A NEW TREATMENT OF THE STEADY STATE WAVE PROBLEM—I

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A steady state investigation into the two-dimensional linearized problem of wave propagation in an infinitely deep fluid due to a harmonically oscillating pressure distribution concentrated in a single point of the free surface of the fluid is made by the generalized function treatment.

1. INTRODUCTION

Surface waves which are simple harmonic in time are usually generated and maintained by the application of pressure (which is also simple harmonic in time with a fixed frequency) over the vicinity of a single point or of certain finite region or over the whole region of the free surface of the fluid. This kind of problem of wave propagation is generally known as 'steady state (or stationary) problem' in which the wave motions are simple harmonic in time. This is one of the basic and interesting problem in the theory of surface wave phenomena. It is important to notice that the solution of the steady state problem, as is well known, is, in general, not unique on mathematical grounds. But in any physical situation only one solution is of interest.

Several methods have been employed in the treatment of this steady state problem, all of which are aimed at deriving a unique solution of physical interest by introducing some artificial device. A brief review stating important features of these methods is made below to understand them clearly and to indicate our motivation.

In order to derive a unique solution of physical interest, the problem is, in general, investigated by imposing a condition—usually known as Sommerfeld's radiation condition (Stoker 1957) which says that the waves behave like outgoing progressing waves at infinity. More precisely, it strictly excludes the possibility of any waves crossing the field from one side to the other or of simultaneously any incoming waves generated at infinity. The

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radiation condition is, indeed, on mathematical and physical reasons essential to achieve a unique solution of the 'stationary problem'. With the aid of complicated complex variable treatment, Stoker (1957, § 4.3, p. 58) examined the problem by imposing the radiation condition at infinity and obtained a unique solution of physical interest.

In this connection, Lighthill's (1960, 1964) method, an alternative way of applying the radiation condition, deserves mention. This method is also useful in deriving a unique solution of physical interest of the general wave problems. Lighthill suggested physical arguments to subtantiate the success of his method.

As an alternative to the radiation condition or to Lighthill's method, mention may be made of a fictitious damping force (Lamb 1905) which has been imposed to obtain a solution of the stationary wave problem appropriate to physical situation. All these methods described above lead, in general, to the same solution.

As an alternative to the above methods and, particularly, to Stoker's complicated complex variable treatment, we propose to use the generalized function treatment in deriving a unique solution of physical interest of the steady state problem by imposing the radiation condition at infinity. The main reason in favour of this new approach is that this is very simple, elegant and straightforward which produces the desired solution of physical interest.

2. Mathematical Formulation of the Problem

We consider two-dimensional linearized problem of wave propagation in inviscid, incompressible and homogeneous fluid with a free surface due to the application of a harmonically oscillating pressure distribution

$$P(X, T) = P\,\delta(X)\,e^{i\omega T} \qquad (2.1)$$

acting on the free surface of the fluid (initially at rest) for all times $T > 0$, where $\delta(X)$ is the Dirac function of distribution and ω is the fixed frequency.

Let us assume that X-Z plane be the undisturbed horizontal free surface and Y-axis be vertical positive upwards. As the motion is irrotational, there exists a wave potential $\Phi(X, Y; T)$ which is governed by the Laplace equation (Lamb 1932)

$$\frac{\partial^2\Phi}{\partial X^2}+\frac{\partial^2\Phi}{\partial Y^2} = 0 \qquad (2.2)$$

$$-\infty < Y \leqslant 0, \quad -\infty < X < \infty.$$

The dynamic condition at the free surface is given by

$$gE+\frac{\partial\Phi}{\partial T} = -\frac{1}{\rho}P(X, T), \quad Y = 0,\ T > 0 \qquad (2.3)$$

where $E(X, T)$ represents the vertical surface elevation, g the acceleration due to gravity and ρ the density of the fluid.

The kinematical surface condition is given by

$$E_T = \Phi_Y, \quad Y = 0, \; T > 0. \qquad \ldots \quad \ldots \quad \ldots \quad (2.4)$$

The condition at the bottom surface is given by

$$\Phi_Y \to 0 \quad \text{as } Y \to -\infty. \qquad \ldots \quad \ldots \quad \ldots \quad (2.5)$$

In addition to eqns. (2.1)–(2.5), it would be necessary to assume the Sommerfeld radiation condition at infinity. This would be mentioned later at the appropriate place.

3. Solution of the Problem

As a trial, let us assume that the functions Φ and E possess the Fourier transform with respect to X defined by integrals like

$$\bar{\Phi}(k, Y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikX}\, \Phi(X, Y)\, dX.$$

The Fourier transform enables us to rewrite eqns. (2.1)–(2.5) into the following form

$$\bar{\Phi}_{YY} = k^2 \bar{\Phi} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

$$\bar{\Phi}_T + g\bar{E} = -\frac{P\, e^{i\omega T}}{\rho\sqrt{2\pi}} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

$$\bar{\Phi}_Y = \bar{E}_T \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

$$\bar{\Phi}_Y \to 0 \quad \text{as } Y \to -\infty. \qquad \ldots \quad \ldots \quad (3.4)$$

The bounded solution of eqn. (3.1) with the transformed bottom condition (3.4) is obtained in the form

$$\bar{\Phi}(k, Y; T) = \bar{A}(k)\, e^{(i\omega T + |k| Y)} \qquad \ldots \quad \ldots \quad \ldots \quad (3.5)$$

where $\bar{A}(k)$ is an arbitrary function of k to be determined by eqns. (3.2) and (3.3).

Having determined the value of $\bar{A}(k)$ by eqns. (3.2) and (3.3), the solution (3.5) can be put into the form

$$\bar{\Phi}(k, Y; T) = \frac{P\, i\omega\, e^{(i\omega T + |k| Y)}}{\rho\sqrt{2\pi}\,(\omega^2 - g|k|)}. \qquad \ldots \quad \ldots \quad (3.6)$$

The expression for the surface elevation $\bar{E}(k, T)$ has, therefore, the form

$$\bar{E}(k, T) = \frac{P\, e^{i\omega T}\, |k|}{\rho\sqrt{2\pi}\,(\omega^2 - g|k|)}. \qquad \ldots \quad \ldots \quad \ldots \quad (3.7)$$

Making reference to the inversion theorem for the Fourier transform, we obtain the expression for the wave potential $\Phi(X, Y; T)$ and the surface elevation $E(X, T)$ in the form

$$\Phi(X, Y; T) = \frac{Pi\omega\, e^{i\omega T}}{2\pi\rho}\int_{-\infty}^{+\infty}\frac{e^{(|k|Y+ikX)}}{(\omega^2-g|k|)}dk$$

$$= \frac{Pi\omega\, e^{i\omega T}}{\pi\rho}\int_0^{\infty}\frac{e^{kY}\cos kX\, dk}{(\omega^2-gk)} \quad \ldots \quad (3.8)$$

$$E(X, T) = \frac{P\, e^{i\omega T}}{2\pi\rho}\int_{-\infty}^{\infty}\frac{e^{ikX}|k|\,dk}{(\omega^2-g|k|)}$$

$$= \frac{P\, e^{i\omega T}}{\pi\rho}\int_0^{\infty}\frac{k\cos kX\, dk}{(\omega^2-gk)}. \quad \ldots \quad (3.9)$$

It may be noticed that the inversion integrals (3.8) and (3.9) are divergent because of the fact that the integrands have a singularity on the axis of integration. Consequently, the original assumption that the functions $\Phi(X, Y; T)$ and $E(X, T)$ possess the Fourier transform in the ordinary sense was invalid. This is logical because a function which has a Fourier transform must tend to zero at infinity, but we expect the potential Φ to have wave-like behaviour, i.e. bounded at infinity but not tending to zero at infinity.

To overcome this difficulty, it is possible to treat the functions Φ and E as generalized functions (Lighthill 1958) in which case they do have the Fourier transform. And then (3.8) and (3.9) are valid inversion integrals in the generalized sense. So, one can perform evaluation of integrals (3.8) and (3.9) by the use of the generalized Fourier analysis.

With the aid of the generalized Fourier analysis (Lavoine 1959, 1963), we readily obtain the solution for the surface elevation $E(X, T)$ in the form

$$E(X, T) = \frac{iP\omega^2}{\rho g^2}\left[e^{i\omega\left(T-\frac{\omega}{g}X\right)} - e^{i\omega\left(T+\frac{\omega}{g}X\right)}\right] - \frac{P\omega^2}{\rho g^2}\delta(X)\, e^{i\omega T}. \quad \ldots \quad (3.10)$$

This expression (3.10) for the surface elevation $E(X, T)$ contains two exponential terms—one corresponds to outgoing progressive waves from the source of disturbance and the other to incoming waves from infinity.

Now, we introduce the Sommerfeld radiation condition in order to eliminate the incoming wave term present in the above solution (3.10). Having done this, we readily obtain the unique solution of physical interest as

$$E(X, T) = \frac{iP\omega^2}{\rho g^2}e^{i\omega\left(T-\frac{\omega}{g}X\right)}, \quad X \neq 0 \quad \ldots \quad (3.11)$$

which is Lamb's (1905) steady state solution obtained independently by the various methods stated above. Stoker (1957) obtained the same solution (3.11) by using a very complicated complex variable treatment.

This solution corresponds to progressive surface waves advancing with the phase velocity g/ω and the group velocity $g/2\omega$.

4. Concluding Remarks

Debnath (1967) has also obtained the solution (3.11) as a limiting case from the most physically realistic and mathematically rigorous initial value investigation by asymptotic methods.

Debnath (1967) has verified the solution (3.11) by Lighthill's method. The success of the Lighthill method lies in the fact that the substitution of $\omega - i\epsilon$ ($\epsilon > 0$ and small) for ω displaces one of the poles of (3.9) into the lower half plane and the other into the upper half plane. Consequently, the contour evaluation of the integral (3.9) by the residue theory combined with the limit operation $\epsilon \to 0$ leads to the same result (3.11). More precisely, the semi-circular contour in the upper half plane encloses only one of these poles of (3.9). The contribution from this pole gives the outgoing progressive waves, while the other pole, which contributes nothing, represents the incoming wave.

Finally, we remark that the use of the generalized function treatment enables us to achieve the desired solution without any need for essentially complex variable treatment suggested by Lighthill and Stoker.

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