x p}{4\mu}\left[(1-2\eta)\left(\frac{I_0}{2h^2}-1\right)-\frac{(1+\eta)}{4h^2}+\frac{L_0}{h^3}\frac{(1-\eta)}{}(1-x^2)^{\frac{1}{2}}+O(h^{-4})\right] \quad \ldots \quad (10.3)$$

$$[v(x,0)]_{0<x<1} = \frac{\pi p}{16\mu}\left[(1-\eta)(1-x^2)^{\frac{1}{2}}\left\{8-\frac{4I_0}{h^2}+\frac{2}{h^4}\left(I_0^2-\frac{4I_1}{3}\right)-\frac{2I_1}{3h^4}x^2\right\}\right.$$
$$\left.-(5-4\eta)\left(\frac{1}{h}-\frac{I_0}{h^3}\right)+\frac{7-4\eta}{2h^3}(x^2+\tfrac{1}{4})+\frac{L_0}{4h^3}(1-2\eta)(1-2x^2)\right]. \quad (10.4)$$

11. Crack Energy

The energy required to open the crack is given by

$$W = 2\int_0^1 pv(x,0)\, dx. \quad \ldots \quad \ldots \quad \ldots \quad (11.1)$$

Substituting the value of $v(x, 0)$, after some calculation, we get

$$W = \frac{\pi^2 p^2}{8\mu}\left[(1-\eta)\left\{1-\frac{I_0}{2h^2}+\frac{I_0^2}{4h^2}-\frac{I_1}{2h^2}\right\}-(5-4\eta)\left(\frac{1}{h}-\frac{I_0}{h^3}\right)\right.$$
$$\left.+\frac{7(7-4\eta)}{24h^3}+\frac{L_0}{12h^3}(1-2\eta)\right]. \quad \ldots \quad \ldots \quad \ldots \quad (11.2)$$

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