ON TRANSIENT DEVELOPMENT OF AXISYMMETRIC WAVES

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An initial value investigation into the axisymmetric wave motions in an infinitely deep fluid with a free surface produced by harmonically oscillating disturbances acting on the free surface of the fluid is made in this paper. An asymptotic analysis of the problem has been carried out in considerable detail for a deeper understanding of the steady state and transient wave motions. The limiting behaviour of the asymptotic solution as time T tends to infinity has been given due attention. A careful consideration has been given to examine the validity of Lighthill's prediction regarding the steady state solution. Finally, the steady state solution is obtained as a limiting case without any fictitious damping force.

1. INTRODUCTION

It is well known that the solution of the steady state (or stationary) wave problem in which the wave motions are simple harmonic in time is, in general, not unique on mathematical grounds. But in any physical situation only one solution is of interest. In order to remove this mathematical difficulty in deriving a solution of physical interest, several methods have been employed in the treatment of this steady state problem, all of which are aimed at deriving a solution of physical interest by introducing some artificial device. A brief review stating some important features of these methods is made below to understand them clearly and to indicate our motivation.

Firstly, mention may be made of a fictitious damping force (Lamb 1905, 1932) which is introduced to avoid indeterminateness and to obtain a solution of the stationary wave problem appropriate to physical situation. Although this method works, it is probably pertinent to ask a physical justification of the artificial force imposed on the fluid system. A satisfactory answer to this question has not yet been obtained.

Secondly, in order to find out a solution of physical interest, the problem is, in general, investigated by imposing Sommerfeld's radiation condition at infinity. The radiation condition is, indeed, on mathematical

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and physical reasons, essential to achieve a unique solution of interest of the steady state problem.

As an alternative to a fictitious damping force or to the radiation condition, Lighthill (1960) suggested an alternative approach of applying the radiation condition in solving the general wave problems. This method is also useful in finding out a solution of physical interest of the problems.

As a matter of fact, the radiation condition or Lighthill's alternative approach of applying the radiation condition or the fictitious damping force is indispensable for deriving a solution of physical interest of the steady state problem. All these methods lead, in general, to the same solution.

These formulations, all of which seek a steady state solution, are open to criticism for various reasons. In the first place, a hypothesis that the motion holds for all times is, of course, an unnatural feature of 'dynamics'. The natural method of generating waves involves transient motions which die out after sufficiently long times, and none of the above methods consider these transient motions at all. Mathematically, a unique solution is not guaranteed unless one considers an initial value formulation. Finally, in some more complicated situations, it is not always immediately clear how the radiation condition should be applied.

From the above considerations, it is probably desirable to pay special attention to the general problem of wave propagation in a fluid as an initial value problem, which appears to be mathematically more sound and physically more realistic in 'mechanics of continuous media'. The first and foremost reason in favour of this new approach is that this is a rigorous way of deriving the unique solution of physical interest of wave problems, without the need for any of the essentially physical assumptions of the three methods described above. Secondly, it gives the behaviour of wave formation, in principle, at all times, instead of only in the limit. Thirdly, the steady state solution, if it at all exists, can be derived under certain circumstances as a limiting case, and this limiting solution satisfies the radiation condition at infinity.

Some recent progress, probably not adequate, has been made in the development of the linearized theory of transient wave motions in a fluid originated by disturbances on the free surface of the fluid. An account of recent works on the initial value problem by Stoker (1957), Miles (1962) and Sen (1962) can be given in a nutshell to describe the important features of their investigations and indicate our motivation.

Stoker (1957) formulated an initial value problem of wave propagation due to an oscillating pressure distribution concentrated in a single point of the undisturbed free surface of a fluid of unlimited depth. With due investigation of the problem, he arrived at the conclusion that the solution does tend to that of the corresponding steady state problem considered by Lamb (1905) as time T tends to infinity for fixed values of X. And the limiting solution also satisfies the appropriate radiation condition at infinity.

Miles (1962) has pointed out that Stoker's formulation suffers from erroneous initial conditions and has reinvestigated the same problem by the method of stationary phase. Finally, he has arrived at the same conclusion as Stoker had. Unfortunately, a careful scrutiny reveals that, although his asymptotic solution is free from any doubt, his conclusion is wrong.

Sen (1962) has formulated an unsteady wave problem by introducing a fictitious damping force similar to Lamb's and investigated it by a complicated mathematical treatment. Finally, he arrived at the conclusion that the transient component of the asymptotic solution always decays to zero as time T tends to infinity and, consequently, the steady state is set up in the limit. Unfortunately, his formulation like Stoker's suffers from erroneous initial conditions. And he did not suggest any explanation in favour of the fictitious frictional force imposed in the initial value formulation.

In this connection, reference may be made to an excellent account of work on general wave problems considered by Lighthill (1964). He suggested in his work that 'all different ways of switching on a circuit produce identical steady states' but he did not adduce any proof in support of his prediction.

To examine the validity of Miles' conclusion and Lighthill's prediction, Debnath (1967, 1968, *in press*) has proposed a linearized theory of transient development of two-dimensional surface waves in a fluid of finite, infinite and shallow depth produced by harmonically oscillating pressure distributions and sources. With a rigorous investigation by the generalized function treatment combined with the method of steepest descent and stationary phase, our analysis reveals a striking difference between the wave motions generated by oscillating pressure distributions confined over a finite region and at a single point on the free surface of the fluid. This analysis further suggests that Lighthill's prediction regarding the steady state is not always true—its validity can be restricted by it.

Motivated by the above consideration and immense applicability of the initial value approach to the wave problems on its own right, we propose to develop, in this paper, a linearized theory of transient development of axisymmetric waves in an inviscid, incompressible and homogeneous fluid of unlimited depth created by harmonically oscillating pressure distributions on the free surface of the fluid. An asymptotic analysis of the problem related to certain pressure distributions of considerable physical interest is made in some detail. A careful consideration has been given to examine the validity of Lighthill's prediction. The limiting behaviour of the asymptotic solution as time T tends to infinity has also been given attention.

Before we embark on the actual investigation of the problem, we would like to mention that we formulate, unlike Sen's, the axisymmetric wave problem with correct initial conditions. And the fictitious frictional force introduced by Sen is not necessary in deriving the steady state solution in the limit.

2. Mathematical Formulation of the Problem

We propose to investigate the propagation of axisymmetric transient surface waves in an inviscid, incompressible and homogeneous fluid of unlimited depth produced by a harmonically oscillating pressure distribution in the form

$$\left.\begin{aligned} P(R, T) &= P(R)\, e^{i\omega T} H(T), \quad R \leqslant A \\ &= 0 \quad\quad\quad\quad\quad\quad\;\;,\quad R > A \end{aligned}\right\} \qquad \text{(2.1)}$$

acting on the free surface $Y = 0$ of the fluid (initially at rest), where ω is the fixed frequency, T the time, $P(R)$ is an arbitrary function of R and $H(T)$ the Heaviside step function.

We assume that X-Z plane be the undisturbed horizontal free surface and Y-axis be vertical positive upward. We take cylindrical polar coordinate (R, Θ, Y) with cylindrical symmetry about the Y-axis such that the quantity R is equal to the distance $\sqrt{X^2+Z^2}$ from the Y-axis.

As the motion is irrotational, there exists a harmonic wave potential $\Phi(R, Y; T)$ which satisfies the Laplace equation (Sommerfeld 1950)

$$\nabla^2\Phi \equiv \frac{\partial^2\Phi}{\partial R^2}+\frac{1}{R}\frac{\partial\Phi}{\partial R}+\frac{\partial^2\Phi}{\partial Y^2} = 0 \qquad \text{(2.2)}$$
$$-\infty < Y \leqslant 0, \quad 0 \leqslant R < \infty.$$

The linearized boundary conditions at the free surface are given by

$$\Phi_T + gE = -\frac{1}{\rho}P(R, T), \quad Y = 0, \quad T > 0 \qquad \text{(2.3)}$$

$$\Phi_Y = E_T, \quad Y = 0, \quad T > 0 \qquad \text{(2.4)}$$

where suffixes denote partial differentiation and $E(R, T)$ represents the vertical surface elevation of the free surface, ρ the density and g the gravitational acceleration.

The condition at infinity is given by

$$\Phi_Y \to 0 \quad \text{as } Y \to -\infty. \qquad \text{(2.5)}$$

The initial conditions are given by

$$\left.\begin{aligned} \Phi(R, Y; 0) &= E(R, 0) = 0 \\ \left[\frac{\partial}{\partial T}\Phi(R, 0; T)\right]_{T=0} &= -\frac{1}{\rho}P(R, 0) \end{aligned}\right\}. \qquad \text{(2.6)}$$

We further assume that the functions $\Phi(R, Y; T)$ and $E(R, T)$ possess the Hankel transform with respect to R in the generalized sense.

We thus complete an initial value formulation of the problem for the determination of the wave potential $\Phi(R, Y; T)$ and the surface elevation $E(R, T)$.

Remarks: (1) A remark on the corresponding axisymmetric wave problem as a steady state may be inserted here. As in the two-dimensional problem, the axisymmetric problem under consideration may also be investigated as a steady state, just by omitting the initial conditions (2.6) from the above formulation. Here, again, the radiation condition at infinity or the fictitious damping force is indispensable for deriving a solution of physical interest.

(2) Sen (1962) has considered a similar problem in an infinitely deep water with the initial conditions

$$\Phi(R, 0; 0) = \Phi_T(R, 0; 0) = 0$$

which seems to be erroneous because it means that $E(R, 0) = 0$ implies

$$\Phi_T(R, 0; 0+) = 0$$

only if $P(R, 0+) = 0$.

Our formulation, stated above, is free from the error.

(3) On account of various reasons suggested in the introduction, we prefer the initial value approach to the axisymmetric wave problem without introducing any fictitious damping force as suggested by Sen in his analysis. This makes the present formulation much more convincing.

3. Solution of the Problem

We first introduce, for convenience, non-dimensional variables r, x, y, z, a, t, η and ϕ defined by the relations

$$(r, x, y, z, a) = \frac{\omega^2}{g}(R, X, Y, Z, A), \quad t = \omega T$$

$$\eta = \frac{P\omega^4}{g^3\rho}E, \quad \phi = \frac{P\omega^5}{\rho g^4}\Phi.$$

With the aid of these relations, the basic equations can be rewritten in the following form:

$$\phi_{rr}+\frac{1}{r}\phi_r+\phi_{yy} = 0 \quad \ldots \quad (3.1)$$

$$-\infty < y \leqslant 0, \quad 0 \leqslant r < \infty$$

$$\phi_t+\eta = -p(r, t), \quad y = 0, \quad t > 0 \quad \ldots \quad (3.2)$$

$$\phi_y = \eta_t, \quad y = 0, \quad t > 0 \quad \ldots \quad (3.3)$$

$$\phi_y \to 0 \quad \text{as } y \to -\infty \quad \ldots \quad (3.4)$$

$$\left.\begin{aligned} &\phi = \eta = 0, \quad t = 0 \\ &\phi_t = -p(r, t), \quad t = 0, \quad y = 0 \end{aligned}\right\} \quad \ldots \quad (3.5)$$

In order to obtain a solution for the function ϕ as well as η, we introduce the Laplace transform $\bar{\phi}$, $\bar{\eta}$ (Churchill 1944; Doetsch 1961) of functions ϕ, η respectively with respect to t defined by integral in the form

$$\bar{\phi} = \bar{\phi}(r, y; s) = \int_0^\infty e^{-st}\, \phi(r, y; t)\, dt.$$

We next introduce the Hankel transform $\bar{\bar{\phi}}$, $\bar{\bar{\eta}}$ (Lighthill 1958; Sneddon 1951) of the functions $\bar{\phi}$, $\bar{\eta}$ respectively with respect to the variable r defined by integral like

$$\bar{\bar{\phi}} = \bar{\bar{\phi}}(k, y; s) = \int_0^\infty r\, J_0(kr)\, \bar{\phi}(r, y; s)\, dr,$$

where $J_0(kr)$ represents the Bessel function of the first kind of order zero. A similar Laplace and Hankel transforms are needed for the function $\eta(r, t)$.

Making use of the Laplace and Hankel transforms with respect to t and r respectively and assuming that the joint Laplace-Hankel transforms $\bar{\bar{p}}(k, s)$ of $p(r, t)$ exist, eqns. (3.1)–(3.5) can be transformed into the form

$$\bar{\bar{\phi}}_{yy} = k^2\, \bar{\bar{\phi}} \qquad \text{(3.6)}$$

$$s\bar{\bar{\phi}} + \bar{\bar{\eta}} = -\bar{\bar{p}}, \quad y = 0, \quad s > 0 \qquad \text{(3.7)}$$

$$\bar{\bar{\phi}}_y = s\,\bar{\bar{\eta}}, \quad y = 0, \quad s > 0 \qquad \text{(3.8)}$$

$$\bar{\bar{\phi}}_y \to 0 \quad \text{as } y \to -\infty \qquad \text{(3.9)}$$

$$\left.\begin{aligned} \bar{\bar{\phi}} = \bar{\bar{\eta}} &= 0, \qquad s = 0 \\ s\,\bar{\bar{\phi}} &= -\bar{\bar{p}}(s, k), \quad s = 0, \quad y = 0 \end{aligned}\right\}. \qquad \text{(3.10)}$$

The solution of eqn. (3.6) with the boundary condition (3.9) can be put into the form

$$\bar{\bar{\phi}}(k, y; s) = \bar{\bar{A}}(k, s)\, e^{ky}, \qquad \text{(3.11)}$$

where $\bar{\bar{A}}(k, s)$ is an arbitrary function of k, s to be determined soon.

Substituting the value for $\bar{\bar{\phi}}$ into eqns. (3.7)–(3.8) and eliminating $\bar{\bar{\eta}}$, we can determine $\bar{\bar{A}}(k, s)$. Having done this, we obtain the expression for $\bar{\bar{\phi}}$ as well as $\bar{\bar{\eta}}$ into the form

$$\bar{\bar{\phi}}(k, y; s) = -\frac{s\,\bar{\bar{p}}(k, s)\, e^{ky}}{(s^2 + k)}, \qquad \text{(3.12)}$$

$$\bar{\bar{\eta}}(k, s) = -\frac{k\,\bar{\bar{p}}(k, s)}{(s^2 + k)}. \qquad \text{(3.13)}$$

Making reference to the inversion theorem for the Laplace transform, we obtain

$$\bar{\phi}(k, y; t) = -e^{ky} \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{st}\, \frac{s\,\bar{\bar{p}}(k, s)}{(s^2 + k)}\, ds, \qquad \text{(3.14)}$$

$$\bar{\eta}(k, t) = -\frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{st}\, \frac{k\,\bar{\bar{p}}(k, s)}{(s^2 + k)}\, ds, \qquad \text{(3.15)}$$

where $\alpha > 0$.

Now the use of the Faltung (or convolution) theorem for the Laplace transformation gives us

$$\bar{\phi}(k, y; t) = -e^{ky}\int_0^t \bar{p}(k, t-\xi) \cos\sqrt{k}\xi \, d\xi, \qquad \text{.. .. (3.16)}$$

$$\bar{\eta}(k, t) = -\sqrt{k}\int_0^t \bar{p}(k, t-\xi) \sin\sqrt{k}\xi \, d\xi. \qquad \text{.. .. (3.17)}$$

It may be observed that in certain cases of physical interest the function $\bar{p}(k, t-\xi)$ is usually simple in nature and the above integrals (3.16) and (3.17) can, therefore, be easily worked out in exact form. Then the inversion theorem of the Hankel transform enables us to obtain the form of $\phi(r, y; t)$ and $\eta(r, t)$ as

$$\phi(r, y; t) = -\int_0^\infty k e^{ky} J_0(kr)\, dk \int_0^t \bar{p}(k, t-\xi) \cos\sqrt{k}\xi \, d\xi, \qquad \text{.. (3.18)}$$

$$\eta(r, t) = -\int_0^\infty k^{\frac{3}{2}} J_0(kr)\, dk \int_0^t \bar{p}(k, t-\xi) \sin\sqrt{k}\xi \, d\xi. \qquad \text{.. (3.19)}$$

These integrals (3.18) and (3.19) cannot, in general, be evaluated exactly. Hence one has to use asymptotic methods with care to obtain the asymptotic solution of physical interest.

4. Some Pressure Distributions of Physical Interest

The non-dimensional form of certain harmonically oscillating pressure distributions are stated below for due consideration by asymptotic methods. They are of considerable physical interest and given by

(4a) .. .. $p(r, t) = e^{it}\dfrac{\delta(r)}{r}$,

where $\delta(r)$ is the Dirac function of distribution.
And we take

$$\begin{aligned} p(r, t) &= e^{it}p(r), \quad r \leqslant a \\ &= 0 \quad\quad\;\; , \quad r > a \end{aligned}$$

with the prescribed form of $p(r)$ as

(4b) .. .. $p(r) = 1$,

(4c) .. .. $p(r) = e^{-\left(r^2/r_0^2\right)}$,

(4d) .. .. $p(r) = J_0(\lambda r)$.

5. Asymptotic Analysis of the Solution

In case 4(a), we have

$$p(r, t) = e^{it}\frac{\delta(r)}{r}.$$

Hence the function $\bar{p}(k, t-\xi)$ is given by

$$\bar{p}(k, t-\xi) = e^{i(t-\xi)}.$$

The integral representation (3.19) for the surface elevation $\eta(r, t)$ assumes the following form

$$\eta(r, t) = \int_0^\infty k^{\frac{3}{2}}(i \sin \sqrt{k}\, t + \sqrt{k} \cos \sqrt{k}\, t - \sqrt{k}\, e^{it})(k-1)^{-1} J_0(kr)\, dk. \quad \text{.. (5.1)}$$

A similar expression for the wave potential $\phi(r, y; t)$ can be obtained without any difficulty.

It may be noticed that the integral (5.1) has no singularities on the real axis $(0, \infty)$, hence the path of integration can be deformed into a path L (say) in the complex $s = k+i\mu$ plane, which coincides with the real axis $(0, \infty)$ except that it is diverted round the zeros of the denominator of the integrand. We can then break up the integral into a sum of two components where the integrals do become singular at the zeros of the denominator.

We thus obtain

$$\eta(r, t) = \int_L s^{\frac{3}{2}}(i \sin \sqrt{s}\, t + \sqrt{s} \cos \sqrt{s}\, t - \sqrt{s}\, e^{it})(s-1)^{-1} J_0(sr)\, ds, \quad (5.2)$$

where the path L is indicated in Fig. 1.

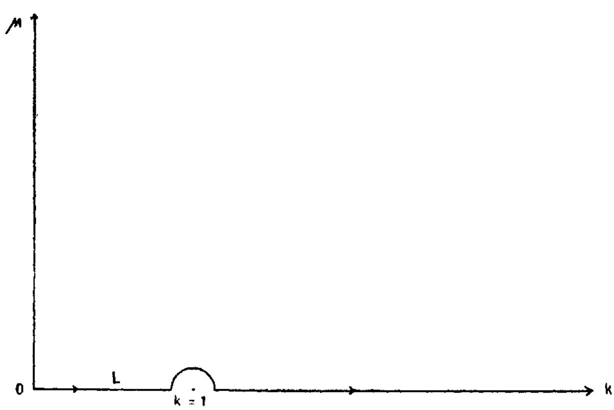

FIG. 1. The $s = k+i\mu$ plane.

We next write $\eta(r, t)$ in the form

$$\eta(r, t) = (I_1 - I_2),$$

where I_1 and I_2 are given by the integrals in the form

$$I_1 = \int_L s^{\frac{3}{2}}(i \sin \sqrt{s}\, t + \sqrt{s} \cos \sqrt{s}\, t) J_0(kr)(s-1)^{-1}\, ds$$

$$I_2 = e^{it} \int_L s^2 (s-1)^{-1} J_0(kr)\, ds$$

where $-\pi < \arg s \leqslant \pi$.

Evaluation of the Wave Integrals:

To evaluate the steady state wave integral I_2, we replace $J_0(sr)$ (Whittaker and Watson 1920) by a pair of Hankel functions $H_0^{(1)}(sr)$ and $H_0^{(2)}(sr)$. This gives us

$$I_2 = \frac{e^{it}}{2}(I_2' + I_2''),$$

where I_2' and I_2'' are given by the integrals

$$I_2' = \int_L s^2 H_0^{(1)}(sr)(s-1)^{-1}\, ds,$$

$$I_2'' = \int_L s^2 H_0^{(2)}(sr)(s-1)^{-1}\, ds.$$

For the integrals I_2' and I_2'', we take contours Γ_1 in the first quadrant and Γ_2 in the fourth quadrant respectively. They are bounded by the path L, μ-axis and the circular arcs C_1 and C_2 lying in the first and fourth quadrants respectively. Making reference to Cauchy's theorem of residues, we find that the integrals along the μ-axis are $O\left(\frac{1}{r}\right)$, $r > 1$, by partial integration. For evaluating the integrals along the arcs C_1 and C_2, we replace the Hankel functions by their asymptotic value for large sr and can show easily that these integrals tend to zero as the radii of the circular arcs tend to infinity.

Finally, it turns out that

$$I_2 = -i\left(\frac{2\pi}{r}\right)^{\frac{1}{2}} e^{i\left(t-r+\frac{\pi}{4}\right)} + O\left(\frac{1}{r}\right). \qquad \text{(5.3)}$$

In order to perform integration of the transient integral I_1, we first replace the Bessel function $J_0(sr)$ by its integral representation

$$J_0(sr) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos\,(sr\cos\theta)\, d\theta.$$

This gives us

$$I_1 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} I_1'\, d\theta, \qquad \text{(5.3a)}$$

where I_1' is given by the integral in the form

$$I_1' = \int_L s^{\frac{1}{2}}(i\sin\sqrt{s}\,t + \sqrt{s}\cos\sqrt{s}\,t)(s-1)^{-1}\cos\,(sr\cos\theta)\, ds.$$

We now introduce a new variable p defined by $p^2 = s$, $R(p) > 0$ and $p = u+iv$ and transform the s-integral to p-integral (Fig. 2).

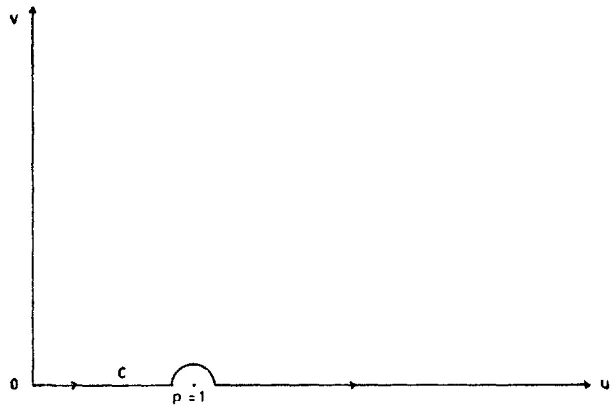

FIG. 2. The $p = u+iv$ plane.

A simple rearrangement of the p-integral gives us the following:

$$I_1' = \tfrac{1}{2}(L_1+L_2+L_3+L_4),$$

where L_1, L_2, L_3 and L_4 are given by the integrals

$$L_1 = \int_C g_+(p)\, e^{-itf_-(p)}\, dp,$$

$$L_2 = \int_C g_-(p)\, e^{itf_-(p)}\, dp,$$

$$L_3 = \int_C g_-(p)\, e^{-itf_+(p)}\, dp,$$

$$L_4 = \int_C g_+(p)\, e^{-itf_+(p)}\, dp,$$

where $g_\pm(p) = \dfrac{p^4}{(p\pm 1)}$ and $f_\pm(p) = \left(p \pm \dfrac{p^2 r}{t} \cos\theta\right)$.

We first consider the integral L_1. This integral has no poles but it has a stationary point in the range of integration $(0, \infty)$. Therefore, it can be integrated along the real axis. We thus find

$$L_1 = \int_0^\infty g_+(u)\, e^{-itf_-(u)}\, du.$$

The stationary point is given by $u = u_1 = \dfrac{t}{2r} \sec\theta$ so that $f_-'(u) = 0$.

With the aid of the stationary phase method (Copson 1965; Jeffreys and Jeffreys 1946) for large values of t, the integral L_1 can be approximated as

$$L_1 \sim g_+(u_1)\left\{\frac{2\pi}{t\left|f''_-(u_1)\right|}\right\}^{\frac{1}{2}} e^{-itf_-(u_1)+i\frac{\pi}{4}}$$

$$= \frac{u_1^4}{(u_1+1)}\left\{\frac{\pi}{r\cos\theta}\right\}^{\frac{1}{2}} e^{i\left(\frac{\pi}{4}-\frac{t^2}{4r}\sec\theta\right)}.$$

In order to evaluate the integral L_2, it may be noticed that it has both a pole at $p = 1$ and a saddle point $p = p_1$ so that p_1 is the root of the equation (Copson 1965; Courant and Hilbert 1953, 1962)

$$f'_-(p) = 0.$$

Then we can write L_2 in the form

$$L_2 = \int_C g_-(p)\, e^{tf(p)}\, dp,$$

where the function $f(p)$ is given by $f(p) = if_-(p)$.
The saddle point

$$p = p_1 = \left(\frac{t}{2r}\right)\sec\theta \text{ and } f''(p) = -2i\left(\frac{r}{t}\right)\cos\theta.$$

Now, by Taylor's expansion of $f(p)$ about $p = p_1$, we obtain

$$f(p) = f(p_1)+\tfrac{1}{2}(p-p_1)^2 f''(p_1)+\ldots$$

$$= \frac{it}{4r\cos\theta}-\frac{ir}{t}(p-p_1)^2\cos\theta+\ldots$$

This suggests that the path of integration must be a straight line through the saddle point $p = p_1$ such that the quantity

$$-\frac{ir}{t}(p-p_1)^2\cos\theta$$

is real and negative.
Suppose the equation to the line is given by

$$p = p_1+l\, e^{i\alpha}.$$

Then it turns out that

$$-\frac{ir}{t}(p-p_1)^2\cos\theta = -\frac{ir}{t}l^2 e^{2i\alpha}\cos\theta$$

whence it follows that α must be such that

$$\cos 2\alpha = 0, \quad \sin 2\alpha < 0.$$

This gives $\alpha = -\pi/4$, $3\pi/4$ which would fix up the slope of the steepest path.

There are two distinct cases to be considered:

(I) $p_1 > 1$ and (II) $p_1 < 1$.

Case I: When $p_1 > 1$—The path is indicated in Fig. 3, the angle of approach of the path of steepest descent through the saddle point $p = p_1$ is $-\pi/4$.

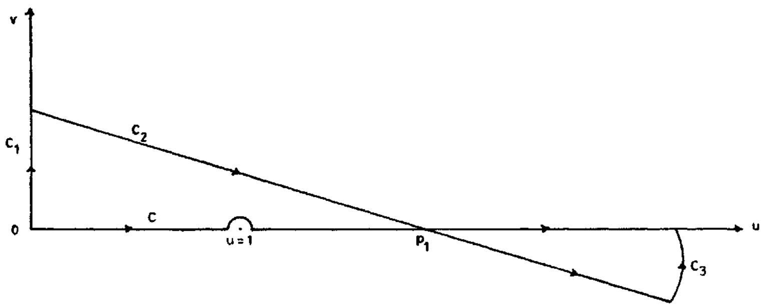

FIG. 3.

We have by Cauchy's theorem

$$\int_C g_-(p)\, e^{tf(p)}\, dp = \left[\int_{C_1} + \int_{C_2} + \int_{C_3}\right] g_-(p)\, e^{tf(p)}\, dp.$$

It can be shown without any difficulty that the integrals along the paths C_1 and C_3 have no significant contribution. In fact, the integral along the path C_3 (on putting $p = p_1 + Re^{i\alpha}$) tends to zero exponentially as $R \to \infty$. The integral along the path C_1 has a contribution which is $O\left(\frac{1}{t}\right)$.

Then it turns out that

$$\begin{aligned} &\int_C g_-(p)\, e^{tf(p)}\, dp \\ &= \int_{C_2} g_-(p)\, e^{tf(p)}\, dp, \quad \text{with } p = p_1 + le^{i\alpha} \text{ on } C_2. \\ &\sim g_-(p_1) \left\{\frac{2\pi}{t\,|f''(p_1)|}\right\}^{\frac{1}{2}} e^{tf(p_1) - i\frac{\pi}{4}} \\ &= g_-(p_1) \left(\frac{\pi}{r\cos\theta}\right)^{\frac{1}{2}} e^{i\left(\frac{t^2}{4r}\sec\theta - \frac{\pi}{4}\right)}. \end{aligned}$$

Case II: When $p_1 < 1$ in this case, making reference to Fig. 4 and applying Cauchy's theorem, we obtain

$$\begin{aligned} &\int_C g_-(p)\, e^{tf(p)}\, dp \\ &= \left[\int_{C_1} + \int_{C_2} + \int_{C_3}\right] g_-(p)\, e^{tf(p)}\, dp - 2\pi i \quad (\text{residue at } p = 1). \end{aligned}$$

A similar argument can be made to the integrals C_1 and C_3 and it turns out that they give no significant contribution to the main integral.

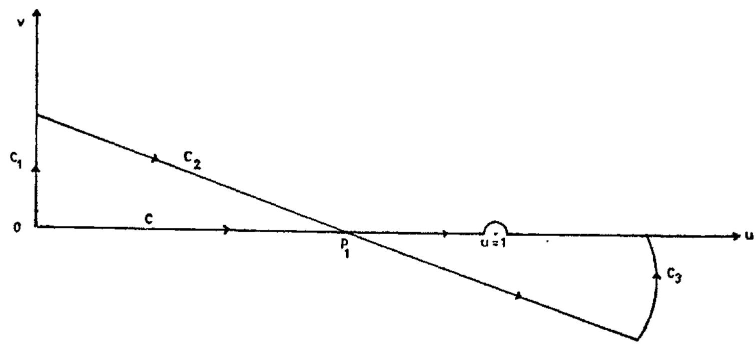

FIG. 4.

Finally, we arrive at the result

$$\int_C g_-(p)e^{tf(p)}\,dp$$

$$\sim -2\pi i\; e^{i(t-r\cos\theta)} + g_-(p_1)\left(\frac{\pi}{r\cos\theta}\right)^{\frac{1}{2}} e^{i\left(\frac{t^2}{4r}\sec\theta-\frac{\pi}{4}\right)}.$$

To compute the integral L_3, it may be noted that it has a pole at $p = 1$, hence we can take a contour Γ_3 bounded by the path L, v-axis and the circular arc C_3, all lying in the first quadrant of the complex p-plane. Again, taking the help of Cauchy's theorem, we easily find that the contribution to L_3 from the circular arc C_3 tends to zero as the radius of C_3 approaches to infinity and from the v-axis is $O\left(\frac{1}{t}\right)$. This leads us to the result

$$L_3 \sim O\left(\frac{1}{t}\right).$$

Lastly, it may be noticed that the integral L_4 has neither poles nor stationary points. Hence it can be treated like L_3. So, by the similar argument, it readily follows that the integral L_4 has no significant contribution.

We thus obtain

$$I_1' \sim \begin{bmatrix} -2\pi i\; e^{i(t-r\cos\theta)}, & p_1 < 1 \\ 0 \quad , & p_1 > 1 \end{bmatrix} + g_-(p_1)\left(\frac{\pi}{r\cos\theta}\right)^{\frac{1}{2}} e^{i\left(\frac{t^2}{4r}\sec\theta-\frac{\pi}{4}\right)}.$$

We then substitute the asymptotic value of I_1' into (5.4) and make appeal to the method of stationary phase to work out θ-integral. Having done this, we obtain an asymptotic representation of I_1 in the form

$$I_1 \sim \begin{bmatrix} -i\left(\dfrac{2\pi}{r}\right)^{\frac{1}{2}} e^{i\left(t-r+\frac{\pi}{4}\right)}, & t \ll 2r \\ 0 \quad , & t \gg 2r \end{bmatrix}$$

$$+\frac{t^3}{4\sqrt{2}\, r^4\left(\dfrac{t^2}{4r^2}-1\right)}\left\{\frac{t}{2r}\cos\left(\frac{t^2}{4r}\right)+i\sin\left(\frac{t^2}{4r}\right)\right\}.$$

Finally, combining the value of I_1 as well as I_2, an asymptotic representation for the surface elevation $\eta(r, t)$ takes the form

$$\eta(r, t) \sim \begin{bmatrix} i\left(\dfrac{2\pi}{r}\right)^{\frac{1}{2}} e^{i\left(t-r+\frac{\pi}{4}\right)}, & t \gg 2r \\ 0 \quad , & t \ll 2r \end{bmatrix}$$

$$+\frac{t^3}{4\sqrt{2}\, r^4\left(\dfrac{t^2}{4r^2}-1\right)}\left[\frac{t}{2r}\cos\left(\frac{t^2}{4r}\right)+i\sin\left(\frac{t^2}{4r}\right)\right]+O\left(\frac{1}{t^2}\right). \quad .. \quad (5.5)$$

Remarks: (1) The above solution (5.5) is not valid at $t = 2r$. As we are essentially interested in the asymptotic solution for large t, the computation of the valid solution at $t = 2r$ is not so important in the present discussion. However, this can be done by a special device advanced in the thesis (Debnath 1967).

(2) It is very important to notice that the transient term involved in solution (5.5) does not tend to zero as $t \to \infty$ for fixed values of r. Consequently, the solution (5.5) for the surface elevation $\eta(r, t)$ does not tend to the corresponding steady state solution in the limit. One of the reasons, from mathematical point of view, is that this is probably due to the strong singularity of the Dirac function at the origin. However, we intend to discuss this point further on a future occasion.

(3) In this connection, it may be suggested that Lighthill's prediction 'all different methods of switching on produce identical steady states after sufficiently long time' is not true. At least, the present solution (5.5) does not support it.

Motivated by the strange behaviour of the asymptotic solution for $\eta(r, t)$, due to an oscillating pressure distribution concentrated in a 'single point' of the free surface, we propose to consider some oscillating pressure distributions confined over a 'finite' region of the free surface, stated before in cases 4(b)–4(d).

In case 4(b), we have

$$p(r) = 1$$

so that

$$\bar{p}(k, t-\xi) = \frac{a}{k} J_1(ak)\, e^{i(t-\xi)},$$

where $J_1(ak)$ represents the Bessel function of the first kind of order unity.

By virtue of this result and making an argument similar to case 4(a), we readily obtain an integral representation for the surface elevation $\eta(r, t)$ in the form

$$\eta(r, t) = a \int_L J_1(as)(s \cos \sqrt{s}\, t + i \sqrt{s} \sin \sqrt{s}\, t - s\, e^{it}) \frac{J_0(sr)}{s-1}\, ds. \qquad (5.6)$$

This integral is very much similar to that of (5.2). Therefore, an identical asymptotic analysis can be carried out to obtain an asymptotic solution for $\eta(r, t)$. This leads us to the asymptotic solution for $\eta(r, t)$ in the form

$$\eta(r, t) \sim \begin{bmatrix} ai\left(\frac{2\pi}{r}\right)^{\frac{1}{2}} J_1(a)\, e^{i\left(t-r+\frac{\pi}{4}\right)}, & t \geqslant 2r \\ 0 & , \; t \leqslant 2r \end{bmatrix}$$

$$+ \frac{at\, J_1\left(\frac{at^2}{4r^2}\right)}{\sqrt{2}\, r^2\left(\frac{t^2}{4r^2}-1\right)} \left[\frac{t}{2r} \cos\left(\frac{t^2}{4r}\right) + i \sin\left(\frac{t^2}{4r}\right)\right] + O\left(\frac{1}{t^2}\right). \quad .. \quad (5.7)$$

Remarks: (1) A similar remark, as before, can be made regarding the validity of the solution (5.7) at $t = 2r$. Making reference to a method similar to that advanced in the thesis (Debnath 1967), the solution of the surface elevation $\eta(r, t)$ at the critical point $t = 2r$ can be obtained as

$$\eta(r, t) \approx -\left(\frac{3a}{2}\right)\left(\frac{\pi}{r}\right)^{\frac{1}{2}} J_1(a)(\cos r + \sin r).$$

(2) It is very interesting to notice that the transient term involved in solution (5.7) does, in fact, tend to zero as $t \to \infty$, for fixed r. Consequently, the solution for the surface elevation $\eta(r, t)$ does approach to the corresponding steady state in the limit $t \to \infty$ as expected.

(3) Proceeding to the limit $r \to \infty$, for fixed t, the surface elevation $\eta(r, t)$ has the following asymptotic representation:

$$\eta(r, t) \sim \frac{at\, J_1\left(\frac{at^2}{4r^2}\right)}{\sqrt{2} r^2\left(\frac{t^2}{4r^2}-1\right)} \left\{\frac{t}{2r} \cos\left(\frac{t^2}{4r}\right) + i \sin\left(\frac{t^2}{4r}\right)\right\}.$$

(4) It may be interesting to observe that the asymptotic solution (5.5) for $\eta(r, t)$ can be deduced from that of (5.7) as a limit $P \to \infty$, $a \to 0$ provided Pa^2 tends to a finite constant.

We next turn our attention to the Gaussian pressure distribution stated in case 4(c) in order to examine the wave motions.

In this case, we readily obtain

$$\bar{p}(k, t-\xi) = 2m^2 e^{i(t-\xi)-k^2m^2}, \quad m^2 = \frac{r_0^2}{4}.$$

An argument analogous to that advanced in case 4(a), with the above-stated value of $\bar{p}(k, t-\xi)$, enables us to obtain the following result for the surface displacement $\eta(r, t)$:

$$\eta(r, t) = 2m^2 \int_L e^{-(sm)^2} (\sqrt{s} \cos \sqrt{s}\, t + i \sin \sqrt{s}\, t - \sqrt{s}\, e^{it}) \frac{s^{\frac{3}{2}} J_0(rs)}{(s-1)}\, ds$$

$$= \frac{4m^2}{\pi} \int_0^{\frac{\pi}{2}} (I_1 - I_2)\, d\theta \qquad \ldots \quad (5.8)$$

where I_1 and I_2 are given by

$$I_1 = \int_L s^{\frac{3}{2}} e^{-s^2m^2} (\sqrt{s} \cos \sqrt{s}\, t + i \sin \sqrt{s}\, t)(s-1)^{-1} \cos (sr \cos \theta)\, ds$$

$$I_2 = e^{it} \int_L e^{-s^2m^2} s^2 (s-1)^{-1} \cos (sr \cos \theta)\, ds.$$

These integrals are similar in nature to those obtained before in case 4(a). Hence an asymptotic treatment analogous to case 4(a) can be given to them. More explicitly, the method of steepest descent and of stationary phase combined with the calculus of residues can be employed to evaluate the wave integrals I_1 and I_2. Finally, the θ-integral involved in (5.8) can also be worked out by the stationary phase approximation.

This leads us to obtain the following asymptotic expansion for $\eta(r, t)$:

$$\eta(r, t) \sim \begin{bmatrix} 2im^2 \sqrt{\frac{2\pi}{r}}\, e^{i\left(t-r+\frac{\pi}{4}\right)-m^2}, & t \gg 2r \\ 0, & t \ll 2r \end{bmatrix}$$

$$+ \frac{m^2 r^3 e^{-\left(\frac{mt^2}{4r^2}\right)^2}}{2\sqrt{2}\, r^4 \left(\frac{t^2}{4r^2} - 1\right)} \left\{ \frac{t}{2r} \cos \left(\frac{t^2}{4r}\right) + i \sin \left(\frac{t^2}{4r}\right) \right\}. \qquad \ldots \quad (5.9)$$

Remarks: The above solution (5.9) suggests that the transient term decays very rapidly as $t \to \infty$, for fixed values of r. As a result, the asymptotic solution tends to the corresponding steady state in the limit.

In fact, the steady state solution for $\eta(r, t)$ is given by

$$\eta(r, t) \sim 2im^2 \left(\frac{2\pi}{r}\right)^{\frac{1}{2}} e^{i\left(t-r+\frac{\pi}{4}\right)-m^2}.$$

Lastly, we consider case 4(d). In this case, we readily find

$$\bar{p}(k, t-\xi) = e^{i(t-\xi)} \int_0^a x\, J_0(\lambda x)\, J_0(kx)\, dx,$$

where the integral on the right is a standard one (Watson 1922).

Without repeating the calculation similar to that encountered earlier, we write down the asymptotic representation for the surface elevation $\eta(r, t)$ related to this case in the form

$$\eta(r, t) \sim \begin{bmatrix} iN_1\left(\frac{2\pi}{r}\right)^{\frac{1}{2}} e^{i\left(t-r+\frac{\pi}{4}\right)}, & t \gg 2r \\ 0 \qquad , & t \ll 2r \end{bmatrix}$$
$$+\frac{N_2\, t^3}{\sqrt{2}\, r^4\left(\frac{t^2}{4r^2}-1\right)}\left\{\frac{t}{2r}\cos\left(\frac{t^2}{4r}\right)+i\sin\left(\frac{t^2}{4r}\right)\right\}+O\left(\frac{1}{t^2}\right), \qquad .. \quad (5.10)$$

where N_1 and N_2 are given by known integrals involving the product of Bessel functions (Watson 1922) as

$$N_1 = \int_0^a x\, J_0(\lambda x)\, J_0(x)\, dx,$$

$$N_2 = \int_0^a x\, J_0(\lambda x)\, J_0\left(\frac{xt^2}{4r}\right) dx.$$

A conclusion similar to cases 4(b) and 4(c) can be drawn regarding the limiting behaviour of the solution (5·10) as $t \to \infty$ for fixed r. In other words, the solution for the surface elevation $\eta(r, t)$ does tend to an ultimate steady state in the limit.

6. Discussion of the Wave Motions

The above analysis reveals a striking difference between the wave motions generated by oscillating pressure distributions confined over 'a finite' region and at 'a single point' on the free surface of the fluid. The relation between case 4(a) with cases 4(b)–4(d) is that the former results from taking the double limit $a \to 0$, $t \to \infty$ in this order while, in the latter three cases, we have taken the limit $t \to \infty$ first. The fact that the answers are different suggests that the order of the double limit taking is not commutative.

The reason why case 4(a) does not tend to the corresponding steady state solution is, of course, as stated earlier from mathematical point of view that there is a strong singularity of the Dirac delta function at the origin. This singularity, corresponding to an infinite pressure applied over a vanishingly small region, violates the small amplitude assumption of the linearized theory.

The situation described by case 4(a) is purely mathematical and would never arise in practice, since any physical pressure distribution would be spread over a finite region, however small, of the free surface and so the ultimate steady state described by cases 4(b)–4(d) would be reached.

In fact, the surface elevation $\eta(r, t)$ related to cases 4(b)–4(d) assumes the limiting form

$$\eta(r, t) \sim \frac{iB}{\sqrt{r}} e^{i\left(t-r+\frac{\pi}{4}\right)},$$

where B is a fixed known constant, of course, different for different pressure distributions. This solution corresponds to the circular surface waves advancing with the phase velocity g/ω and the group velocity $g/2\omega$; and the amplitude of the waves decays like $r^{-\frac{1}{2}}$.

So the significant conclusion is that the solution of the initial value problem would always tend to the ultimate steady state as $t \to \infty$ for fixed r, provided the pressure distribution is distributed over a finite region on the undisturbed free surface.

This analysis further suggests that the ultimate steady state can be obtained as a limit from the transient solution without imposing any fictitious damping force as suggested by Sen. So, it is probably fair to say that the present initial value treatment is more convincing.

Lastly, we can say that the axisymmetric analysis, presented above, further embolden us to put forward exactly the same comment, as in the two-dimensional case (Debnath 1968), on Lighthill's prediction regarding the steady state. More precisely, it is probably fair to suggest that Lighthill's prediction is not always true—its validity can also be restricted by this axisymmetric analysis.

7. Concluding Remarks

In this paper, our discussion has been restricted to the transient development of axisymmetric surface waves in a fluid of unlimited depth. The investigation into the same problem in a fluid of finite and very shallow depth may remain open for future communication.

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