THE OSCILLATIONS OF A VISCOUS INFINITE CYLINDER

by K. M. Srivastava *and* R. S. Kushwaha, *Department of Mathematics, University of Jodhpur, Jodhpur*

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The method of normal mode analysis is applied to study the oscillations of a gravitating cylinder in the presence of uniform axial magnetic field when viscous forces are present for non-axisymmetric perturbations. The dispersion relation has been obtained and in particular the case of high viscosity is discussed. The case of zero azimuthal number is derived which is in agreement with the form for $m = 0$. It is found that the damping constants for different wave numbers are diminished due to viscosity. The damping constant has been calculated for the case of $m = 1$ and high viscosity.

1. Introduction

The oscillations of a homogeneous fluid cylinder in the presence of magnetic field parallel to the axis have been considered by several authors. Chandrasekhar and Fermi (1953) have studied the stability of the fluid cylinder in the presence of an axial magnetic field which vanished outside the cylinder. Simon (1958) considered the same problem of stability taking a continuous uniform axial field. It does not vanish in the vacuum and has a stabilizing effect over the inside magnetic field. Lyttkens (1954) has studied the radial oscillations of a compressible adiabatic gas. The problem has also been considered by Gurm (1961) and Tondon and Talwar (1958) in the presence of suitable variable axial magnetic fields for radial oscillations. The authors (Srivastava and Kushwaha, *in press*[1]) studied the hydromagnetic oscillations of a viscous fluid cylinder for axisymmetric perturbations. The aim of the present paper is to study the small oscillations of a viscous incompressible fluid cylinder in the presence of a uniform axial magnetic field. Here we have studied the joint effect of the axial magnetic field and the dissipative viscous forces on gravitational instability of the cylinder for non-axisymmetric perturbations.

2. The Dispersion Relation

Consider the static state of an infinite cylinder of uniform density ρ and of radius R, infinitely conducting and in the presence of dissipative forces. The cylinder is surrounded by infinite vacuum and a uniform magnetic field is supposed to be present throughout the space parallel to the axis of the cylinder. The material under consideration is characterized by a scalar pressure p. In the equilibrium state p, V_i, V_e, respectively, denote gravitational pressure, potential inside and outside which are all functions of r only.

To obtain the linearized equations, suppose that the boundary of the cylinder is deformed from $r = R$ to the one given by

$$r = R+\xi_R \exp\,[i(kz+m\phi)+\omega t] \qquad (1)$$

where ξ_R is the radial displacement at the surface of the cylinder, k is the wave number of the disturbance along the z-axis, m the azimuthal number which is an integer or zero and $\frac{\xi_R}{R} \ll 1$. Let the various perturbed quantities resulting from deformation be denoted by

$$\tilde{u},\ p+\Delta p \text{ and } \bar{H}+\bar{h} \qquad (2)$$

where $\tilde{u}$, Δp and $\bar{h}$ are the perturbations in the velocity field (initially assumed to be zero) pressure and the magnetic field respectively.

Applying the following boundary conditions to the solutions of linearized equations of motion: (1) the radial component of velocity u_r at $r = R$ must be compatible with the assumed form of the deformed boundary given by eqn. (1), (2) the tangential viscous stresses must vanish on the boundary, (3) the normal component of magnetic field $\bar{H}.\,\bar{n}$ should be continuous across the boundary, and (4) the (r, r) component of the total stress tensor must be continuous across the deformed boundary, we get the dispersion relation as (Srivastava and Kushwaha 1967, *in press*[2])

$$\begin{aligned} Jl[k_m(x)I_m(x)-\tfrac{1}{2}]+SlQ_m(x) = (y^2-x^2)\Bigg[4\left\{-1+\frac{y^2-x^2}{f}\,(1-m^2P_m(x)P_m(y))\right\} \\ \times\left(1-\frac{1}{P_m(y)}\right)-\frac{4x^2}{f}\,(1-m^2P_m(x)P_m(y))\{(m^2+y^2)P_m(y)-1\} \\ +\left\{\frac{2y^2}{f}\,(1-m^2P_m(x)P_m(y))-1\right\}\{-2+2P_m(x)(m^2+x^2)+(y^2-x^2)P_m(x)\}\Bigg] \qquad (3)\end{aligned}$$

where

$$x = kR, \qquad y = bR \qquad (4)$$

$$J = \frac{4\pi G\rho R^4}{\nu^2}, \qquad S = \frac{\Omega_A^2 R^4}{\nu^2} \qquad (5)$$

$$l = \left[1+\frac{\Omega_A^2}{\omega^2}\right], \quad \Omega_A^2 = \frac{\mu k^2 H^2}{4\pi\rho} \qquad (6)$$

$$f = (y^2-x^2)+m^2y^2P_m(y)\left[P_m(y)\,\frac{x^2+y^2}{y^2}-2P_m(x)\right] \qquad (7)$$

$$P_m(x) = I_m(x)/xI_m'(x) \qquad (8)$$

$$Q_m(x) = \frac{K_m(x)}{xK_m'(x)}, \quad b^2 = a^2+k^2 \qquad (9)$$

$$\omega R^2 = \frac{(y^2-x^2)}{l}, \quad a^2 = \frac{1}{\omega\nu}\left(\omega^2+\Omega_A^2\right). \qquad (10)$$

The oscillations are stable for $m \neq 0$ in absence of viscosity and are stable and damped for the case of high viscosity [cf. eqn. (24)]. The damping constants are less than those of non-viscous case.

Solving (10) as a quadratic equation in ω and with the help of (6) we have

$$\omega = \frac{\nu}{2R^2}[(y^2-x^2) \pm \{(y^2-x^2)-4S\}^{\frac{1}{2}}] \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

and

$$l = \frac{(y^2-x^2)[(y^2-x^2) \pm \sqrt{(y^2-x^2)^2-4S}]}{(y^2-x^2)[(y^2-x^2) \pm \sqrt{(y^2-x^2)-4S}]-2S}. \quad \ldots \quad \ldots \quad (12)$$

Two cases of eqn. (11) are discussed.

Case I:

$$2\sqrt{S} > y^2-x^2, \quad y < x \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

then ω is complex of the form $\alpha \pm i\beta$. The oscillations are stable and damped.

Case II:

$$2\sqrt{S} \leqslant y^2-x^2, \quad y < x \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

then ω is real and negative. The oscillations are damped.

If $y > x$, J from eqn. (3) for $m \geqslant 1$ comes out negative which is not admissible.

The dispersion relation (3) in the limit when $m \to 0$ transforms as (Watson 1922)

$$\left. \begin{aligned} & f \to (y^2-x^2)\,; \quad I''_m(x) = -\frac{I'_1}{x}+I_0\,; \\ & P'_m(x) = \frac{I_1}{xI'_1}; \quad P_m(x) = \frac{I_0(x)}{xI_1(x)} \end{aligned} \right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

and

$$\frac{I'_1(x)}{I_0(x)} 2x^2(x^2+y^2)\left[1-\frac{2xy}{(x^2+y^2)}\frac{I'_1(y)}{I_1(y)}\frac{I_1(x)}{I'_1(x)}\right]-(x^4-y^4)$$
$$= Jl[K_0(x)I_0(x)-\tfrac{1}{2}]+Sl\frac{K_0(x)}{xK'_0(x)}. \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

This equation is in complete agreement with the equation obtained separately for $m = 0$ [Srivastava and Kushwaha (*in press*[1])].

The case of high viscosity can be treated by writing

$$y = x+\delta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

where δ is a small quantity whose second and higher powers can be neglected.

Now other approximations by Taylor's theorem are

$$\left.\begin{aligned} I_m(x+\delta) &= I_m(x)+\delta I'_m(x) \\ I'_m(x+\delta) &= I'_m(x)+\delta I''_m(x) \end{aligned}\right\} \quad \ldots \quad \ldots \quad . \quad (18)$$

$$P_m(y) = P_m(x)\left(1-\frac{\delta}{x}g\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

where

$$g = 1-1/P_m(x)+\frac{1}{P'_m(x)} = (m^2+x^2)P_m(x)-1/P_m(x) \quad \ldots \quad \ldots \quad (20)$$

also

$$\left.\begin{aligned} f &= 2x\delta[1-m^2P_m(x)(g+1)] \\ P_m(x)P_m(y) &= P_m^2(x)\left(1-\frac{\delta}{x}g\right) \end{aligned}\right\} . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

Making use of (17) to (21), the dispersion relation (3) reduces to (Chandrasekhar 1961)

$$\begin{aligned} &\left[1-m^2P_m^2(x)(g+1)\right][J(K_m(x)I_m(x)-\tfrac{1}{2})+SQ_m(x)] \\ &= \frac{2x\delta}{l}\left[2m^2P_m(x)\{(1-g-m^2P_m^2)+P_m(x)(g+1)\}+2x^2P_m(x)g-2\right]. \quad \ldots \quad (22) \end{aligned}$$

In the present approximation, we have [cf. (10)]

$$\frac{2x\delta}{l} = \frac{\omega R^2}{\nu}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

Making use of eqns. (23) in (22), we get

$$\frac{\omega}{4\pi G\rho R^2/\nu} = \frac{\left[1-m^2P_m^2(g+1)\right]\left[K_mI_m-\tfrac{1}{2}+\left(\dfrac{H}{H_G}\right)^2\dfrac{xK_m}{K'_m}\right]}{P_mDr}. \quad \ldots \quad (24)$$

where the argument is always x and

$$H_G = 4\pi\rho R\sqrt{\frac{G}{\mu}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

$$Dr = \left[2m^2P_m\{(1-g-m^2P_m^2)P_m(g+1)\}+2x^2gP_m-2\right]. \quad \ldots \quad (26)$$

Here, again in the limit of $m \to 0$, the dispersion relation (24) becomes

$$\frac{\omega}{4\pi G\rho R^2/\nu} = \frac{(K_0I_0-\tfrac{1}{2})+\left(\dfrac{H}{H_G}\right)^2\dfrac{xK_0}{K'_0}}{2\left[x^2\dfrac{I_0^2}{I_1^2}-(1+x^2)\right]} \quad \ldots \quad \ldots \quad (27)$$

and the eqn. (27) completely agrees with the one obtained for $m = 0$ separately [Srivastava and Kushwaha (*in press*[1])].

In the limit, when $\nu \to \infty$, eqn. (24) gives the frequency of oscillation in the units of $4\pi G\rho R^2/\nu$.

It is inferred from (24) that when the magnetic field is present and $m \neq 0$ the modes of oscillation are stable and damped. ω is purely real and negative in the case of high viscosity. The damping constants for $m = 1$ when viscosity is paramount are given in Table I. The magnetic field increases the damping constant but its effect is decreased due to viscosity.

From eqn. (24) we calculate the damping constants for $m = 1$ when viscosity is dominant. These are given in Table I for five values of H/H_G.

TABLE I

The damping constants in the units of $4\pi G\rho R^2/\nu$ when viscosity is predominant

x	H/H_G				
	0	0·25	0·50	0·75	1·00
0·1	0·00362	0·00395	0·00495	0·00605	0·00891
0·2	0·00830	0·00927	0·01217	0·01700	0·02377
0·3	0·01464	0·01662	0·02260	0·03256	0·04650
0·4	0·02122	0·02068	0·03440	0·05072	0·07356
0·5	0·02772	0·03238	0·04634	0·06922	0·10217
0·6	0·03354	0·03963	0·05789	0·08832	0·13093
0·7	0·03862	0·04614	0·06866	0·10619	0·15874
0·8	0·04295	0·05182	0·07844	0·12280	0·18491
0·9	0·04651	0·05666	0·08714	0·13794	0·20906
1·0	0·04938	0·06070	0·09476	0·15152	0·23100
1·1	0·05154	0·06401	0·10141	0·16372	0·25097

For assigned values of x and S and $m = 1$, we can obtain (ω, J) relation by evaluating ω and l from eqns. (11) and (12) and J from eqn. (10) with the help of (7) for a sequence of values of x and a given value of J and S.

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