COMPOSITE THRUST PROGRAMMING OF A VERTICALLY ASCENDING ROCKET

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The thrust programme consisting of a constant thrust phase followed successively by a constant acceleration phase and a coasting phase in the case of a vertically ascending rocket has been studied. It is shown that the burnout velocity, the burnout altitude and the peak altitude depend not only on the total consumption of fuel in the flight, but also on the distribution of the same in the first two phases. Also the burnout velocity and the peak altitude gradually increase with the increase of fuel consumption in the first phase (i.e. constant thrust) and ultimately attain their maximum values when the total fuel is utilized fully in this phase alone; burnout altitude on the other hand decreases monotonically with the increase of fuel consumption in the first phase except when thrust-to-weight ratio is unity.

Nomenclature

x, h = horizontal distance and altitude respectively of the rocket (regarded as a particle) from the starting point
V = velocity of the rocket at any time
t = time
m = mass of the rocket at any time
m_i = initial mass of the rocket
β = mass flow rate
V_E = equivalent exit velocity
$T = \beta V_E$ = rocket thrust
g = acceleration due to gravity
F_1 = fuel expenditure for the first phase, i.e. the constant thrust phase
F = total fuel expenditure

$$\theta = \frac{tg}{V_E}, \quad \eta = \frac{hg}{V_E^2}, \quad u = \frac{V}{V_E}, \quad \mu = \frac{m}{m_i}, \quad \tau = \frac{\beta V_E}{mg}, \quad \zeta_1 = \frac{F_1}{m_i}, \quad \zeta = \frac{F}{m_i}$$

and the dot sign represents differentiation with respect to time.

Subscripts

i = initial values for the constant thrust phase
1 = final values for the constant thrust phase
= initial values for the constant acceleration phase

2 = final values for the constant acceleration phase
= initial values for the coasting phase
f = final values for the coasting phase

Introduction

Composite thrust programmes consisting of (i) constant thrust phase followed by a coasting phase and (ii) constant acceleration phase followed by a coasting phase have been studied by Miele (1962). It has been stated there that 'for the same propellant mass ratio and the initial thrust-to-weight ratio, the peak altitude of the constant thrust programme is higher than that of the constant acceleration programme'.

The object of the present study is to consider a thrust programme consisting of a constant thrust phase, followed successively by a constant acceleration phase and a coasting phase. Expressions for the burnout velocity, the burnout altitude and the peak altitude in terms of fuel expenditure in the first two phases have been obtained. The expression for peak altitude reduces to the simple case of Miele (1962) for constant thrust phase followed by a coasting phase when the fuel is fully consumed in the constant thrust phase. This gives the value of peak altitude for a specified amount of fuel—the peak altitude in this case being designated hereafter as the best peak altitude.

Results and Discussion

The motion of a vertically ascending rocket is given by the following equations:

$$\dot{x} = 0 \quad \quad (1)$$

$$\dot{h} = V \quad \quad (2)$$

$$m\dot{V} = T - mg \quad \quad (3)$$

$$\dot{m} + \beta = 0 \quad \quad (4)$$

Equation (1) gives x = constant. Replacing m as the independent variable, eqns. (2)–(4) can be written in the form

$$\left.\begin{aligned} \frac{dh}{dm} &= -\frac{V}{\beta} \\ \frac{dV}{dm} &= \frac{g}{\beta} - \frac{V_E}{m} \\ \frac{dt}{dm} &= -\frac{1}{\beta} \end{aligned}\right\} \quad \quad (5)$$

Introducing the dimensionless variables θ, η, u, μ and τ already defined, we get

$$\left.\begin{aligned} &\frac{d\eta}{d\mu}+\frac{u}{\tau\mu} = 0 \\ &\frac{du}{d\mu}+\frac{1}{\mu} = \frac{1}{\tau\mu} \\ &\frac{d\theta}{d\mu}+\frac{1}{\tau\mu} = 0 \end{aligned}\right\} \quad \ldots \quad (6)$$

Now for constant thrust phase,

$$\beta V_E = \text{constant}.$$

Therefore, $$\frac{\tau}{\tau_i} = \frac{m_i}{m} = \frac{1}{\mu} \text{ or } \tau\mu = \tau_i.$$

Therefore, equations in (6) reduce to

$$\left.\begin{aligned} &\frac{d\eta}{d\mu}+\frac{u}{\tau_i} = 0 \\ &\frac{du}{d\mu}+\frac{1}{\mu} = \frac{1}{\tau_i} \\ &\frac{d\theta}{d\mu}+\frac{1}{\tau_i} = 0 \end{aligned}\right\} \quad \ldots \quad (7)$$

Solution of these equations are

$$\left.\begin{aligned} &\mu+\tau_i\theta = c_1 \\ &u+\frac{c_1-\mu}{\tau_i}+\log\mu = c_2 \\ &\eta+\frac{c_2}{\tau_i}\mu+\frac{1}{2}\left(\frac{c_1-\mu}{\tau_i}\right)^2+\frac{\mu}{\tau_i}(1-\log\mu) = c_3 \end{aligned}\right\} \quad \ldots \quad (8)$$

where c_1, c_2 and c_3 are constants of integration.
For this phase (i.e. constant thrust),

initially, $\theta = u = \eta = 0, \quad \mu = 1$

and finally, $\theta = \theta_1, \quad u = u_1, \quad \eta = \eta_1, \quad \mu = \mu_1.$

Putting initial values in (8), we get

$$c_1 = 1, \quad c_2 = 0, \quad c_3 = \frac{1}{\tau_i}.$$

Therefore, equations in (8) reduce to

$$\left.\begin{aligned} &\mu+\tau_1\theta = 1 \\ &u+\frac{1-\mu}{\tau_i}+\log\mu = 0 \\ &\eta+\frac{1}{2}\left(\frac{1-\mu}{\tau_i}\right)^2+\frac{\mu}{\tau_i}(1-\log\mu) = \frac{1}{\tau_i} \end{aligned}\right\} \quad \ldots \quad (9)$$

Putting final values in (9), we get

$$\left.\begin{aligned}\theta_1 &= \frac{1-\mu_1}{\tau_i}\\ u_1 &= -\frac{1-\mu_1}{\tau_i}-\log\mu_1\\ \eta_1 &= \frac{\mu_1\log\mu_1}{\tau_i}+\frac{1-\mu_1}{\tau_i}-\frac{1}{2}\left(\frac{1-\mu_1}{\tau_i}\right)^2\end{aligned}\right\} \quad \ldots \quad (10)$$

For constant acceleration phase,

$$\tau = \tau_1 = \frac{\tau_i}{\mu_1} = \text{constant}.$$

Then equations in (6) reduce to

$$\left.\begin{aligned}\frac{d\eta}{d\mu}+\frac{u}{\tau_1\mu} &= 0\\ \frac{du}{d\mu}+\frac{1}{\mu} &= \frac{1}{\tau_1\mu}\\ \frac{d\theta}{d\mu}+\frac{1}{\tau_1\mu} &= 0\end{aligned}\right\} \quad \ldots \quad (11)$$

Solution of these equations are

$$\left.\begin{aligned}&\log\mu+\tau_1\theta = c_4\\ &u+\frac{\tau_1-1}{\tau_1}\log\mu = c_5\\ &\eta+\frac{c_5}{\tau_1}\log\mu-\frac{\tau_1-1}{2\tau_1^2}(\log\mu)^2 = c_6\end{aligned}\right\} \quad \ldots \quad (12)$$

where c_4, c_5 and c_6 are constants of integration.

In this phase,

initially, $\theta = \theta_1,\quad u = u_1,\quad \eta = \eta_1,\quad \mu = \mu_1$

and finally, $\theta = \theta_2,\quad u = u_2,\quad \eta = \eta_2,\quad \mu = \mu_2.$

Putting initial values in (12), we get

$$c_4 = \log\mu_1+\tau_1\theta_1 = \log\mu_1+\frac{1-\mu_1}{\mu_1}$$

$$c_5 = u_1+\frac{\tau_1-1}{\tau_1}\log\mu_1 = \frac{1}{\tau_i}\{\mu_1(1-\log\mu_1)-1\}$$

$$\begin{aligned}c_6 &= \eta_1+\frac{c_5}{\tau_1}\log\mu_1-\frac{\tau_1-1}{2\tau_1^2}(\log\mu_1)^2\\ &= \frac{\mu_1\log\mu_1}{\tau_i}+\frac{1-\mu_1}{\tau_i}-\frac{1}{2}\left(\frac{1-\mu_1}{\tau_i}\right)^2+\frac{\mu_1}{\tau_i^2}\{\mu_1(1-\log\mu_1)-1\}\log\mu_1\\ &\quad -\frac{\mu_1(\tau_i-\mu_1)}{2\tau_i^2}(\log\mu_1)^2.\end{aligned}$$

From final values, we get

$$\left.\begin{aligned} \theta_2 &= \frac{c_4-\log\mu_2}{\tau_1} \\ u_2 &= c_5-\frac{\tau_1-1}{\tau_1}\log\mu_2 \\ \eta_2 &= c_6-\frac{c_5}{\tau_1}\log\mu_2+\frac{\tau_1-1}{2\tau_1^2}(\log\mu_2)^2 \end{aligned}\right\} . \qquad \ldots \quad \ldots \quad (13)$$

For coasting phase, the differential equations are

$$\left.\begin{aligned} \dot{h} &= V \\ m\dot{V} &= -mg \\ \dot{m} &= 0 \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

or in terms of dimensionless variables

$$\left.\begin{aligned} \frac{d\eta}{du}+u &= 0 \\ \frac{du}{d\theta}+1 &= 0 \\ \frac{d\mu}{d\theta} &= 0 \end{aligned}\right\} . \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

Solution of these equations are

$$\left.\begin{aligned} \mu &= c_7 \\ u+\theta &= c_8 \\ \eta+\tfrac{1}{2}u^2 &= c_9 \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

where c_7, c_8 and c_9 are constants of integration.
In this phase,

initially, $u = u_2, \quad \eta = \eta_2$

and finally, $u = 0, \quad \eta = \eta_f.$

Putting initial values in the third equation of (16), we get

$$\eta_2+\tfrac{1}{2}u_2^2 = c_9.$$

Putting final values in the same equation, we get

$$\eta_f = c_9.$$

Therefore,

$$\begin{aligned} \eta_f &= \eta_2+\tfrac{1}{2}u_2^2 \\ &= c_6-\frac{c_5}{\tau_1}\log\mu_2+\frac{\tau_1-1}{2\tau_1^2}(\log\mu_2)^2+\frac{1}{2}\left(c_5-\frac{\tau_1-1}{\tau_1}\log\mu_2\right)^2 . \qquad \ldots \quad (17) \end{aligned}$$

Replacing τ_1 by τ_i/μ_1 and substituting the values of c_5 and c_6 in the last two equations of (13) and in (17), we get

$$u_2 = -\frac{1}{\tau_i}(1-\mu_1)-\log \mu_2-\frac{\mu_1}{\tau_i}\log\frac{\mu_1}{\mu_2} \qquad (18)$$

$$\eta_2 = \frac{\mu_1 \log \mu_1}{\tau_i}+\frac{1-\mu_1}{\tau_i}-\frac{1}{2}\left(\frac{1-\mu_1}{\tau_i}\right)^2+\frac{\mu_1}{\tau_i^2}\left\{\mu_1(1-\log \mu_1)-1-\frac{\tau_i-\mu_1}{2}\log(\mu_1\mu_2)\right\}\log\frac{\mu_1}{\mu_2} \qquad (19)$$

$$\eta_f = \frac{1}{\tau_i}(1-\mu_1)+\frac{1}{\tau_i}\log \mu_2+\tfrac{1}{2}(\log \mu_2)^2+\frac{\mu_1}{\tau_i}\left\{\log\frac{\mu_1}{\mu_2}-\frac{1}{2}\left(\log\frac{\mu_1}{\mu_2}\right)^2\right\}. \qquad (20)$$

Now

$$\mu_1 = \frac{m_1}{m_i} = \frac{m_i-F_1}{m_i} = 1-\zeta_1$$

$$\mu_2 = \frac{m_2}{m_i} = \frac{m_i-F}{m_i} = 1-\zeta.$$

Therefore, the relations (18), (19) and (20) reduce to

$$u_2 = -\frac{\zeta_1}{\tau_i}-\log(1-\zeta)-\frac{1-\zeta_1}{\tau_i}\log\frac{1-\zeta_1}{1-\zeta} \qquad (21)$$

$$\eta_2 = \frac{\zeta_1+(1-\zeta_1)\log(1-\zeta_1)}{\tau_i}-\frac{1}{2}\left(\frac{\zeta_1}{\tau_i}\right)^2$$
$$-\frac{1-\zeta_1}{\tau_i^2}\left\{\zeta_1+(1-\zeta_1)\log(1-\zeta_1)+\frac{\tau_i-(1-\zeta_1)}{2}\log\left(\overline{1-\zeta_1}\,\overline{1-\zeta}\right)\right\}\log\frac{1-\zeta_1}{1-\zeta} \qquad (22)$$

$$\eta_f = \frac{\zeta_1+\log(1-\zeta)}{\tau_i}+\tfrac{1}{2}\{\log(1-\zeta)\}^2+\frac{1-\zeta_1}{\tau_i}\left\{\log\frac{1-\zeta_1}{1-\zeta}-\frac{1}{2}\left(\log\frac{1-\zeta_1}{1-\zeta}\right)^2\right\}. \qquad (23)$$

Equation (23) shows that peak altitude depends not only on the total consumption of fuel in the flight but also on the distribution of the same in the first two phases.

Putting $\zeta_1 = \zeta$, which means that the total fuel is consumed in the constant thrust phase, we get from (23)

$$\eta_f = \frac{\zeta+\log(1-\zeta)}{\tau_i}+\tfrac{1}{2}\{\log(1-\zeta)\}^2. \qquad (24)$$

This agrees with eqn. (23) given by Miele (1962, page 341).

Some curves are drawn showing different values of the burnout velocity, the burnout altitude, the peak altitude for varying distribution of the total fuel (which is 80 per cent of the initial mass of rocket) in the first two phases (Figs. 1–3) and best peak altitude for different amount of fuel expenditures ($0 \leqslant \zeta \leqslant 0{\cdot}8$) in the case of rockets having different values of initial thrust-to-weight ratio (τ_i) (Fig. 4).

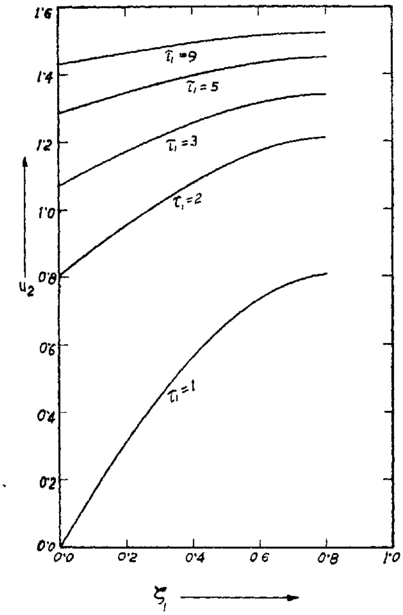

FIG. 1. Burnout velocity for different values of ζ_1 (fuel expenditure ratio in the constant thrust phase) when the total fuel expenditure is 80 per cent of the initial mass of the rocket.

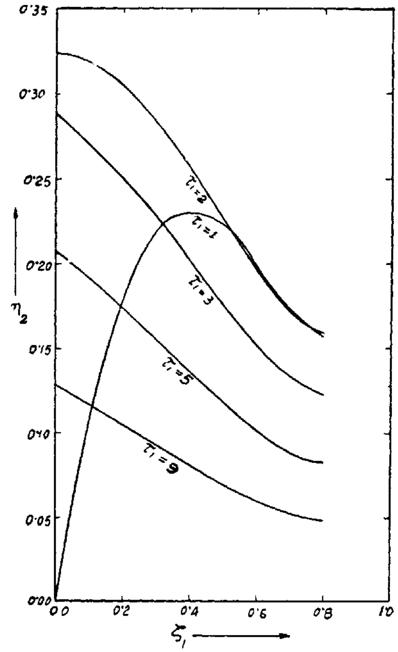

FIG. 2. Burnout altitude for different values of ζ_1 when the total fuel expenditure is 80 per cent of the initial mass of the rocket.

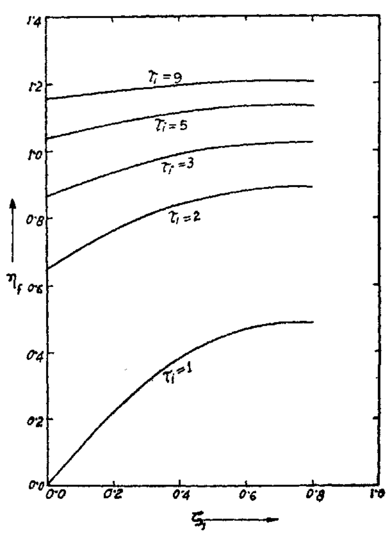

FIG. 3. Peak altitude for different values of ζ_1 when the total fuel is 80 per cent of the initial mass of the rocket.

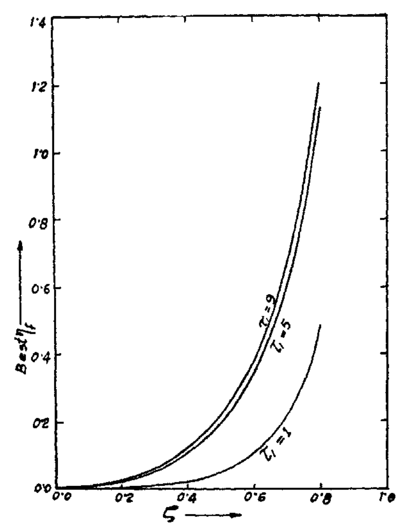

FIG. 4. Best peak altitudes for different fuel expenditures.

It is found that the peak altitudes depend not only on the total fuel consumption but also on the distribution of fuel in the first two phases.

From Figs. 1 and 3, it is evident that the burnout velocity and the peak altitude rapidly increase with increase of fuel expenditure in the first phase when the latter is sufficiently small in comparison with the total fuel expenditure; while Fig. 2 shows that the burnout altitude decreases monotonically with the increase of fuel expenditure in the first phase except when τ_t is unity.

From Fig. 4, it is evident that the peak altitude rapidly increases with a small increase of total fuel expenditure when the latter is sufficiently large.

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Reference

Miele, A. (1962). Flight Mechanics, Vol. 1. Addison-Weseley Publishing Co., Inc., Massachusetts, 335–342.