FLOW IN A ROTATING STRAIGHT ANNULAR PIPE

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A straight annular pipe of circular cross-section through which liquid is flowing under a pressure gradient is rotated about an axis perpendicular to its own. The flow pattern has been studied in detail, assuming narrow gap approximation for small values of angular velocity.

1. Introduction

Barua (1954) has studied the secondary flow of a viscous fluid passing through a straight pipe which is rotating about an axis perpendicular to its own. We consider in this paper the stationary circulatory flow of an incompressible viscous fluid in an annular pipe (with radius r_1 and r_2, $r_2 > r_1$) which rotates with an angular velocity Ω about an axis perpendicular to its own. For simplicity, we restrict to the case of narrow gap solution for which the width gap $\epsilon = r_2 - r_1$ is a finite quantity, but negligible in comparison with r_1. Substituting in eqns. (11) and (12) of Barua,

$$w = \frac{\nu}{\epsilon}\omega, \quad f = \nu\phi, \quad r = r_1 + \epsilon\zeta$$

we have, on neglecting terms of order ϵ/r_1 and replacing ∇_1^2 by $\partial^2/\partial\zeta^2$ (Chandrasekhar 1961),

$$-R\cos\theta\frac{\partial\omega}{\partial\zeta} = \frac{\partial^4\phi}{\partial\zeta^4} \quad \ldots \quad (1)$$

$$R\cos\theta\frac{\partial\phi}{\partial\zeta} = -c + \frac{\partial^2\omega}{\partial\zeta^2} \quad \ldots \quad (2)$$

where

$$R = \frac{2\Omega\epsilon^2}{\nu}, \quad c = c'\frac{\epsilon^3}{\nu^2}. \quad \ldots \quad (3)$$

We seek the solutions of the foregoing equations subject to the conditions for no slip at both the edges of the pipe, i.e.

$$\frac{\partial\phi}{\partial\zeta} = 0, \quad \frac{\partial\phi}{\partial\theta} = 0, \quad \omega = 0 \text{ at } \zeta = 0, 1 \quad \ldots \quad (4)$$

2. Solution of the Problem

In the absence of rotation, the problem reduces to the flow in the annulus under a pressure gradient. It is well known that the velocity distribution is the same as that of the Poiseuille flow between parallel planes (Chandrasekhar 1961) and the solution of (1) and (2) can be satisfied by

$$\phi = 0, \quad \omega = \tfrac{1}{2}c(\zeta^2-\zeta). \qquad (5)$$

When the rotation is present, for small values of the angular velocity Ω, we expand ω and ϕ in ascending powers of R:

$$\omega = \omega_0+R\omega_1+R^2\omega_2+\ldots \qquad (6)$$

$$\phi = \phi_0+R\phi_1+R^2\phi_2+\ldots \qquad (7)$$

where

$$\phi_0 = 0, \quad \omega_0 = \tfrac{1}{2}c(\zeta^2-\zeta). \qquad (8)$$

Substituting ω and ϕ in (1) and (2) and equating the coefficients of equal powers of R, we get a set of relations

$$\frac{\partial^4\phi_1}{\partial\zeta^4} = -\cos\theta\frac{\partial\omega_0}{\partial\zeta}, \quad \frac{\partial^4\phi_2}{\partial\zeta^4} = -\cos\theta\frac{\partial\omega_1}{\partial\zeta} \qquad (9)$$

$$\frac{\partial^2\omega_0}{\partial\zeta^2} = c, \quad \cos\theta\frac{\partial\phi_0}{\partial\zeta} = \frac{\partial^2\omega_1}{\partial\zeta^2}, \quad \cos\theta\frac{\partial\phi_1}{\partial\zeta} = \frac{\partial^2\omega_2}{\partial\zeta^2}. \qquad (10)$$

Solving relations (9) and (10) subject to the appropriate boundary conditions given by (4), we obtain to the second order in R,

$$\phi = -\frac{Rc\cos\theta}{240}(\zeta-1)^2(2\zeta-1)\zeta^2 \qquad (11)$$

$$\omega = \tfrac{1}{2}c(\zeta^2-\zeta)-\frac{R^2c}{720}\cos^2\theta\zeta^3(\zeta-1)^3 \qquad (12)$$

whence

$$u = -\frac{\nu}{r_1}\frac{Rc}{240}\sin\theta\zeta^2(\zeta-1)^2(2\zeta-1) \qquad (13)$$

$$v = -\frac{\nu}{\epsilon}\frac{Rc}{120}\cos\theta\zeta(\zeta-1)(5\zeta^2-5\zeta+1). \qquad (14)$$

3. Discussion

We now examine the nature of the streamlines to the first order of approximation in R. The differential equation for any streamline is given by

$$\frac{dr}{u} = \frac{r\,d\theta}{v} = \frac{dz}{w}. \qquad (15)$$

The central plane perpendicular to the axis of rotation corresponds to $\theta = \pi/2$ and $\theta = 3\pi/2$ and gives both halves of the annular pipe. In both cases from (14) $v = 0$. Hence a particle of fluid once in this plane does not leave it in the subsequent motion. On integrating (15), the streamlines in the central plane are given by

$$z - z_0 = \pm \frac{120 r_1}{R} \log \frac{(2\zeta-1)^2}{\zeta(1-\zeta)} \quad \text{for } \theta = \tfrac{1}{2}\pi, \tfrac{3}{2}\pi \qquad \ldots \quad \ldots \quad (16)$$

where z_0 is a constant of integration and differs from streamline to streamline. The streamlines are symmetric about the line $\zeta = \frac{1}{2}$ (the line through the middle of the annulus) as shown in Fig. 1. None of these lines reach either edges of the annular pipe as well as $\zeta = \frac{1}{2}$. Further the pattern is the same for both halves of the annular pipe $\theta = \pi/2$ and $3\pi/2$ unlike in Barua's problem. For a given value of ζ, z varies inversely as R. As the angular velocity Ω increases, the distance which must be covered by the streamlines to be within a given distance from either edges gets smaller.

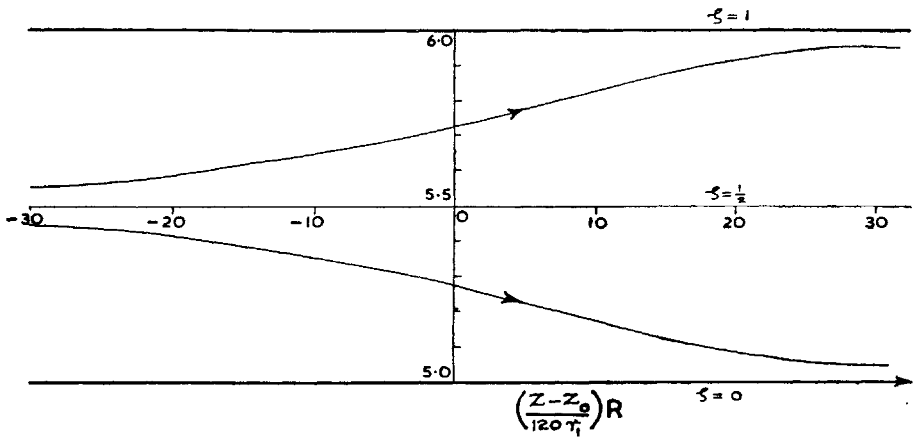

FIG. 1. Streamlines in the plane $\theta = \frac{\pi}{2}$.

For any other plane, the first relation of (15) gives, on integration,

$$\sec \theta = A \zeta^2 (\zeta-1)^2 (2\zeta-1) \qquad \ldots \quad \ldots \quad \ldots \quad (17)$$

where A is a constant of integration. For different values of A, we get a set of curves which are projections of the streamlines on a cross-section of the annular pipe (shown in Fig. 2 for the first quadrant $0 \leqslant \theta \leqslant \pi/2$). For an equal value of A, we get similar curves for the fourth quadrant $3\pi/2 \leqslant \theta \leqslant 2\pi$ being the reflections of the former curves on $\theta = 0$. For an equal and opposite value of A, the curves in the lower half of the cross-section are obtained by the reflection of the above curves on the diameter of the section perpendicular to the axis of rotation. None of the streamlines in each of the quadrants ever reach the edges of the annular pipe or the middle of the pipe, $r = \frac{1}{2}(r_1 + r_2)$.

For all values of θ, v vanishes when $\zeta = \zeta_1 = 0{\cdot}276$ or $\zeta = \zeta_2 = 0{\cdot}724$ approximately and for all values of ζ, u vanishes when $\theta = 0$ or π. We thus see the streamlines of motion through the points ($\zeta = \zeta_1$, $\theta = 0$), ($\zeta = \zeta_2$, $\theta = 0$), ($\zeta = \zeta_1$, $\theta = \pi$), ($\zeta = \zeta_2$, $\theta = \pi$) are straight lines and the motion of the fluid may be regarded as screw motion in opposite directions about two pairs of straight lines. The dashed line in Fig. 2 represents the section of the surface of zero transverse velocity.

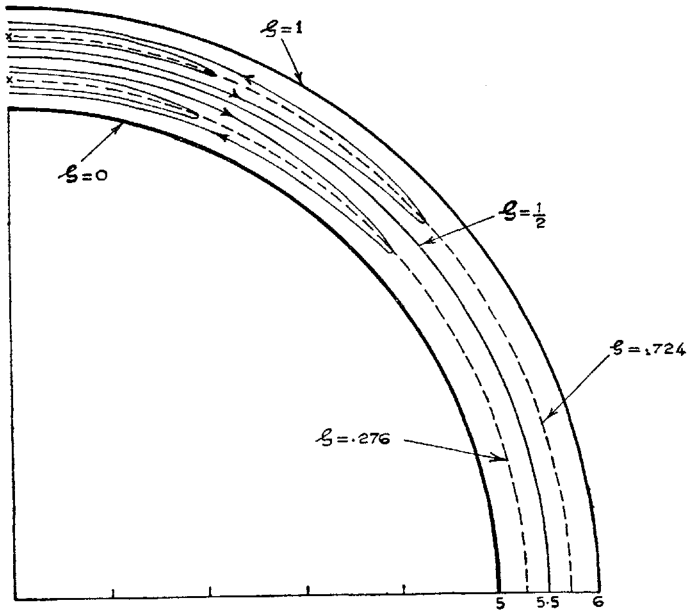

FIG. 2. Projection of the streamlines on a cross-section of the annular pipe in the first quadrant.

From (15) the relation between θ and z is given by

$$dz = -\frac{60r_1}{R(5\zeta^2-5\zeta+1)\cos\theta}\,d\theta. \qquad (18)$$

For points very near to either edges of the pipe this gives, on integration,

$$z-z_1 = -\frac{60r_1}{R}\log\tan\left(\tfrac{1}{2}\theta+\tfrac{1}{4}\pi\right) \qquad (19)$$

where z_1 is the value of z when $\theta = 0$. The distance $z-z_1$ covered by a fluid particle in going from $\theta = 0$ to $\theta = \theta_1$ is proportional to $\log\tan(\frac{1}{2}\theta+\frac{1}{4}\pi)$ and inversely proportional to R.

The flux Q_r through the annular pipe is

$$Q_r = -\int_{r_1}^{r_2}\int_0^{2\pi} wr\,dr\,d\theta = \frac{c\pi\nu r_1}{6}\left\{1-\frac{R^2}{16800}\right\}. \qquad (20)$$

The resistance coefficient becomes approximately (Goldstein 1938)

$$\frac{\gamma_r}{\gamma} = \left(\frac{Q}{Q_r}\right)^2 = 1 + \frac{R^2}{8400} \quad \ldots \quad (21)$$

where γ and γ_r are the resistance coefficients for flows without and with rotation and Q and Q_r stand for the flux in those cases.

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