ON PERTURBATION THEORY—II

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Let $p(\mathbf{x}) = p(x_1, x_2)$ and $q(\mathbf{x}) = q(x_1, x_2)$ and λ be a variable parameter. Let λ_n and ψ_n be the eigenvalue and eigenfunction associated with the differential equation

$$\Delta\psi+(\lambda p-q)\psi = 0.$$

If p is perturbed to $p+\epsilon t$, where ϵ is a small positive constant and $t = t(x_1, x_2)$, the boundary conditions remaining unaltered, the perturbed eigenvalue Λ_n and the eigenfunction $\tilde{\psi}_n(\mathbf{x})$ under suitable conditions admit of the convergent infinite series expansions

$$\Lambda_n = \lambda_n+\epsilon\lambda_n^{(1)}+\epsilon^2\lambda_n^{(2)}+\ldots;$$

$$\tilde{\psi}_n(\mathbf{x}) = \psi_n(\mathbf{x})+\epsilon\psi_n^{(1)}(\mathbf{x})+\epsilon^2\psi_n^{(2)}(\mathbf{x})+\ldots.$$

The result holds also in the one-dimensional case as well as in three and higher dimensions.

§ 1. Consider the differential equation

$$\Delta\psi+(\lambda p-q)\psi = 0 \qquad \ldots \quad (1.1)$$

over a two-dimensional region E, which may be finite or infinite.

$$\psi = \psi(\mathbf{x}) = \psi(x_1, x_2), \;\; p = p(\mathbf{x}) = p(x_1, x_2), \;\; q = q(\mathbf{x}) = q(x_1, x_2)$$

and λ is a variable parameter, real or complex. $p(\mathbf{x})$ is assumed strictly positive for all $\mathbf{x} \in E$.

The differential equation

$$\Delta\psi+(\lambda P-q)\psi = 0, \qquad \ldots \quad (1.2)$$

obtained by slightly changing p to $p = p+\epsilon t, t \equiv t(\mathbf{x}) = t(x_1, x_2)$ and ϵ, a small positive constant, is the 'p-perturbed' equation.

We assume that the perturbed as well as the unperturbed eigenvalues are all discrete. This happens if $q(\mathbf{x})/p(\mathbf{x}) \to \infty$, as $\mathbf{x} \to \infty$ and $t(\mathbf{x}) \leqslant Lp(\mathbf{x})$, where L is a positive constant. For, the first condition implies that the unperturbed spectrum in (1.1) is discrete [Titchmarsh (1958), p. 153] and by the second condition $P(\mathbf{x}) = p+\epsilon t \leqslant p(1+\epsilon L)$. Thus $\dfrac{q(\mathbf{x})}{P(\mathbf{x})} \geqslant \dfrac{q(\mathbf{x})}{(1+\epsilon L)p(\mathbf{x})}$ and hence the perturbed spectrum in (1.2) is also discrete.

We assume further that the boundary conditions remain unaltered for the perturbed as well as the unperturbed equations. We confine our attention to the non-degenerate case only.

Let Λ_n and $\tilde{\psi}_n(\mathrm{x})$ be the perturbed eigenvalues and eigenfunctions, the corresponding unperturbed ones being represented, respectively, by λ_n and $\psi_n(\mathrm{x})$.

We obtained asymptotic formulae of the type [Chakravarty (1966)]

$$\Lambda_n = \lambda_n+\epsilon\lambda_n^{(1)}+\ldots+\epsilon^m\lambda_n^{(m)}+O(\epsilon^{m+1}) \qquad (1.3)$$

and

$$\tilde{\psi}_n(\mathrm{x}) = \psi_n(\mathrm{x})+\epsilon\psi_n^{(1)}(\mathrm{x})+\ldots+\epsilon^m\psi_n^{(m)}(\mathrm{x})+O(\epsilon^{m+1}), \qquad (1.4)$$

m fixed and $\epsilon \to 0$, expressing the perturbed eigenvalues and eigenfunctions in terms of the unperturbed ones.

Our object in the present paper is to obtain series expansions of the form

$$\Lambda_n = \lambda_n+\epsilon\lambda_n^{(1)}+\epsilon^2\lambda_n^{(2)}+\ldots \qquad (1.5)$$

and

$$\tilde{\psi}_n(\mathrm{x}) = \psi_n(\mathrm{x})+\epsilon\psi_n^{(1)}(\mathrm{x})+\epsilon^2\psi_n^{(2)}(\mathrm{x})+\ldots \qquad (1.6)$$

convergent for sufficiently small ϵ.

The two problems are different. For, it is not always possible to obtain convergent series expansions for $\tilde{\psi}_n(\mathrm{x})$ and Λ_n but the theory of asymptotic perturbations is applicable to these problems. (Compare Titchmarsh 1958, p. 218).

§ 2. We assume the validity of the expansions (1.5) and (1.6) and derive formally the coefficients in these expansions.

Let

$$\psi_n^{(m)}(\mathrm{x}) = \sum_{r=0}^{\infty} \alpha_{n,r}^{(m)}\psi_r(\mathrm{x}), \qquad (2.1)$$

$$m = 0, 1, 2, \ldots, \psi_n^{(0)}(\mathrm{x}) = \psi_n(\mathrm{x}).$$

We write $P = p+\epsilon t$ and $\lambda = \Lambda_n$ in (1.2), so that $\psi = \tilde{\psi}_n(\mathrm{x})$. Then substituting for Λ_n and $\tilde{\psi}_n(\mathrm{x})$ as given by (1.5) and (1.6) and equating powers of ϵ to zero, we obtain

$$\Delta\psi_n^{(1)}+(\lambda_n p-q)\psi_n^{(1)}+(\lambda_n^{(1)}p+\lambda_n t)\psi_n = 0 \qquad (2.2)$$

$$\Delta\psi_n^{(2)}+(\lambda_n p-q)\psi_n^{(1)}+(\lambda_n^{(1)}p+\lambda_n t)\psi_n^{(1)}+(\lambda_n^{(2)}p+\lambda_n^{(1)}t)\psi_n = 0 \qquad (2.3)$$

and generally,

$$\Delta\psi_n^{(m)}+(\lambda_n p-q)\psi_n^{(m)}+(\lambda_n^{(1)}p+\lambda_n t)\psi_n^{(m-1)}+(\lambda_n^{(2)}p+\lambda_n^{(1)}t)\psi_n^{(m-2)}+\ldots$$
$$+(\lambda_n^{(m)}p+\lambda_n^{(m-1)}t)\psi_n = 0, \qquad (2.4)$$

$m = 3, 4, \ldots$

Using (2.1), it follows formally that

$$\begin{aligned}\Delta\psi_n^{(m)}(\mathbf{x})+(\lambda_n p-q)\psi_n^{(m)}(\mathbf{x}) &= \sum_{r=0}^{\infty}\alpha_{n,r}^{(m)}\,\Delta\psi_r(\mathbf{x})+(\lambda_n p-q)\sum_{r=0}^{\infty}\alpha_{n,r}^{(m)}\,\psi_r(\mathbf{x}) \\ &= \sum_{r=0}^{\infty}(\lambda_n-\lambda_r)\alpha_{n,r}^{(m)}\,\psi_r(\mathbf{x})p(\mathbf{x}), \qquad (2.5)\end{aligned}$$

$m = 1, 2, 3, \ldots$

Substituting (2.5) (with $m = 1$) in (2.2), multiplying the resulting equation by $\psi_n(\mathbf{x})$ and integrating over E, we obtain, by using the orthogonal relation

$$\int_E \psi_r(\mathbf{x})\psi_s(\mathbf{x})p(\mathbf{x})\,d\mathbf{x} = \delta_{r,s}$$

$\delta_{r,s}$ being the Kronecker delta, the relation

$$\lambda_n^{(1)} = -\lambda_n b_{n,n} \qquad (2.6)$$

where

$$b_{m,n} = \int_E \psi_m(\mathbf{x})\psi_n(\mathbf{x})t(\mathbf{x})\,d\mathbf{x}. \qquad (2.7)$$

Also, multiplying the same by $\psi_M(\mathbf{x})$, $M \neq n$, and integrating over E, we obtain

$$\alpha_{n,M}^{(1)} = -\frac{\lambda_n}{\lambda_n-\lambda_M}\,b_{M,n}. \qquad (2.8)$$

Further, we have

$$1 = \int_E \tilde{\psi}_n^2(\mathbf{x})P(\mathbf{x})\,d\mathbf{x} = \int_E \left(\psi_n+\epsilon\psi_n^{(1)}+\epsilon^2\psi_n^{(2)}+\ldots\right)^2(p+\epsilon t)\,d\mathbf{x}.$$

Hence equating like powers of ϵ from both sides,

$$\int_E \psi_n^2 t\,d\mathbf{x}+\sum_{r+s=1}\int_E \psi_n^{(r)}\psi_n^{(s)}p\,d\mathbf{x} = 0 \qquad (2.9)$$

$$\sum_{r+s=1}\int_E \psi_n^{(r)}\psi_n^{(s)}t\,d\mathbf{x}+\sum_{r+s=2}\int_E \psi_n^{(r)}\psi_n^{(s)}p\,d\mathbf{x} = 0 \qquad (2.10)$$

and generally,

$$\sum_{r+s=m}\int_E \psi_n^{(r)}\psi_n^{(s)}t\,d\mathbf{x}+\sum_{r+s=m+1}\int_E \psi_n^{(r)}\psi_n^{(s)}p\,d\mathbf{x} = 0. \qquad (2.11)$$

From (2.9) and (2.1) (with $m = 1$),

$$\alpha_{n,n}^{(1)} = -\tfrac{1}{2}b_{n,n}. \qquad (2.12)$$

Substituting (2.5) (with $m = 2$) and (2.1) (with $m = 1$) in (2.3), multiplying by $\psi_M(\mathbf{x})$ and integrating over E, we have

$$\lambda_n^{(2)} = \lambda_n\left(b_{n,n}^2-\lambda_n\sum_{r\neq n}\frac{b_{n,r}^2}{\lambda_r-\lambda_n}\right) \qquad (2.13)$$

and

$$\alpha^{(2)}_{n,\,M} = \frac{\lambda_n}{\lambda_n-\lambda_M}\left\{\left(\frac{1}{2}+\frac{\lambda_M}{\lambda_M-\lambda_n}\right)b_{n,\,n}b_{n,\,M}+\lambda_n\sum_{r\neq n}\frac{b_{n,r}\,b_{M,\,r}}{\lambda_n-\lambda_r}\right\} \quad .. \ (2.14)$$

$n \neq M$.

Also from (2.10) and (2.1) (with $m = 1, 2$),

$$\alpha^{(2)}_{n,\,n} = \tfrac{1}{2}\left\{\tfrac{3}{4}b^2_{n,\,n}-2\lambda_n\sum_{r\neq n}\frac{b^2_{n,\,r}}{\lambda_r-\lambda_n}-\lambda_n^2\sum_{r\neq n}\frac{b^2_{n,\,r}}{(\lambda_r-\lambda_n)^2}\right\}. \quad .. \ (2.15)$$

Hence in the non-degenerate case, the formal expansions of Λ_n and $\tilde{\psi}_n(\mathbf{x})$, as far as terms involving ϵ^2, are given by

$$\Lambda_n = \lambda_n-\epsilon\lambda_n b_{n,\,n}+\epsilon^2\lambda_n\left(b^2_{n,\,n}-\lambda_n\sum_{r\neq n}\frac{b^2_{n,\,r}}{\lambda_r-\lambda_n}\right)+\ldots \quad .. \ (2.16)$$

and

$$\begin{aligned}\tilde{\psi}_n(\mathbf{x}) &= \psi_n(\mathbf{x})+\epsilon\left(-\tfrac{1}{2}b_{n,n}\,\psi_n(\mathbf{x})+\lambda_n\sum_{r\neq n}\frac{b_{n,\,r}\,\psi_r(\mathbf{x})}{\lambda_r-\lambda_n}\right)\\ &\quad+\epsilon^2\left[\tfrac{1}{2}\left\{\tfrac{3}{4}b^2_{n,\,n}-2\lambda_n\sum_{r\neq n}\frac{b^2_{n,\,r}}{\lambda_r-\lambda_n}-\lambda_n^2\sum_{r\neq n}\frac{b^2_{n,\,r}}{(\lambda_r-\lambda_n)^2}\right\}\psi_n(\mathbf{x})\right.\\ &\quad\left.+\lambda_n\sum_{R\neq n}(\lambda_n-\lambda_R)^{-1}\left\{\left(\frac{1}{2}+\frac{\lambda_R}{\lambda_R-\lambda_n}\right)b_{n,n}\,b_{n,R}+\lambda_n\sum_{r\neq n}\frac{b_{n,\,r}b_{R,\,r}}{\lambda_n-\lambda_r}\right\}\psi_r(\mathbf{x})\right]+\ldots\end{aligned} \quad .. \ (2.17)$$

§ 3. We now investigate the conditions under which the infinite series (1.5) and (1.6) converge and represent respectively the perturbed eigenvalues and eigenfunctions. We state our results in the form of the following theorem. The result holds also in the one-dimensional case and in higher dimensions than two.

Theorem. (i) *Let* $p(\mathbf{x}) \geqslant \eta > 0$ *and* $|t(\mathbf{x})| \leqslant p(\mathbf{x})$ *in* E, *which may be finite or infinite.*

(ii) $G(\mathbf{x}, \xi, \lambda)$ *be the unperturbed Green's function and*

$$G_n(\mathbf{x}, \xi, \lambda) = G(\mathbf{x}, \xi, \lambda)-\frac{\psi_n(\mathbf{x})\psi_n(\xi)}{\lambda_n-\lambda} = \sum_{r\neq n}\frac{\psi_r(\mathbf{x})\psi_r(\xi)}{\lambda_r-\lambda} \quad .. \ (3.1)$$

be the 'modified Green's function'.

(iii) $F(\mathbf{x})$ *and* $\tilde{F}(\mathbf{x})$ *be two functions such that* $F^2(\mathbf{x})\,p(\mathbf{x})$, $\tilde{F}^2(\mathbf{x})\,p(\mathbf{x})$ *belong to* L, $\tilde{F}(\mathbf{x})$ *has for its Fourier coefficient* $\tilde{C}_n$ *and* $F(\mathbf{x})$ *satisfies the differential equation*

$$\Delta F+(\lambda p-q)F = -\tilde{F}p+\tilde{C}_n\psi_n(\mathbf{x})p(\mathbf{x}). \quad .. \quad .. \ (3.2)$$

If E *is finite,* $F(\mathbf{x})$ *and* $\tilde{F}(\mathbf{x})$ *satisfy the prescribed boundary conditions.*

(iv) *$F(\mathbf{x})$ and $\tilde{F}(\mathbf{x})$ satisfy the inequality*

$$\int_E F^2 t\, d\mathbf{x} \leqslant C_1 \int_E F^2 p\, d\mathbf{x} + C_2 \int_E \tilde{F}^2 p\, d\mathbf{x} \qquad (3.3)$$

where C_1, C_2, are positive constants independent of $F(\mathbf{x})$ and $\tilde{F}(\mathbf{x})$.

(v) *Further, let*

$$(\psi_n^{(m)})^2 p \text{ and } (\psi_n^{(m)})^2 t$$

belong to L for every positive integral value of m; $\psi_n^{(0)}$ being equal to $\psi_n(\mathbf{x})$;

(vi) $\lambda_n^{(1)} = -\lambda_n \int_E \psi_n^2(\xi) t(\xi)\, d\xi$;

(vii) $\psi_n^{(1)}(\mathbf{x}) = -\lambda_n \int_E G_n(\mathbf{x}, \xi, \lambda_n)\psi_n(\xi) t(\xi)\, d\xi - \frac{1}{2}\psi_n(\mathbf{x}) \int_E \psi_n^2(\xi) t(\xi)\, d\xi$

and for all integral values of $m > 1$, let

(viii) $\chi_n^{(m)}(\mathbf{x}) = -\{\lambda_n \psi_n^{(m-1)} t + \lambda_n^{(1)}(\psi_n^{(m-1)} p + \psi_n^{(m-2)} t) + \lambda_n^{(2)}(\psi_n^{(m-2)} p + \psi_n^{(m-3)} t) + \dots + \lambda_n^{(m-1)}(\psi_n^{(1)} p + \psi_n t)\}$;

(ix) $\lambda_n^{(m)} = \int_E \chi_n^{(m)}(\xi)\psi_n(\xi)\, d\xi$;

(x) $\psi_n^{(m)}(\mathbf{x}) = \int_E G_n(\mathbf{x}, \xi, \lambda_n)\chi_n^{(m)}(\xi)\, d\xi - \frac{1}{2}\psi_n \int_E \{\psi_n^{(m-1)}(\psi_n^{(1)} p + \psi_n t) + \psi_n^{(m-2)}(\psi_n^{(2)} p + \psi_n^{(1)} t) + \dots + \psi_n^{(1)}(\psi_n^{(m-1)} p + \psi_n^{(m-2)} t) + \psi_n^{(m-1)}\psi_n t\}\, d\mathbf{x}$.

Then the infinite series in (1.5) *and* (1.6) *are both convergent and represent respectively the perturbed eigenvalue and eigenfunction.*

The coefficients $\{\lambda_n^{(m)}\}$ and $\{\psi_n^{(m)}(\mathbf{x})\}$, $m = 1, 2, \dots$, have the values as formally determined in § 2.

Proof. Let

$$\alpha_m = \left\{ \int_E (\psi_n^{(m)})^2 p\, d\mathbf{x} \right\}^{\frac{1}{2}} \text{ and } \beta_m = \left\{ \int_E (\psi_n^{(m)})^2 |t|\, d\mathbf{x} \right\}^{\frac{1}{2}}.$$

Then by the Schwartz inequality and the orthogonality relations for $\psi_n(\mathbf{x})$ it follows that

$$\left| \int_E \psi_n^{(m)} \psi_n p\, d\mathbf{x} \right| \leqslant \alpha_m. \qquad (3.4)$$

Since

$$\left| \int_E \psi_n^2 t\, d\mathbf{x} \right| \leqslant \int_E \psi_n^2 |t|\, d\mathbf{x} \leqslant \int_E \psi_n^2 p\, d\mathbf{x} = 1,$$

we obtain

$$\left| \int_E \psi_n^{(m)} \psi_n t\, d\mathbf{x} \right| \leqslant \beta_m. \qquad (3.5)$$

Hence by the Schwartz inequality,

$$\left| \int_E \psi_n^{(r)} \psi_n^{(s)} p\, d\mathbf{x} \right| \leqslant \alpha_r \alpha_s \qquad (3.6)$$

and

$$\left| \int_E \psi_n^{(r)} \psi_n^{(s)} t \, d\mathbf{x} \right| \leqslant \beta_r \beta_s. \quad \ldots \quad \ldots \quad \ldots \quad (3.7)$$

The unperturbed Green's function $G(\mathbf{x}, \xi, \lambda)$ satisfies the relation (*see* Titchmarsh 1958, p. 81)

$$(\lambda-\lambda') \int_E G(\mathbf{x}, \mathbf{u}, \lambda) G(\mathbf{u}, \xi, \lambda') p(\mathbf{u}) \, d\mathbf{u} = G(\mathbf{x}, \xi, \lambda) - G(\mathbf{x}, \xi, \lambda').$$

Hence as given by Titchmarsh (1958, p. 58), it follows that

$$\int_E G(\mathbf{x}, \xi, \lambda) \psi_m(\xi) p(\xi) \, d\xi = \frac{\psi_m(\mathbf{x})}{\lambda_m - \lambda}. \quad \ldots \quad \ldots \quad \ldots \quad (3.8)$$

If E is infinite we ensure the uniqueness of the Green's function by imposing the conditions:
$q(\mathbf{x}) \geqslant -Q(r)$, $p(\mathbf{x}) \geqslant S(r)$, where $Q(r)/S(r)$ is non-decreasing and $S(2r) = O\{S(r)\}$, r, being the radius vector to the point $\mathbf{x}$ (*see* Titchmarsh 1958, p. 83).

Put

$$\Phi_n(\mathbf{s}, \lambda) = \int_E G_n(\mathbf{x}, \mathbf{s}, \lambda) f(\mathbf{x}) p(\mathbf{x}) \, d\mathbf{x} \quad \ldots \quad \ldots \quad \ldots \quad (3.9)$$

where f is any function with Fourier coefficient C_n such that $f^2 p$ belongs to L.

Multiply (3.8) by $f(\mathbf{x})\, p(\mathbf{x})$, integrate and then replace G in terms of G_n as given by (3.1). Then it follows that the Fourier coefficients of Φ_n $(\mathbf{s}, \lambda)$ are

$$\frac{C_m}{\lambda_m - \lambda}, \text{ if } m \neq n\text{; and } 0, \text{ if } m = n.$$

Then by Parseval's theorem

$$\int_E |\Phi_n(\mathbf{s}, \lambda)|^2 p(\mathbf{s}) \, d\mathbf{s} = \sum_{m \neq n} \frac{|C_m|^2}{|\lambda_m - \lambda|^2}.$$

Making $\lambda \to \lambda_n$, we obtain by using Fatou's theorem,

$$\int_E |\Phi_n(\mathbf{s}, \lambda_n)|^2 p(\mathbf{s}) \, d\mathbf{s} \leqslant \sum_{m \neq n} \frac{|C_m|^2}{|\lambda_m - \lambda_n|^2} \leqslant \frac{1}{\delta^2} \sum_m C_m^2 = \frac{1}{\delta^2} \int_E f^2 p \, d\mathbf{x} \quad \ldots \quad (3.10)$$

where

$$|\delta| = \min_{m \neq n} |\lambda_m - \lambda_n|.$$

Using (3.1) and (3.8), (3.9) can be written as

$$\Phi_n(\mathbf{s}, \lambda) = \int_E G(\mathbf{x}, \mathbf{s}, \lambda) \left\{ f(\mathbf{x}) p(\mathbf{x}) - \psi_n(\mathbf{x}) p(\mathbf{x}) \int_E f(\mathbf{u}) \psi_n(\mathbf{u}) p(\mathbf{u}) \, d\mathbf{u} \right\} d\mathbf{x}.$$

Hence Φ_n satisfies the equation

$$\Delta \Phi_n(\mathbf{x}, \lambda) + (\lambda p - q) \Phi_n(\mathbf{x}, \lambda) = -fp + \psi_n(\mathbf{x}) p(\mathbf{x}) \int_E f(\mathbf{u}) \psi_n(\mathbf{u}) p(\mathbf{u}) \, d\mathbf{u}. \quad \ldots \quad (3.11)$$

It follows from (3.11), that (3.2) is satisfied if $F(\mathbf{x}) = \Phi_n(\mathbf{x}, \lambda)$ and $\tilde{F}(\mathbf{x}) = f(\mathbf{x})$.

Hence from (3.3) with $F = \Phi_n(\mathbf{x}, \lambda)$ and $\tilde{F} = f(\mathbf{x})$, and the relation (3.10), we have

$$\int_E \Phi_n^2(\mathbf{x}, \lambda_n)t(\mathbf{x})\,d\mathbf{x} \leqslant C' \int_E f^2 p\,d\mathbf{x} \quad \ldots \quad (3.12)$$

where the constant C' is independent of f and Φ_n.

Let $g(\mathbf{x})$ be a function such that $g^2(\mathbf{x})\,p(\mathbf{x})$ belongs to L. Then $g^2(\mathbf{x})\,t(\mathbf{x})$ also belongs to L.

Then

$$\left| \int_E \int_E G_n(\mathbf{x}, \xi, \lambda_n) f(\mathbf{x}) g(\xi) p(\mathbf{x}) t(\xi)\,d\mathbf{x}\,d\xi \right| = \left| \int_E \Phi_n(\xi, \lambda_n) g(\xi) t(\xi)\,d\xi \right|$$
$$\leqslant K' \left\{ \int_E f^2 p\,d\mathbf{x} \int_E g^2 t\,d\mathbf{x} \right\}^{\frac{1}{2}} \quad \ldots \quad (3.13)$$

by the Schwartz inequality and the inequality (3.12).

In particular,

$$\left| \int_E \int_E G_n(\mathbf{x}, \xi, \lambda_n) \psi_n^{(r)}(\mathbf{x}) \psi_n^{(s)}(\xi) p(\mathbf{x}) t(\xi)\,d\mathbf{x}\,d\xi \right| \leqslant K_2 \alpha_r \beta_s. \quad \ldots \quad (3.14)$$

Similarly,

$$\left| \int_E \int_E G_n(\mathbf{x}, \xi, \lambda_n) f(\mathbf{x}) g(\xi) p(\mathbf{x}) p(\xi)\,d\mathbf{x}\,d\xi \right| \leqslant K'' \left\{ \int_E f^2 p\,d\mathbf{x} \int_E g^2 p\,d\mathbf{x} \right\}^{\frac{1}{2}}. \quad \ldots \quad (3.15)$$

In particular,

$$\left| \int_E \int_E G_n(\mathbf{x}, \xi, \lambda_n) \psi_n^{(r)}(\mathbf{x}) \psi_n^{(s)}(\xi) p(\mathbf{x}) p(\xi)\,d\mathbf{x}\,d\xi \right| \leqslant K_1 \alpha_r \alpha_s. \quad \ldots \quad (3.16)$$

Put

$$b_r = \left| \lambda_n^{(r)} \right|, \; r = 1, 2, \ldots, m-1, \quad \text{and} \quad b_m = \left| \lambda_n^{(m)} \right| - |\lambda_n| \beta_{m-1}.$$

Then by (3.4) and (3.5), it follows from the condition (ix) of the theorem that

$$b_m \leqslant b_1(\alpha_{m-1}+\beta_{m-2}) + b_2(\alpha_{m-2}+\beta_{m-3}) + \ldots + b_{m-1}(\alpha_1+\beta_0). \quad \ldots \quad (3.17)$$

Multiplying the condition (x) by $\psi_n^{(m)}\,p$, integrating over E and making use of the inequalities (3.6), (3.7), (3.14) and (3.16), we obtain

$$\alpha_m \leqslant \tfrac{1}{2}(\alpha_{m-1}\alpha_1 + \beta_{m-1}\beta_0 + \alpha_{m-2}\alpha_2 + \beta_{m-2}\beta_1 + \ldots + \alpha_1\alpha_{m-1} + \beta_1\beta_{m-2} + \beta_{m-1}\beta_0)$$
$$+ K|\lambda_n|\beta_{m-1} + K\{b_1(\alpha_{m-1}+\beta_{m-2}) + b_2(\alpha_{m-2}+\beta_{m-3}) + \ldots + b_{m-1}(\alpha_1+\beta_0)\}, \quad \ldots \quad (3.18)$$

where

$$K = \max(K_1, K_2).$$

Now,

$$(\alpha_{m-1}+\beta_{m-1})(\alpha_1+\beta_0) = \alpha_{m-1}\alpha_1 + \beta_{m-1}\beta_0 + \alpha_1\beta_{m-1} + \beta_0\alpha_{m-1}$$
$$\geqslant \alpha_{m-1}\alpha_1 + \beta_{m-1}\beta_0\,; \text{ etc.}$$

Hence (3.18) reduces to

$$\alpha_m \leqslant \tfrac{1}{2}\{(\alpha_1+\beta_0)(\alpha_{m-1}+\beta_{m-1})+(\alpha_2+\beta_1)(\alpha_{m-2}+\beta_{m-2})+\ldots+(\alpha_{m-1}+\beta_{m-2})(\alpha_1+\beta_1)\}$$
$$+\tfrac{1}{2}\beta_{m-1}\beta_0+K|\lambda_n|\beta_{m-1}+K\{b_1(\alpha_{m-1}+\beta_{m-2})+b_2(\alpha_{m-2}+\beta_{m-3})$$
$$+\ldots+b_{m-1}(\alpha_1+\beta_0)\}. \quad \ldots (3.19)$$

Let

$$\alpha'_m = \alpha_m,\ \beta'_m = \beta_m,\ \mathscr{A}_m = \alpha'_m+\beta'_m,\ \mathscr{C}_m = \alpha'_m+\beta'_{m-1},\ \text{and } Kb_m = \tfrac{1}{2}b'_m = \tfrac{1}{2}\mathscr{D}_m,$$
$$m = 1, 2, \ldots.$$

Also, let

$$\tfrac{1}{2}\beta'_0 = \tfrac{1}{2}\alpha_1+\beta_0+K|\lambda_n|.$$

Then since

$$\tfrac{1}{2}(\alpha_1+\beta_0)(\alpha_{m-1}+\beta_{m-1})+\tfrac{1}{2}\beta_{m-1}\beta_0+K|\lambda_n|\beta_{m-1}$$
$$\leqslant \tfrac{1}{2}(\alpha_1+2\beta_0)\alpha_{m-1}+\tfrac{1}{2}\beta_{m-1}(\alpha_1+\beta'_0)$$
$$\leqslant \tfrac{1}{2}(\alpha'_1+\beta'_0)(\alpha'_{m-1}+\beta'_{m-1}),$$

the inequalities (3.19) and (3.17) can be written in the forms

$$\alpha'_m \leqslant \tfrac{1}{2}(\mathscr{A}_{m-1}\mathscr{C}_1+\mathscr{A}_{m-2}\mathscr{C}_2+\mathscr{A}_{m-3}\mathscr{C}_3+\ldots+\mathscr{A}_1\mathscr{C}_{m-1})$$
$$+\tfrac{1}{2}(\mathscr{C}_{m-1}\mathscr{D}_1+\mathscr{C}_{m-2}\mathscr{D}_2+\ldots+\mathscr{C}_1\mathscr{D}_{m-1}) \quad \ldots (3.20)$$

and

$$b'_m \leqslant \mathscr{D}_1\mathscr{C}_{m-1}+\mathscr{D}_2\mathscr{C}_{m-2}+\ldots+\mathscr{D}_{m-1}\mathscr{C}_1. \quad \ldots (3.21)$$

Hence substituting $A_m = \max(\mathscr{A}_m, \mathscr{C}_m)$ and $B_m = \mathscr{D}_m$, the relations (3.20) and (3.21) reduce to the forms

$$\alpha'_m \leqslant \tfrac{1}{2}(A_{m-1}A_1+A_{m-2}A_2+\ldots+A_{m-1}A_1)+\tfrac{1}{2}(B_1A_{m-1}+B_2A_{m-2}+\ldots+B_{m-1}A_1)$$

and

$$b'_m \leqslant B_1A_{m-1}+B_2A_{m-2}+\ldots+B_{m-1}A_1.$$

Hence by following the analysis of Titchmarsh (1958, pp. 226-27), we can show that the series for Λ_n given by (1.5) is convergent for $|\epsilon| < (4M)^{-1}$ where

$$M = \max(A_1, B_1) = \max(\alpha'_1+\beta'_1,\ \alpha'_1+\beta'_0,\ b'_1).$$

We next establish the convergence of the series for $\tilde{\psi}_n(\mathbf{x})$ given by (1.6). Following Titchmarsh (1958, p. 227), we have

$$[\Delta+(\lambda_n p-q)]\psi_n^{(1)}(\mathbf{x}) = -(\lambda_n t+\lambda_n^{(1)}p)\psi_n \quad \ldots (3.21)$$

and for $m > 1$,

$$[\Delta+(\lambda_n p-q)]\psi_n^{(m)}(\mathbf{x}) = \chi_n^{(m)}(\mathbf{x})-\lambda_n^{(m)}\psi_n(\mathbf{x})p(\mathbf{x}). \quad \ldots (3.22)$$

Also

$$\Phi(\mathbf{u}) = \frac{1}{\pi R^2}\int_{r\leqslant R}\Phi\, d\mathbf{x}+\frac{1}{2\pi}\int_{r\leqslant R}\left\{\tfrac{1}{2}\left(1-\frac{r^2}{R^2}\right)-\log\frac{R}{r}\right\}\{f+(q-\lambda p)\Phi\}\, d\mathbf{x}$$
$$\ldots (3.23)$$

where $\Phi(\mathrm{x})$ have continuous partial derivatives up to the second order and f, p, q are continuous, satisfies the differential equation (Titchmarsh 1958, p. 330)

$$[\Delta+\lambda p-q]\Phi=f. \quad \ldots \quad (3.24)$$

By the Schwartz inequality

$$\left|\int_{r\leqslant R}\psi_n^{(m)}(\mathrm{x})\,d\mathrm{x}\right|\leqslant\left|\int_{r\leqslant R}\{\psi_n^{(m)}(\mathrm{x})\}^2p(\mathrm{x})\,d\mathrm{x}\right|^{\frac{1}{2}}\left|\int_{r\leqslant R}\{p(\mathrm{x})\}^{-1}\,d\mathrm{x}\right|^{\frac{1}{2}}.$$

Hence, if R is sufficiently small,

$$\int_{r\leqslant R}\psi_n^{(m)}(\mathrm{x})\,d\mathrm{x}=O(\alpha_m),$$

α_m being defined as before.

Again, by Minkowski's inequality, it follows by the condition (viii) of the theorem, that

$$\begin{aligned}\Omega_{m,n}\equiv\left|\int_{r\leqslant R}\{\chi_n^{(m)}(\mathrm{x})\}^2\,d\mathrm{x}\right|^{\frac{1}{2}}&\leqslant|\lambda_n|\left|\int_{r\leqslant R}\{\psi_n^{(m-1)}t\}^2\,d\mathrm{x}\right|^{\frac{1}{2}}\\&+|\lambda_n^{(1)}|\left\{\left|\int_{r\leqslant R}(\psi_n^{(m-1)}p)^2\,d\mathrm{x}\right|^{\frac{1}{2}}+\left|\int_{r\leqslant R}(\psi_n^{(m-2)}t)^2\,d\mathrm{x}\right|^{\frac{1}{2}}\right\}+\ldots\\&+|\lambda_n^{(m-1)}|\left\{\left|\int_{r\leqslant R}(\psi_n^{(1)}p)^2\,d\mathrm{x}\right|^{\frac{1}{2}}+\left|\int_{r\leqslant R}(\psi_nt)^2\,d\mathrm{x}\right|^{\frac{1}{2}}\right\}\\&\leqslant|\lambda_n|\beta_{m-1}+|\lambda_n^{(1)}|(\alpha_{m-1}+\beta_{m-2})+\ldots+|\lambda_n^{(m-1)}|(\alpha_1+\beta_0)\end{aligned} \quad \ldots \quad (3.25)$$

$p(\mathrm{x})$ and so $t(\mathrm{x})$ being bounded in $r\leqslant R$.

Hence taking $\Phi(\mathrm{x})=\psi_n^{(m)}(\mathrm{x})$ in (3.23), we can argue as in Titchmarsh (1958, 227-28) to establish the convergence of the series for $\tilde{\psi}_n$ (x) defined in (1.6). The analysis applies to the one-dimensional case also; the formula corresponding to (3.23) is now

$$(\xi-x)^2\Phi(x)=\int_x^{\xi}(6y-2x-4\xi)\Phi(y)\,dy-\int_x^{\xi}(\xi-y)^2(y-x)\{f+(q-\lambda p)\Phi\}\,dy$$

a formula due to Titchmarsh (1946, p. 34) and used extensively in a paper by the author (1968).

To verify that $\tilde{\psi}_n(\mathrm{x})$ and Λ_n represent respectively the perturbed eigenfunction and eigenvalue, we observe that

$$(\Delta+\lambda_np-q)\psi_n=0.$$

Hence operating on (1.6) by the operator $\Delta+\lambda_np-q$ and making use of the relations (3.21) and (3.22), we obtain, on slight reduction, by using (1.5), that

$$(\Delta+\lambda_np-q)\tilde{\psi}_n=-(\Lambda_n-\lambda_n)p\tilde{\psi}_n-\epsilon\Lambda_nt\tilde{\psi}_n.$$

Therefore, $\tilde{\psi}_n$ satisfies (1.2) with $\lambda=\Lambda_n$.

In one dimension the infinite series expansion of $G_n(\mathbf{x}, \xi, \lambda_n)$ as derived from (3.1) converges absolutely and uniformly but in two dimensions this is not necessarily the case. However, in two dimensions the weaker relation

$$\lim_{m\to\infty}\int_E\left\{G_n(\mathbf{x}, \xi, \lambda_n)-\sum_{r=0,\,(r\neq n)}^{m}\frac{\psi_r(\mathbf{x})\psi_r(\xi)}{\lambda_n-\lambda_r}\right\}^2 d\mathbf{x}=0$$

holds (Courant and Hilbert 1953).
Therefore,

$$\int_E G_n(\mathbf{x}, \xi, \lambda_n)\psi_n(\mathbf{x})p(\mathbf{x})\,d\mathbf{x}=0. \quad \ldots \quad (3.26)$$

Multiplying the condition (x) of the theorem by $\psi_n(\mathbf{x})\,p(\mathbf{x})$ and then integrating over E, it follows by (3.26) that

$$\int_E \psi_n^{(m)}(\mathbf{x})\psi_n(\mathbf{x})p(\mathbf{x})\,d\mathbf{x}$$

$$=-\tfrac{1}{2}\int_E\{\psi_n^{(m-1)}(\psi_n^{(1)}p+\psi_n t)+\ldots+\psi_n^{(1)}(\psi_n^{(m-1)}p+\psi_n^{(m-2)}t)+\psi_n^{(m-1)}\psi_n t\}\,d\mathbf{x}. \quad \ldots \quad (3.27)$$

Now

$$\int_E \tilde{\psi}_n^2(\mathbf{x})P(\mathbf{x})\,d\mathbf{x}$$

$$=\int_E(\psi_n+\epsilon\psi_n^{(1)}+\epsilon^2\psi_n^{(2)}+\ldots)^2(p+\epsilon t)\,d\mathbf{x}.$$

The term independent of ϵ on the right-hand side of the above is equal to

$$\int_E \psi_n^2 p\,d\mathbf{x}=1.$$

Also the coefficient of ϵ^m, $m=1, 2, \ldots$, is equal to

$$2\int_E \psi_n^{(m)}\psi_n p\,d\mathbf{x}$$

$$+\int_E\{\psi_n^{(m-1)}(\psi_n^{(1)}p+\psi_n t)+\ldots+\psi_n^{(1)}(\psi_n^{(m-1)}p+\psi_n^{(m-2)}t)+\psi_n^{(m-1)}\psi_n t\}\,d\mathbf{x},$$

which is zero by (3.27).

Hence the set $\{\tilde{\psi}_n(\mathbf{x})\}$ is orthonormal.

Thus $\tilde{\psi}_n(\mathbf{x})$ is the perturbed eigenfunction, the corresponding eigenvalue being Λ_n.

It remains now to show that the coefficients $\{\lambda_n^{(m)}\}$ and $\{\psi_n^{(m)}\}$ have the values as determined formally in § 2. We have

$$\lambda_n^{(1)}=-\lambda_n\int_E\psi_n^2 t\,d\mathbf{x}=-\lambda_n b_{n,n}$$

$$\psi_n^{(1)}(\mathbf{x}) = -\lambda_n \int_E G_n(\mathbf{x}, \xi, \lambda_n)\psi_n(\xi)t(\xi)\, d\xi - \tfrac{1}{2}\psi_n(\mathbf{x}) \int_E \psi_n^2(\xi)t(\xi)\, d\xi$$

$$= -\tfrac{1}{2}b_{n,\,n}\,\psi_n + \lambda_n \sum_{r \neq n} \frac{\psi_r(\mathbf{x})}{\lambda_n - \lambda_r} \int_E \psi_r(\xi)\psi_n(\xi)t(\xi)\, d\xi$$

$$= -\tfrac{1}{2}b_{n,\,n}\,\psi_n + \lambda_n \sum_{r \neq n} \frac{b_{n,\,r}\psi_r(\mathbf{x})}{\lambda_n - \lambda_r}$$

$$\lambda_n^{(2)} = \int_E \chi_n^{(2)}(\xi)\psi_n(\xi)\, d\xi$$

$$= -\int_E \{\lambda_n \psi_n^{(1)} t + \lambda_n^{(1)}(\psi_n^{(1)} p + \psi_n t)\}\psi_n(\xi)\, d\xi$$

$$= \lambda_n \left(b_{n,\,n}^2 - \lambda_n \sum_{r \neq n} \frac{b_{n,\,r}^2}{\lambda_r - \lambda_n} \right)$$

and so for the higher coefficients. The theorem, therefore, is completely established. A similar argument is used to prove the result in three dimensions and for higher dimensions an iterated formula is required.

It may be remarked in connection with the theorem established before that conditions (iii) and (iv) are only necessary to establish the inequalities (3.14) and (3.16) and hence can be dispensed with if these inequalities are found to be satisfied for a particular problem.

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