EFFECT OF SMALL HUB RADIUS CHANGE ON FREQUENCIES OF COUPLED VIBRATIONS OF A SLENDER BEAM IN A CENTRIFUGAL FORCE FIELD

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A simple relation has been formulated by perturbation method for determining the change of frequencies for coupled vibrations of a slender rotating beam due to a small change of hub radius. The first order solution shows that a frequency parameter is a linear function of the hub radius change and the constant of proportionality is obtained from the known parameters.

The analysis presented in this paper considers vibration of a slender beam that could represent a turbine blade of simple geometry. The shear centre of each cross-section does not coincide with the centre of gravity, consequently the torsional and bending oscillations are 'coupled'. The beam is attached to a hub of radius r, rotating at a constant angular velocity Ω, as shown in Fig. 1. The beam is allowed to vibrate in a plane making an angle $(\pi/2-\psi)$ with the plane of rotation.

The frequencies of the coupled vibrations and the modal shapes of bending and torsional vibrations can be determined from the solutions of the following differential equations [Tomar (*in press*)] with proper conditions:

$$\left.\begin{aligned} &EI\frac{\partial^4 v}{\partial x^4}-m\Omega^2\left[\frac{d}{dx}\left\{N\frac{\partial v}{dx}\right\}+\sin^2\psi v\right]+m\frac{\partial^2}{\partial t^2}(v+x_\theta\,.\,\theta)=0\\ &GJ\frac{\partial^2\theta}{\partial x^2}-C_1\frac{\partial^4\theta}{\partial x^4}-mx_\theta\frac{\partial^2}{\partial t^2}(v+x_\theta\,.\,\theta)+I_\theta\frac{\partial^2\theta}{\partial t^2}=0\\ &N=\int_x^l(r+x)\,dx \end{aligned}\right\} \quad \ldots \quad (1)$$

where EI, GJ and C_1 are flexural, torsional and warping rigidities respectively; m and I_θ are mass and mass moment of inertia about shear centre axis per unit length; x_θ is the distance between the shear centre axis and the centroidal axis.

When the equations are put in dimensionless variable $\xi = x/l$ and the substitutions

$$\beta^2=\frac{EI}{ml^4},\ \ \alpha^2=\frac{GJ}{ml^2},\ \ c_2=\frac{c_1}{ml^4},\ \ I'_\theta=\frac{I_\theta}{m}\ \text{and}\ \bar{r}=\frac{r}{l}$$

are used, the equations become

$$\left.\begin{aligned}&\beta^2\frac{\partial^4 v}{\partial \xi^4}-\Omega^2\frac{d}{d\xi}\left\{\bar{N}\frac{\partial v}{\partial \xi}\right\}-\Omega^2\sin^2\psi\,.\,v+\frac{\partial^2}{\partial t^2}(v+x_\theta\,.\,\theta)=0\\&\alpha^2\frac{\partial^2\theta}{\partial \xi^2}-c_2\frac{\partial^4\theta}{\partial \xi^4}-x_\theta\frac{\partial^2}{\partial t^2}(v+x_\theta\,.\,\theta)+I_\theta'\frac{\partial^2\theta}{\partial t^2}=0\\&\bar{N}=\int_\xi^1(\bar{r}+\xi)\,d\xi\end{aligned}\right\}. \qquad (2)$$

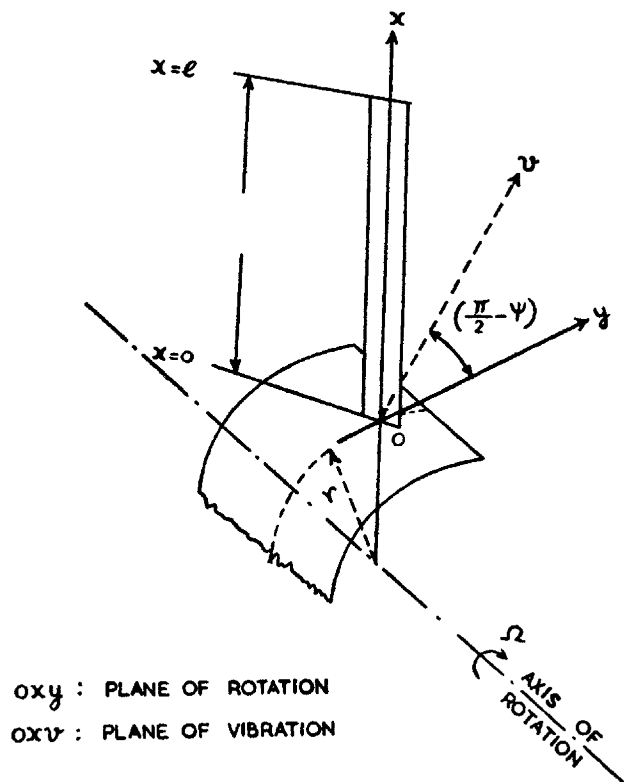

FIG. 1. The system under consideration.

The solution of the equations are of the form

and
$$\left.\begin{aligned}v&=Af(\xi)e^{i\omega t}\\\theta&=B\phi(\xi)e^{i\omega t}\end{aligned}\right\} \qquad .. \quad .. \quad .. \quad .. \quad (3)$$

where A and B are constants which are not independent; and $f(\xi)$ and $\phi(\xi)$ are functions of ξ only and which are also respectively the mode shapes of bending and torsional vibrations.

$$\left.\begin{aligned}
&\beta^2 \frac{d^4 f}{d\xi^4} - \Omega^2 \frac{d}{d\xi}\left\{N \frac{df}{d\xi}\right\} - \gamma^2 . f - x_\theta \omega^2 \phi . \frac{B}{A} = 0 \\
&\alpha^2 \frac{d^2\phi}{d\xi^2} - c_2 \frac{d^4\phi}{d\xi^4} + (x_\theta^2 + I'\theta)\omega^2\phi + x_\theta \omega^2 f . \frac{A}{B} = 0 \\
&\bar{N} = \int_\xi^l (\bar{r}+\xi)\, d\xi, \quad \gamma^2 = (\omega^2 + \Omega^2 \sin^2 \psi)
\end{aligned}\right\} \qquad (4)$$

For the beam clamped at the root the boundary conditions can be written as

$$\left.\begin{aligned}
&f = \frac{df}{d\xi} = \phi = \frac{d^2\phi}{d\xi^2} = 0, \qquad \text{at } \xi = 0 \\
&\frac{d^2 f}{d\xi^2} = \frac{d^3 f}{d\xi^3} = \frac{d\phi}{d\xi} = \frac{d^3\phi}{d\xi^3} = 0, \quad \text{at } \xi = 1
\end{aligned}\right\} \qquad (5)$$

when there is a small change in the hub radius from r to $r_1 = r + \Delta r$, while all other conditions remain the same equations (4) become

$$\left.\begin{aligned}
&\beta^2 \frac{d^4 f_1}{d\xi^4} - \Omega^2 \frac{d}{d\xi}\left\{N_1 \frac{df_1}{d\xi}\right\} - \gamma_1^2 f_1 - x_\theta \omega_1^2 \phi_1 \frac{B}{A} = 0 \\
&\alpha^2 \frac{d^2\phi_1}{d\xi^2} - c_2 \frac{d^4\phi_1}{d\xi^4} + (x_\theta^2 + I_\theta')\omega_1^2 \phi_1 + x_\theta \omega_1^2 f_1 \frac{A}{B} = 0 \\
&\bar{N}_1 = \int_\xi^l (\bar{r}_1 + \xi)\, d\xi, \quad \bar{r}_1 = \bar{r} + \frac{\Delta r}{l} = \bar{r} + \delta \\
&\gamma_1^2 = (\omega_1^2 + \Omega^2 \sin^2 \psi)
\end{aligned}\right\} \qquad (6)$$

Here ω_1 and f_1, ϕ_1 are the new frequency and mode shapes corresponding to new hub radius r_1. The boundary conditions are not affected by the change of hub radius. They remain the same as given by (5) except f and ϕ replaced by f_1 and ϕ_1.

When first and second equations of (4) are multiplied respectively by f_1 and ϕ_1, and integrated from 0 to 1, one gets

$$\left.\begin{aligned}
&\beta^2 \int_0^1 \frac{d^2 f}{d\xi^2} \cdot \frac{d^2 f_1}{d\xi^2}\, d\xi + \Omega^2 \int_0^1 \bar{N} \frac{df}{d\xi} \cdot \frac{df_1}{d\xi}\, d\xi \\
&- \gamma^2 \int_0^1 f f_1\, d\xi - \omega^2 x_\theta \cdot \frac{B}{A} \int_0^1 \phi f_1\, d\xi = 0 \\
&- \alpha^2 \int_0^1 \frac{d\phi}{d\xi} \cdot \frac{d\phi_1}{d\xi}\, d\xi - c_2 \int_0^1 \frac{d^2\phi}{d\xi^2} \cdot \frac{d^2\phi_1}{d\xi^2} \cdot d\xi \\
&+ \omega^2 \left\{ (x_\theta^2 + I_\theta') \int_0^1 \phi\phi_1\, d\xi + x_\theta \frac{A}{B} \int_0^1 f\phi_1\, d\xi \right\} = 0
\end{aligned}\right\} \qquad (7)$$

where the following identities have been used

$$\int_0^1 \frac{d^4 f}{d\xi^4} \cdot f_1 \, d\xi = \int_0^1 \frac{d^2 f}{d\xi^2} \cdot \frac{d^2 f_1}{d\xi^2} \cdot d\xi$$

$$\int_0^1 \frac{d}{d\xi} \left\{ N \frac{df}{d\xi} \right\} f_1 \, d\xi = - \int_0^1 N \frac{df}{d\xi} \cdot \frac{df_1}{d\xi} \cdot d\xi$$

$$\int_0^1 \frac{d^2 \phi}{d\xi^2} \cdot \phi_1 \, d\xi = - \int_0^1 \frac{d\phi}{d\xi} \cdot \frac{d\phi_1}{d\xi} \cdot d\xi$$

$$\int_0^1 \frac{d^4 \phi}{d\xi^4} \cdot \phi_1 \, d\xi = \int_0^1 \frac{d^2 \phi}{d\xi^2} \cdot \frac{d^2 \phi_1}{d\xi^2} \cdot d\xi .$$

The above identical relations can be verified by integrating by parts and making use of the boundary conditions. Similarly, when first and second equations of (6) are multiplied respectively by f and ϕ, and integrated from 0 to 1, one has

$$\left.\begin{aligned} &\beta^2 \int_0^1 \frac{d^2 f}{d\xi^2} \cdot \frac{d^2 f_1}{d\xi^2} \, d\xi + \Omega^2 \int_0^1 \bar{N}_1 \frac{df}{d\xi} \cdot \frac{df_1}{d\xi} \cdot d\xi \\ &-\gamma_1^2 \int_0^1 f f_1 \, d\xi - \omega_1^2 x_\theta \frac{B}{A} \int_0^1 \phi_1 f \, d\xi = 0 \\ &-\alpha^2 \int_0^1 \frac{d\phi}{d\xi} \cdot \frac{d\phi_1}{d\xi} \cdot d\xi - c_2 \int_0^1 \frac{d^2 \phi}{d\xi^2} \cdot \frac{d^2 \phi_1}{d\xi^2} \cdot d\xi \\ &+\omega_1^2 \left\{ (x_\theta^2 + I_\theta') \int_0^1 \phi \phi_1 \, d\xi + x \frac{A}{B} \int_0^1 f_1 \phi \, . \, d\xi \right\} = 0 \end{aligned}\right\} \quad \ldots \quad (8)$$

Subtracting eqn. (7) from the respective eqn. (8), we get

$$\left.\begin{aligned} &\Omega^2 \int_0^1 (\bar{N}_1 - \bar{N}) \frac{df}{d\xi} \frac{df_1}{d\xi} \, d\xi - (\gamma_1^2 - \gamma^2) \int_0^1 f f_1 \, d\xi \\ &-x_\theta \frac{B}{A} \left\{ \omega_1^2 \int_0^1 \phi_1 f \, d\xi - \omega^2 \int_0^1 \phi f_1 \, d\xi \right\} = 0 \\ &(\omega_1^2 - \omega^2)(x_\theta^2 + 1_\theta') \int_0^1 \phi \phi_1 \, d\xi + x_\theta \cdot \frac{A}{B} \left\{ \omega_1^2 \int_0^1 f_1 \phi \, d\xi - \omega^2 \int_0^1 f \phi_1 \, d\xi \right\} = 0 \end{aligned}\right\} . \quad (9)$$

Assuming that for small changes of hub radius, the change of modal shape is small and can be neglected. Thus, $f = f_1$ and $\phi = \phi_1$ which after substituting in eqn. (9) gives

$$\left.\begin{aligned} &\Omega^2 \delta \int_0^1 \left(\int_\xi^1 d\xi \right) \left(\frac{df}{d\xi} \right)^2 d\xi - (\gamma_1^2 - \gamma^2) \int_0^1 f^2 \, d\xi - (\omega_1^2 - \omega^2) \frac{B}{A} x_\theta \int_0^1 f \phi \, d\xi = 0 \\ &(\omega_1^2 - \omega^2) \left\{ (x_\theta^2 + I_\theta') \int_0^1 \phi^2 \, d\xi + x_\theta \cdot \frac{A}{B} \int_0^1 f \phi \, d\xi \right\} = 0 \end{aligned}\right\} \quad (10)$$

second of eqn. (10) gives

$$\frac{B}{A} = -\frac{x_\theta \int_0^1 f\phi \, d\xi}{(x_\theta^2 + I_\theta') \int_0^1 \phi^2 \, . \, d\xi}.$$

Substituting in the first of eqn. (10)

$$\Omega^2 \delta \int_0^1 \left(\int_\xi^1 d\xi \right) \left(\frac{df}{d\xi} \right)^2 d\xi - (\gamma_1^2 - \gamma^2) \int_0^1 f^2 \, d\xi$$

$$+ (\omega_1^2 - \omega^2) x_\theta^2 \frac{\left[\int_0^1 f\phi \, . \, d\xi \right]^2}{(x_\theta^2 + I_\theta') \int_0^1 \phi^2 \, d\xi} = 0$$

or

$$\Omega^2 \delta \int_0^1 \left(\int_\xi^1 d\xi \right) \left(\frac{df}{d\xi} \right)^2 d\xi - (\gamma_1^2 - \gamma^2) \left\{ \int_0^1 f^2 \, d\xi \right.$$

$$\left. - \frac{x_\theta^2}{(x_\theta^2 + I_\theta')} \frac{\left[\int_0^1 f\phi \, . \, d\xi \right]^2}{\int_0^1 \phi^2 \, d\xi} \right\} = 0$$

since $\gamma_1^2 - \gamma^2 = \omega_1^2 - \omega^2$

or

$$\Omega^2 \delta K_1 - (\gamma_1^2 - \gamma^2)(k_2 - b k_3) = 0 \quad \ldots \quad (11)$$

where

$$\left. \begin{aligned} k_1 &= \int_0^1 \left(\int_\xi^1 d\xi \right) \left(\frac{df}{d\xi} \right)^2 d\xi, \quad k_2 = \int_0^1 f^2 \, d\xi \\ k_3 &= \frac{\left[\int_0^1 f\phi \, . \, d\xi \right]^2}{\int_0^1 \phi^2 \, d\xi} \quad \text{and } b = \frac{x_\theta^2}{x_\theta^2 + I_\theta'} \end{aligned} \right\} . \quad \ldots \quad (12)$$

Eqn. (11) is the governing equation for the application of the 'perturbation method'.

PERTURBATION METHOD

The method of perturbation as used by Hsu Lo has been extended to this coupled problem. For a small change of hub radius, Δr, the $\delta \ll 1$, and the new frequency parameter γ_1 can be expressed in terms of a power series of δ. Thus

$$\gamma_1 = \gamma \sum_{n=0}^{\infty} a_n \delta^n \quad \ldots \quad (13)$$

where the coefficients a_n are to be determined by the method of perturbation. Substitution of eqn. (13) into eqn. (11) gives

$$\Omega^2\delta k_1-\gamma^2\left[\left(\sum_{n=0}^{\infty} a_n\delta^n\right)^2-1\right](k_2-bk_3)=0.$$

Expanding this equation into a series with ascending powers of δ, we get

$$\delta^0\{-\gamma^2(a_0^2-1)\}+\delta\{\Omega^2k_1-\gamma^2(2a_0a_1)(k_2-bk_3)\}$$
$$+\delta^2\{-\gamma^2(2a_0a_2+a_1^2)\}(k_2-bk_3)+\ldots=0. \quad \ldots \quad \ldots \quad (14)$$

Equating the coefficient of the δ^0- term equal to zero, we get

$$a_0=\pm 1. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

The positive sign should be used so that the condition that at $\delta=0$, $\gamma_1=\gamma$ is satisfied. Equating the coefficient of δ-term to zero, we get

$$a_1=\Omega^2k_1/2\gamma^2(k_2-bk_3). \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

Neglecting the second and higher order terms of δ, the following first order solution is obtained

$$\gamma_1=\gamma(1+a_1\delta). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

This gives a linear relation between the frequency parameter γ_1 and δ, with a_1 as the slope of this linear relation.

Substituting the expressions for γ_1 and γ in eqn. (17) and solving for ω_1, one obtains the relation between the new frequency ω_1 and the parameter δ as given by

$$\omega_1=[(\omega^2+\Omega^2\sin^2\psi)(1+a_1\delta)^2-\Omega^2\sin^2\psi]^{1/2}. \quad \ldots \quad \ldots \quad (18)$$

This relation can be put in a linear form by expanding into a series and discarding terms of higher powers of δ.

$$\omega_1=\omega\left[1+a_1\left(\frac{\omega^2+\Omega^2\sin^2\psi}{\omega^2}\right)\delta\right]. \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

For the particular case $\psi=0$, this reduces to

$$\omega_1=\omega(1+a_1\delta). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

The results (20) and (19) confirm the results previously obtained by Boyce (1956) and Hsu La (1960) if in addition $x_\theta=0$, i.e. for uncoupled case.

The convergence of the infinite series

$$\gamma_1=\gamma\sum_{n=0}^{\infty}a_n\delta^n$$

and the determination of its radius of convergence can be treated as follows:

Let

$$g(\delta)=\sum_{n=0}^{\infty}a_n\delta^n.$$

Eqn. (13) becomes

$$\gamma_1 = \gamma_g(\delta). \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

Substituting this relation into eqn. (11) gives

$$\Omega^2\delta k_1 - \gamma^2[g^2(\delta)-1](k_2 - bk_3) = 0$$

whereas

$$g(\delta) = \left(1+\frac{\Omega^2\delta k_1}{\gamma^2(k_2-bk_3)}\right)^{1/2}$$

when eqn. (16) is substituted it becomes

$$g(\delta) = (1+2a_1\delta)^{1/2}. \qquad \ldots \quad \ldots \quad \ldots \quad (22)$$

Thus the infinite series

$$\sum_{n=0}^{\infty} a_n\delta^n$$

converges to the sum given by eqn. (22). The radius of convergence is

$$|2a_1\delta < 1|$$

or

$$|\delta| < \frac{1}{2a_1}; \; a_1 > 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

Conclusion

It has been shown that for coupled transverse and torsional vibrations of a rotating slender beam, a small change in hub radius causes a corresponding change in the frequency parameter. The relation between them is approximately a linear one and the constant of proportionality can be determined from the known parameters. This conclusion is also valid for beams with variable cross-sections and with variable mass along the beam.

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