LAMINAR FLOW THROUGH A UNIFORM CIRCULAR PIPE WITH SMALL OUTWARD NORMAL SUCTION

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In the present paper an attempt has been made to find the solution of the Navier-Stokes equations for the steady flow of a viscous incompressible fluid through a porous pipe of uniform circular cross-section with small outward normal suction at the porous wall. An exact solution of the equations, reduced to second and third order non-linear differential equations with appropriate boundary conditions, is obtained. It is found that the maximum velocity of the flow exists on the axis of the pipe and that at the centre of the mouth it is greater than the maximum velocity of the Poiseuille flow. The axial pressure gradient and the axial velocity decrease along the length of the pipe and vanish simultaneously giving rise to adverse pressure gradient which causes a back flow. The magnitude of the shearing stress at the wall increases with the increase of the suction parameter defined with reference to suction velocity and pipe radius. For $\sigma = 0$ and $v = 0$ when $y = 0$, the results transform to the known results of Hagen-Poiseuille flow.

§ 1. INTRODUCTION

An analysis has been presented for the flow of a viscous liquid through a porous pipe of circular section with small outward normal suction at the porous wall. An exact solution of the Navier-Stokes equations, reduced to second and third order non-linear differential equations with appropriate boundary conditions, is obtained. A perturbation method is used to solve the dynamical equations for small flow through the porous wall.

The exact solution of the Navier-Stokes equations for the flow of a viscous liquid between two parallel plates, one in uniform motion and the other at rest with uniform suction at the stationary plate, has been recently discussed by Verma and Bansal (1966). The Hagen-Poiseuille flow in a circular pipe is well known. Yuan and Finkelstein (1956) have discussed the laminar flow through a pipe of porous wall with injection and suction under the assumption that the maximum velocity of the Hagen-Poiseuille flow exists at the centre of the mouth of the pipe. But it is found that the pressure gradient along the axis, at the mouth of the pipe, is not the same as the pressure gradient of the Poiseuille flow. Recently, Choudhary and Sinha (1964) have discussed the steady flow of a viscous incompressible fluid through a uniform circular pipe with small outward normal suction under the assumptions that the pressure is uniform over a cross-section and that the axial pressure gradient is uniform throughout the pipe. This assumption on pressure seems to be very restricted. The uniform suction at the boundary may be very small; even then the change in the pressure gradient along the axis of the pipe cannot be neglected. Moreover, their final expressions for the axial velocity and radial velocity distributions do not satisfy the continuity equation. Bansal (1966) has reconsidered the same problem under the assumptions that the flow is due to the pressure gradient of the Hagen-Poiseuille flow at the mouth and that the suction at the wall is uniform and small. Moreover, he has used a perturbation method to determine the motion instead of assuming the form of stream function as done by Yuan and Finkelstein. He has also assumed that the radial velocity, which vanishes in the Poiseuille flow, has a finite magnitude except at the axis of the pipe where it vanishes. In the present paper, another attempt has been made to study the same problem with the same method, but no restriction is imposed on the radial velocity as done by Bansal.

It has been found that the maximum velocity of the flow exists on the axis of the pipe. This is greater than the maximum velocity of the Poiseuille flow at the centre of the mouth of the pipe. The velocity distribution is parabolic. It is found that due to suction an adverse pressure gradient is developed which causes a back flow. In the present case the radial velocity, which vanishes in Poiseuille case, has a finite magnitude. The pressure drop along the axis is parabolic. For $\sigma = 0$ and $v = 0$ when $y = 0$, the results transform to the known results of Hagen-Poiseuille flow.

§ 2. Formulation of the Problem

The Navier-Stokes equations of the motion in cylindrical polar co-ordinates (y, ϕ, x) for a viscous liquid steady flow are (Schlichting 1960)

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{w}{y}\frac{\partial u}{\partial \phi} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u \qquad \text{(2.1)}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{w}{y}\frac{\partial v}{\partial \phi}-\frac{w^2}{y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left[\nabla^2 v-\frac{v}{y^2}-\frac{2}{y^2}\frac{\partial w}{\partial \phi}\right] \qquad \text{(2.2)}$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\frac{w}{y}\frac{\partial w}{\partial \phi}+\frac{vw}{y} = -\frac{1}{\rho}\frac{1}{y}\frac{\partial p}{\partial \phi}+\nu\left[\nabla^2 w+\frac{2}{y^2}\frac{\partial v}{\partial \phi}-\frac{w}{y^2}\right] \qquad \text{(2.3)}$$

and the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{v}{y}+\frac{1}{y}\frac{\partial w}{\partial \phi} = 0, \qquad \text{(2.4)}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial y^2}+\frac{1}{y}\frac{\partial}{\partial y}+\frac{1}{y^2}\frac{\partial^2}{\partial \phi^2}+\frac{\partial^2}{\partial x^2}$$

and x is taken in the direction of the flow, y is the radial direction, ϕ the azimuthal angle and u, v, w the velocity components along x, y, ϕ increasing. For laminar flow through a circular pipe

$$\left.\begin{aligned} & w = 0, \\ & \frac{\partial}{\partial \phi} = 0 \quad \text{for rational symmetry} \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (2.5)$$

Hence eqn. (2.3) identically vanishes and the remaining eqns. (2.1), (2.2) and (2.4) become

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left[\frac{\partial^2 u}{\partial y^2}+\frac{1}{y}\frac{\partial u}{\partial y}+\frac{\partial^2 u}{\partial x^2}\right] \qquad \ldots \quad \ldots \quad (2.6)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{1}{y}\frac{\partial v}{\partial y}-\frac{v}{y^2}\right] \qquad \ldots \quad \ldots \quad (2.7)$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{v}{y} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.8)$$

The boundary conditions are

$$\left.\begin{aligned} & y = y_0\,; \;\; v = v_0,\;\; u = 0 \\ & y = y_0\,; \;\; \frac{\partial v}{\partial y} = 0 \\ & y = 0\,; \;\; v = \text{finite},\;\; u = \text{finite} \end{aligned}\right\}. \qquad \ldots \quad \ldots \quad (2.9)$$

Since there is uniform suction, $\dfrac{\partial v}{\partial x} = 0$, therefore v is a function of y only. Then it is evident from eqn. (2.8) that $\dfrac{\partial^2 u}{\partial x^2} = 0$. Thus eqns. (2.6), (2.7) and (2.8) reduce to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left[\frac{\partial^2 u}{\partial y^2}+\frac{1}{y}\frac{\partial u}{\partial y}\right] \qquad \ldots \quad \ldots \quad (2.10)$$

$$v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left[\frac{\partial^2 v}{\partial y^2}+\frac{1}{y}\frac{\partial v}{\partial y}-\frac{v}{y^2}\right] \qquad \ldots \quad \ldots \quad \ldots \quad (2.11)$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{v}{y} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.12)$$

Let us introduce non-dimensional quantities as follows:

$$\bar{u} = \frac{u}{U},\;\; \bar{v} = \frac{v}{v_0},\;\; \bar{x} = \frac{x}{y_0},\;\; \eta = \frac{y}{y_0},\;\; \bar{p} = \frac{p}{\rho U^2},$$

$$\sigma = \frac{v_0 y_0}{\nu},\;\; R = \frac{U y_0}{\nu}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.13)$$

where y_0 is the radius of the cylinder, v_0 the suction velocity, U the maximum velocity of the Poiseuille flow when there is no suction, and σ and R the suction parameter and Reynolds number for the axial flow respectively.

Eqns. (2.10), (2.11) and (2.12) in non-dimensional form are

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\sigma}{R}\bar{v}\frac{\partial \bar{u}}{\partial \eta} = -\frac{\partial \bar{p}}{\partial x}+\frac{1}{R}\left(\frac{\partial^2 \bar{u}}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial \bar{u}}{\partial \eta}\right) \quad \ldots \quad (2.14)$$

$$\bar{v}\frac{\partial \bar{v}}{\partial \eta} = -\frac{R^2}{\sigma^2}\frac{\partial \bar{p}}{\partial \eta}+\frac{1}{\sigma}\left(\frac{\partial^2 \bar{v}}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial \bar{v}}{\partial \eta}-\frac{\bar{v}}{\eta^2}\right) \quad \ldots \quad (2.15)$$

and

$$\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\sigma}{R}\left(\frac{\partial \bar{v}}{\partial \eta}+\frac{\bar{v}}{\eta}\right) = 0. \quad \ldots \quad (2.16)$$

The boundary conditions (2.9) with the aid of (2.13) are

$$\left.\begin{aligned} &\eta = 1; \;\; \bar{v} = 1,\; \bar{u} = 0 \\ &\eta = 1; \;\; \frac{\partial \bar{v}}{\partial \eta} = 0 \\ &\text{and } \eta = 0; \;\; \bar{v} = \text{finite},\; \bar{u} = \text{finite} \end{aligned}\right\}. \quad \ldots \quad (2.17)$$

Let

$$\left.\begin{aligned} \bar{p}(\bar{x}, \eta) &= p_0+p'(\bar{x}, \eta) \\ \bar{u}(\bar{x}, \eta) &= u_0+u'(\bar{x}, \eta) \\ \bar{v} &= v'(\eta) \end{aligned}\right\}, \quad \ldots \quad (2.18)$$

where the primed quantities are the perturbations caused by the suction and p_0 and u_0 are the known quantities for flow when there is no suction [i.e. the Hagen-Poiseuille flow in a circular pipe (Schlichting 1960)], satisfying the equations

$$\left.\begin{aligned} &\frac{\partial p_0}{\partial \eta} = 0, \quad \frac{\partial u_0}{\partial \bar{x}} = 0 \\ &\frac{\partial p_0}{\partial \bar{x}} = \text{constant } (a_0), \;\; u_0 = A(\eta^2-1) \end{aligned}\right\} \quad \ldots \quad (2.19)$$

where

$$A = \frac{R}{4}\frac{\partial p_0}{\partial \bar{x}}. \quad \ldots \quad (2.20)$$

Substituting (2.18) in (2.14), (2.15) and (2.16), we get

$$(u_0+u')\frac{\partial u'}{\partial \bar{x}}+\frac{\sigma}{R}v'\left[\frac{\partial u_0}{\partial \eta}+\frac{\partial u'}{\partial \eta}\right] = -\frac{\partial p_0}{\partial \bar{x}}-\frac{\partial p'}{\partial \bar{x}}+\frac{1}{R}\left(\frac{\partial^2 u_0}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial u_0}{\partial \eta} +\frac{\partial^2 u'}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial u'}{\partial \eta}\right) \quad \ldots \quad (2.21)$$

$$v'\frac{\partial v'}{\partial \eta} = -\frac{R^2}{\sigma^2}\frac{\partial p'}{\partial \eta}+\frac{1}{\sigma}\left[\frac{\partial^2 v'}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial v'}{\partial \eta}-\frac{v'}{\eta^2}\right] \quad \ldots \quad (2.22)$$

and

$$\frac{\partial u'}{\partial \bar{x}}+\frac{\sigma}{R}\left(\frac{\partial v'}{\partial \eta}+\frac{v'}{\eta}\right) = 0. \quad \ldots \quad (2.23)$$

The boundary conditions (2.17) with the aid of (2.18) are

$$\left.\begin{array}{l} \eta=1\,;\;\; v'=1,\;u'=0 \\ \eta=1\,;\;\; \dfrac{\partial v'}{\partial\eta}=0 \\ \eta=0\,;\;\; v'=\text{finite},\;u'=\text{finite} \end{array}\right\}. \qquad \ldots \quad \ldots \quad (2.24)$$

§ 3. Method of Solution

Let

$$v'=f(\eta) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

therefore, from eqn. (2.23), we have

$$u'=-\frac{\sigma}{R}\bar{x}\left[f'+\frac{1}{\eta}f\right]+F(\eta), \qquad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

where $f(\eta)$ and $F(\eta)$ are the known functions to be determined.

Substituting eqns. (3.1) and (3.2) in (2.21) and (2.22), we get

$$\frac{\partial p'}{\partial\bar{x}}=-a_0+\frac{1}{R}\left[4A+F''+\frac{1}{\eta}F'+\sigma A(\eta^2-1)\left(f'+\frac{1}{\eta}f\right)+\sigma\left(Ff'+\frac{1}{\eta}Ff\right)\right.$$
$$\left.-2\sigma A\eta f-\sigma fF'\right]-\frac{\sigma}{R^2}\bar{x}\left[f'''+\frac{2}{\eta}f''-\frac{1}{\eta^2}f'+\frac{1}{\eta^3}f\right.$$
$$\left.+\sigma\left(f'^2+\frac{1}{\eta}ff'-ff''+\frac{2}{\eta^2}f^2\right)\right] \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

and

$$\frac{\partial p'}{\partial\eta}=-\frac{\sigma^2}{R^2}\left[ff'-\frac{1}{\sigma}\left(f''+\frac{1}{\eta}f'-\frac{1}{\eta^2}f\right)\right]. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.4)$$

At the mouth of the pipe the pressure gradient along the axis of the pipe is the same as the pressure gradient of the Poiseuille flow.

Hence, at $x=0$, $\dfrac{\partial p'}{\partial\bar{x}}=0$.

Therefore, from (3.3), we have

$$F''+\frac{1}{\eta}F'+\sigma\left[A\eta^2f'-A\eta f-Af'-\frac{A}{\eta}f+\frac{Ff}{\eta}+Ff'-F'f\right]=0. \qquad \ldots \quad (3.5)$$

Differentiating (3.4) w.r.t. $\bar{x}$, we have

$$\frac{\partial^2 p'}{\partial\bar{x}\,\partial\eta}=0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.6)$$

Now differentiating (3.3) w.r.t. η and using (3.5) and (3.6), we get

$$\frac{d}{d\eta}\left[f'''+\frac{2}{\eta}f''-\frac{1}{\eta^2}f'+\frac{1}{\eta^3}f-\sigma\left(ff''-\frac{1}{\eta}ff'-f'^2-\frac{2}{\eta^2}f^2\right)\right]=0 \qquad \ldots \quad (3.7)$$

which is to be satisfied for all $\bar{x}$.

Integrating eqn. (3.7), we have

$$f'''+\frac{2}{\eta}f''-\frac{1}{\eta^2}f'+\frac{1}{\eta^3}f-\sigma\left(ff''-\frac{1}{\eta}ff'-f'^2-\frac{2}{\eta^2}f^2\right) = C, \quad .. \quad (3.8)$$

where C is the constant of integration to be determined.

Now we have to obtain the solution of (3.5) and (3.8) with the help of the following boundary conditions:

$$\left.\begin{aligned} &\eta = 1\,;\; f(1) = 1,\; f'(1) = 0,\; F(1) = \frac{\sigma}{R}\bar{x} \\ &\eta = 0\,;\; f(0) = F(0) = \text{finite} \end{aligned}\right\}. \quad .. \quad .. \quad (3.9)$$

§ 4. Solution for Small Suction Parameter

The solution of eqn. (3.8) can be expressed for small values of σ by a power series developed near $\sigma = 0$ as follows:

$$f = f_0+\sigma f_1+\sigma^2 f_2+\ldots+\sigma^n f_n \quad .. \quad .. \quad .. \quad (4.1)$$

and

$$C = C_0+\sigma C_1+\sigma^2 C_2+\ldots+\sigma^n C_n \quad .. \quad .. \quad .. \quad (4.2)$$

where the f_n's and C_n's are taken to be independent of σ.

By substituting eqns. (4.1) and (4.2) in (3.8) and equating coefficients of like powers of σ, we get the following set of equations:

$$f_0'''+\frac{2}{\eta}f_0''-\frac{1}{\eta^2}f_0'+\frac{1}{\eta^3}f_0 = C_0 \quad .. \quad .. \quad .. \quad .. \quad (4.3)$$

$$f_1'''+\frac{2}{\eta}f_1''-\frac{1}{\eta^2}f_1'+\frac{1}{\eta^3}f_1-f_0f_0''+\frac{1}{\eta}f_0f_0'+f_0'^2+\frac{2}{\eta^2}f_0^2 = C_1 \quad .. \quad (4.4)$$

and

$$f_2'''+\frac{2}{\eta}f_2''-\frac{1}{\eta^2}f_2'+\frac{1}{\eta^3}f_2-f_0f_1''-f_1f_0''+\frac{1}{\eta}f_0f_1'$$

$$+\frac{1}{\eta}f_1f_0'+2f_0'f_1'+\frac{4}{\eta^2}f_0f_1 = C_2. \quad .. \quad .. \quad .. \quad .. \quad (4.5)$$

The boundary conditions to be satisfied by f_n's are

$$\left.\begin{aligned} &\eta = 1\,;\; f_n'(1) = 0 && \text{for } n \geqslant 0 \\ &\eta = 1\,;\; f_0(1) = 1,\; f_n(1) = 0 && \text{for } n \geqslant 1 \\ \text{and } &\eta = 0\,;\; f_n(0) = \text{finite} && \text{for } n \geqslant 0 \end{aligned}\right\}. \quad .. \quad .. \quad (4.6)$$

From eqns. (4.3) to (4.5) the second order perturbation solution of eqn. (3.8) is

$$f(\eta) = \left(\tfrac{3}{2}\eta-\tfrac{1}{2}\eta^3\right)+\sigma\left(\frac{7}{144}\eta-\frac{5}{48}\eta^3+\frac{1}{16}\eta^5-\frac{1}{144}\eta^7\right)$$

$$+\sigma^2\left(\frac{103}{9600}\eta-\frac{101}{4320}\eta^3+\frac{13}{864}\eta^5-\frac{5}{1728}\eta^7+\frac{1}{1920}\eta^9-\frac{1}{43200}\eta^{11}\right) \quad (4.7)$$

and

$$C = -8+\frac{22}{3}\sigma+\frac{113}{540}\sigma^2. \qquad (4.8)$$

It is seen from the foregoing equations that the second order perturbation solution is sufficiently accurate even for $\sigma = 1$.

Now, the solution of eqn. (3.5) can be expressed for small values of σ by a power series developed near $\sigma = 0$ as follows:

$$F = F_0+\sigma F_1+\sigma^2 F_2+ \ldots +\sigma^n F_n \qquad (4.9)$$

where the F_n's are taken to be independent of σ. By substituting eqns. (4.1) and (4.9) into eqn. (3.5) and equating to zero the different powers of σ, one obtains the following set of equations:

$$F_0''+\frac{1}{\eta}F_0' = 0 \qquad (4.10)$$

$$F_1''+\frac{1}{\eta}F_1'+A\eta^2 f_0'-A\eta f_0-Af_0'-\frac{A}{\eta}f_0+\frac{1}{\eta}F_0f_0+F_0f_0'-F_0'f_0 = 0 \qquad (4.11)$$

$$F_2''+\frac{1}{\eta}F_2'+A\eta^2 f_1'-A\eta f_1-Af_1'-\frac{A}{\eta}f_1+\frac{1}{\eta}F_0f_1+\frac{1}{\eta}F_1f_0$$
$$+F_0f_1'+F_1f_0'-F_0'f_1-F_1'f_0 = 0. \qquad (4.12)$$

The boundary conditions satisfied by the F_n's are

$$\left.\begin{aligned}&\eta = 1;\ F_0(1) = 0,\ F_1(1) = \frac{\bar{x}}{R},\ F_n(1) = 0 \quad \text{for } n \geqslant 2\\ &\eta = 0;\ F_n(0) = \text{finite} \quad \text{for } n \geqslant 0.\end{aligned}\right\} \qquad (4.13)$$

From eqns. (4.10) to (4.12) the second order perturbation solution of eqn. (3.5) is

$$F(\eta) = \frac{\sigma\bar{x}}{R}+\sigma A\left(\frac{1}{36}\eta^6-\frac{1}{8}\eta^4+\frac{3}{4}\eta^2-\frac{47}{72}\right)$$
$$+\frac{\sigma^2\bar{x}}{R}\left(\frac{5}{8}-\frac{3}{4}\eta^2+\frac{1}{8}\eta^4\right)+\sigma^2\left(\frac{1}{256}\eta^8-\frac{1}{24}\eta^6+\frac{9}{64}\eta^4-\frac{79}{768}\right)$$
$$+\sigma^2 A\left(-\frac{28337}{86400}+\frac{37}{72}\eta^2-\frac{143}{576}\eta^4+\frac{59}{864}\eta^6-\frac{7}{1152}\eta^8+\frac{1}{7200}\eta^{10}\right). \qquad (4.14)$$

The velocity components in the x- and y-directions obtained by the help of (2.18), (2.19), (3.1), (3.2), (4.7) and (4.14) are

$$\bar{u} = -A(1-\eta^2)-\frac{\sigma}{R}\bar{x}\left[3-2\eta^2+\sigma\left(\frac{7}{72}-\frac{5}{12}\eta^2+\frac{3}{8}\eta^4-\frac{1}{18}\eta^6\right)\right.$$
$$\left.+\sigma^2\left(\frac{103}{4800}-\frac{101}{1080}\eta^2+\frac{13}{144}\eta^4-\frac{5}{216}\eta^6+\frac{1}{192}\eta^8-\frac{1}{3600}\eta^{10}\right)\right]+F(\eta), \qquad (4.15)$$

where $F(\eta)$ is given by (4.14), and

$$\bar{v} = \left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right) + \sigma\left(\frac{7}{144}\eta - \frac{5}{48}\eta^3 + \frac{1}{16}\eta^5 - \frac{1}{144}\eta^7\right)$$
$$+\sigma^2\left(\frac{103}{9600}\eta - \frac{101}{4320}\eta^3 + \frac{13}{864}\eta^5 - \frac{5}{1728}\eta^7 + \frac{1}{1920}\eta^9 - \frac{1}{43200}\eta^{11}\right). \quad (4.16)$$

§ 5. The flux across a section at $\bar{x}$ is

$$Q = 2\pi U y_0^2 \int_0^1 \eta\, \bar{u}\, d\eta$$
$$= -\frac{\pi U y_0^2}{2} A\left[\left(1 + \frac{5}{8}\sigma + \frac{2383}{8640}\sigma^2\right) + \frac{2\sigma\bar{x}}{AR}\left(1 - \frac{7\sigma}{24}\right) + \frac{21}{160A}\sigma^2\right]. \quad (5.1)$$

The discharge for solid wall ($\sigma = 0$) is

$$Q_0 = -\frac{\pi U y_0^2}{2} A.$$

The discharge coefficient is

$$C_Q = \frac{Q}{Q_0} = 1 + \frac{5}{8}\sigma + \frac{2383}{8640}\sigma^2 + \frac{2\sigma\bar{x}}{AR}\left(1 - \frac{7\sigma}{24}\right) + \frac{21}{160A}\sigma^2. \quad (5.2)$$

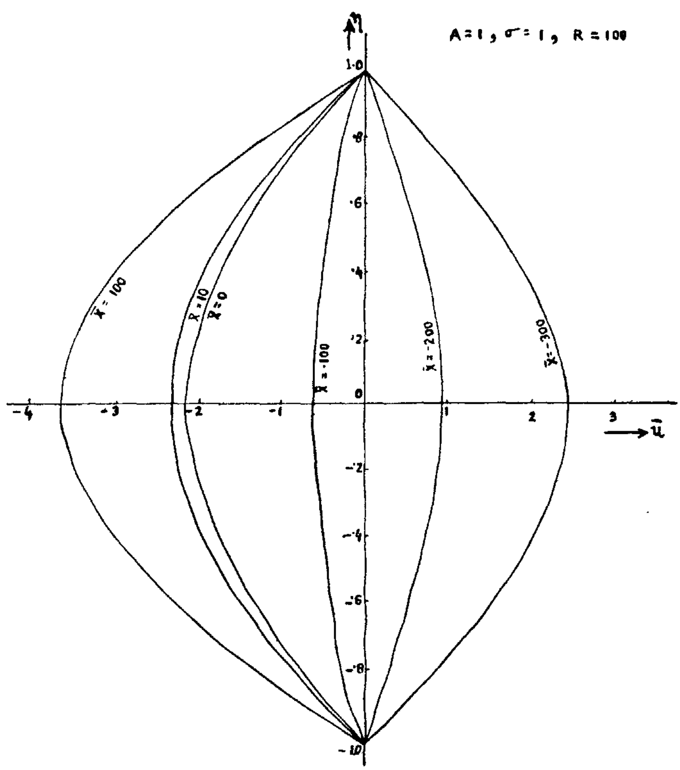

FIG. 1. Longitudinal velocity profiles v. η.

Proceeding to the limit as $\sigma = 0$, we have $C_Q = 1$, as in the case of the Hagen-Poiseuille flow.

The pressure distributions in the axial and radial directions can be obtained from eqns. (3.3) and (3.4) respectively by substituting the values of f and F and their derivatives from eqns. (4.7) and (4.14). Thus, on integration, we have

$$\bar{p}(0,\,0)-\bar{p}(\bar{x},\,\eta) = \frac{2\sigma}{R^2}\left(\frac{3}{2}+\frac{7\sigma}{144}+\frac{103}{9600}\sigma^2\right)+\frac{\sigma^2}{R^2}\left[\frac{f^2}{2}-\frac{1}{\sigma}\left(f'+\frac{1}{\eta}f\right)\right]+\frac{\sigma\bar{x}^2 C}{2R^2}. \quad (5.3)$$

The pressure drop in the flow direction is

$$\bar{p}(0,\,\eta)-\bar{p}(\bar{x},\,\eta) = -\frac{\sigma\bar{x}^2}{R^2}\left(4-\frac{11}{3}\sigma-\frac{113}{1080}\sigma^2\right). \quad (5.4)$$

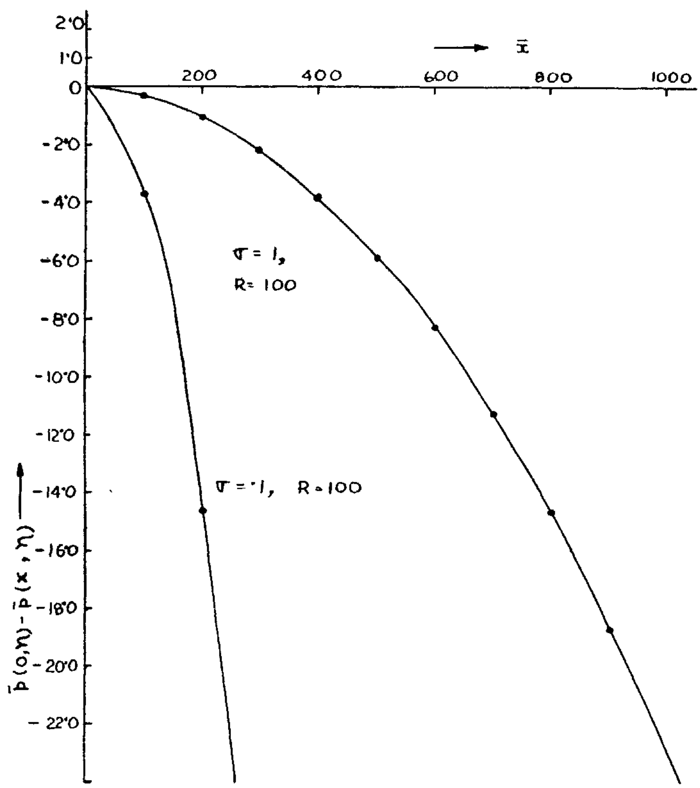

FIG. 2. Axial pressure drop *v.* length in flow direction.

The shearing stress at the wall is

$$\tau_0 = -\frac{\mu U}{R}\left(\frac{\partial \bar{u}}{\partial \eta}\right)_{\eta=1}$$

i.e.

$$\frac{\tau_0 R}{\mu U} = -2A+\sigma\left(\frac{7A}{6}-\frac{4\bar{x}}{R}\right)+\sigma^2\left(\frac{11}{32}+\frac{143}{360}A-\frac{2\bar{x}}{3R}\right)+\sigma^3\frac{97}{540}\frac{\bar{x}}{R}. \quad (5.5)$$

The axial velocity profile for various values of $\bar{x}$ has been plotted in Fig. 1.

This shows that the maximum velocity of the flow exists on the axis of the pipe. This is greater than the maximum velocity of the Poiseuille flow at the centre of the mouth of the pipe. The velocity distribution is parabolic.

$$\bar{u} = -\frac{R}{4}\frac{\partial p_0}{\partial \bar{x}}(1-\eta^2)$$

when $\sigma = 0$, which is equivalent to the maximum velocity on the axis of the Hagen-Poiseuille flow.

From Fig. 1, it is clear that due to suction an adverse pressure gradient is developed which causes a back flow. In the present case, the radial velocity, which vanishes in Poiseuille case, has a finite magnitude, i.e.

$$\bar{v} = (\tfrac{3}{2}\eta - \tfrac{1}{2}\eta^3) \quad \text{when } \sigma = 0.$$

This is maximum when $\eta = 1$.

The pressure drop along the axis is parabolic as shown in Fig. 2.

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