THE GRAVITATIONAL FIELD AND THE TYPE OF MATTER

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In this note a classification of the energy-momentum tensor is given. It has been shown that for a given type of gravitational field the algebraic structure of the energy-momentum tensor may be any one of the three types discussed in this paper or vice versa. In this manner there are nine physical states possible in general relativity.

Notations

Operational space: Normal hyperbolic Riemannian manifold V_4

Signature of space-time: $(+++-)$

Choice of Ricci tensor: $R_{ab} = R^{c}_{acb}$; $R = R^{a}_{a}$

Range of indices: Small Latin indices range from 1 to 4 and those of Greek indices from 1 to 3. Latin indices are used as tensor indices whereas Greek indices shall be used as labels. If an index is given a particular value it should be understood as a label and not a tensor index. Labels as well as tensor indices will follow summation convention

Square brackets for antisymmetrization:

$$A_{[ab]} = \tfrac{1}{2}(A_{ab} - A_{ba})$$

Round brackets for symmetrization:

$$A_{(ab)} = \tfrac{1}{2}(A_{ab} + A_{ba})$$

$$\eta_{abcd} = \eta_{[abcd]} = +1(-1)$$

if $abcd$ form an even (odd) permutation of 1, 2, 3 and 4 and zero otherwise

Metric induced in bivector space by g_{ab}:

$$g_{abcd} = 2g_{a[c}g_{d]b} = g_{ap}g_{bq}\delta^{pq}_{cd}$$

Dual of a second rank skew tensor:

$$^{*}S_{ab} = \tfrac{1}{2}\eta_{abcd}S^{cd}$$

Right dual of a fourth rank tensor:

$$^{*}R_{abcd} = \tfrac{1}{2}\eta_{abpq}R^{pq}_{\ \ cd}$$

Left dual of a fourth rank tensor:

$$R^*_{abcd} = \tfrac{1}{2}\eta_{cdpq}R^{pq}_{ab}$$

1. Introduction

In a recent paper we discussed the classification of the gravitational fields by making use of three-tensors constructed from the Riemann tensor with the help of a time-like tangent vector under the assumption that the space-time is empty (Misra 1967). But for the general case it is desirable to discuss the classification of the field for any possible distribution and motion of matter and radiation. One may ask the question: What are the possible types of the field for a given distribution and motion of matter or, conversely, what physically admissible sources and motions are possible for a given type of field. This problem has been investigated by Petrov (1961, 1962, 1966) who has shown that even in the general case there are only three types of the gravitational field. In this note we propose to discuss the classification of the general gravitational field from the following viewpoint. In the general case where the structure of the energy-momentum tensor is not known we may be interested in investigating the dependence of the classification of the gravitational field on the algebraic structure of the energy-momentum tensor. But the energy-momentum tensor is related to Ricci tensor through the field equations

$$R_{ab}-\tfrac{1}{2}g_{ab}R = T_{ab}. \qquad (1)$$

Hence the algebraic structure of the energy-momentum tensor T_{ab} is the same as that of the Ricci tensor R_{ab}. Now it is well known that the curvature tensor can be decomposed according to the following formula (Debever 1956, Géhéniau 1956 and Géhéniau and Debever 1956):

$$R_{abcd} = C_{abcd}+E_{abcd}+\frac{R}{12}g_{abcd} \qquad (2)$$

where the constituents have the symmetry properties of the curvature tensor and are characterized by their dual symmetries

$$^*C^*_{abcd} = -C_{abcd}, \quad ^*E^*_{abcd} = E_{abcd}. \qquad (3)$$

They have the further properties

$$C^{ab}_{ac} = E^{ab}_{ab} = 0 \qquad (4)$$

$$E_{abcd} = -g_{abe[c}S^e_{d]}, \quad S_{ab} = R_{ab}-\frac{R}{4}g_{ab}. \qquad (5)$$

Here C_{abcd} is Weyl's conformal tensor and has 10 independent components. E_{abcd} is the Ricci part of the curvature tensor and has 9 components. E_{abcd} is algebraically equivalent to the reduced Ricci tensor S_{ab}. On account of eqns. (1) and (2) the Ricci part in E_{abcd} can be equated with the presence of matter. The Weyl tensor is to be thought of as representing the free fields. The Ricci term in eqn. (2) can be equated with the presence of matter. Therefore, a classification of the energy-momentum tensor is equivalent to the classification of E_{abcd}. The canonical forms to which S_{ab} and g_{ab} may be reduced in space-time has been discussed in part by Eisenhart (1949). Hawking (1966) and Estabrook and Wahlquist (1964) have also discussed the relation between E_{abcd} and the Hermitian matrix which may be obtained from the Ricci part of the curvature tensor. In this note we propose to present another method of classification which leads to more physical insight into this problem.

2. Algebraic Relations

As already pointed out we shall consider the non-empty gravitational fields characterized by eqns. (1). We define the following three-tensors with the help of E_{abcd} and its dual:

$$P_{ac} = E_{abcd}u^b u^d \quad \ldots \quad (6)$$

$$Q_{ac} = {}^*E_{abcd}u^b u^d \quad \ldots \quad (7)$$

$$R_{ac} = E^*_{abcd}u^b u^d \quad \ldots \quad (8)$$

$$T_{ac} = {}^*E^*_{abcd}u^b u^d \quad \ldots \quad (9)$$

where u^a is the time-like unit velocity four-vector satisfying the following equation

$$u^a u_a = -1.$$

But all these three-tensors are not independent of each other. In view of (3) we may easily see that P_{ac} and T_{ac} are equivalent to each other. Further the left and right duals of E_{abcd} (Q_{ac} and R_{ac} respectively) are equal to each other with a minus sign. Therefore, we may discuss the algebraic properties of E_{abcd} with the help of only two three-tensors, namely P_{ac} and Q_{ac}. They satisfy the following equations.

$$P_{[ac]} = 0, \quad Q_{(ac)} = 0 \quad \ldots \quad (10)$$

$$P_{ab}u^b = Q_{ab}u^b = 0. \quad \ldots \quad (11)$$

Equation (10) states that P_{ac} is symmetric whereas Q_{ac} is skewsymmetric. Thus P_{ac} has six independent components and Q_{ac} has only three. Both together account for the nine components of E_{abcd}. They further satisfy the following equation

$$(E+i^*E)_{abcd} = (g+i\eta)_{abpq}(g-i\eta)_{cdrs}u^p u^r(P+iQ)^{qs}. \quad \ldots \quad (12)$$

Thus the components of E_{abcd} have been divided into two groups. This procedure is entirely analogous to that of electromagnetic field where the tensor f_{ab} is partitioned into two three-vectors $\vec{E}$ and $\vec{H}$ in the hypersurface

$t = 0$. It is, however, necessary to emphasize that this partitioning has local validity only. In the next section we discuss the classification into different types.

3. Classification of E_{abcd}

The classification of E_{abcd} may be carried out very simply in terms of 3×3 matrices $P_{a'b'}$ and $Q^{\dagger}_{a'b'}$. Then a classification of $P_{a'b'}$ and $Q_{a'b'}$ will be equivalent to a classification of E_{abcd}. The required classification is obtained by reducing the matrices $P_{a'b'}$ and $Q_{a'b'}$ to their canonical forms. Let us introduce the complex matrix

$$D_{a'b'} = P_{a'b'}+iQ_{a'b'}. \qquad (13)$$

Evidently $D_{a'b'}$ is a Hermitian matrix, with zero trace. The eigenvalues and eigenvectors of $D_{a'b'}$ may be defined with the help of the following equation:

$$D_{a'b'}e^{a'}_{x} = \lambda_x e_{xb'}. \qquad (14)$$

Since the trace of $D_{a'b'}$ is zero the sum of three eigenvalues must vanish

$$\sum_{x=1}^{3} \lambda_x = 0.$$

Depending on the number of independent eigenvectors $e_{x_{a'}}$ we arrive at the following classification of the tensor E_{abcd}.

Type I.—There are three independent eigenvectors. By a suitable rotation we can bring the matrix $D_{a'b'}$ to its diagonal form. Therefore, the following form is obtained for $P_{a'b'}$ and $Q_{a'b'}$:

$$P_{a'b'} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad Q_{a'b'} = (0) \qquad (15)$$

$$\Sigma \lambda_i = 0.$$

In this case there are only two invariants of E_{abcd}. If any two of the eigenvalues coincide the tensor will be degenerate.

Type II.—There are only two independent eigenvectors. In this case we get the following canonical type:

$$P_{a'b'} = \begin{pmatrix} \lambda+B & 0 & 0 \\ 0 & \lambda-B & 0 \\ 0 & 0 & -2\lambda \end{pmatrix} \quad Q_{a'b'} = \begin{pmatrix} 0 & B & 0 \\ -B & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (16)$$

Type III.—In this case there is just one eigenvector which is null. It is easy to see that $P_{a'b'}$ and $Q_{a'b'}$ may be transformed to the following form:

$$P_{a'b'} = \begin{pmatrix} 0 & 0 & A \\ 0 & 0 & 0 \\ A & 0 & 0 \end{pmatrix} \quad Q_{a'b'} = (0). \qquad (17)$$

† The discussion of section 2 and reference 1 clearly shows that P_{ab} and Q_{ab} are, in fact, 3×3 matrices whereas the notation 'P_{ab}' implies that it is 4×4 matrix. In order to avoid this confusion we write every index with a stroke which will mean a tensor index taking values 1, 2 and 3.

The above classification may be further specialized by considering the equality of eigenvalues. It is interesting to note that $Q_{a'b'}$ is a null matrix if it has an odd rank.

4. Discussion and Conclusion

From the discussion of the previous section we conclude that algebraically there are three types of energy-momentum tensor for physically meaningful gravitational fields. Further, from eqn. (2), it is evident that at any point of space-time the Ricci tensor and Weyl tensor are completely independent but in a region they are related through the Bianchi identities. Bianchi identities are, therefore, interpreted to represent the interaction between matter and the gravitational field. However, here we are concerned with the structure of the energy-momentum tensor at a point. Thus the point of central importance here from our viewpoint is that for a given conform tensor (given type of gravitational field) the type of energy-momentum tensor may be any one of the three types discussed in § 3 or, conversely, for a given type of the energy-momentum tensor the gravitational field may be any one of the three possible types. This may be stated in a different way in the following manner. Suppose, that in a problem of given specification different types of the gravitational field and energy-momentum tensor are possible then the above discussion states that in general relativity there are no more than nine possible physical states, viz. three of the gravitational field and three of the energy momentum tensor. Here it should be made clear that the structure of the energy-momentum tensor depends not only on the type of sources but also on the motion of these sources. Therefore, in a particular problem even if the type of the field and the nature of the sources are known it will not be possible to determine the type of energy-momentum tensor unless the motion of the sources is also known. The following example will help to clear the above discussion.

Let us consider a space filled with incoherent matter. The energy-momentum tensor for this problem is

$$T_{ij} = \mu u_i u_j \qquad \qquad (18)$$

where μ is the rest energy of the particle and u_i is the four-velocity vector of the particle. In this case the classification of T_{ij} is simple; it has one non-zero real eigenvalue in a given coordinate frame and three being equal to zero. Suppose now that the line element of the space under consideration is given by

$$ds^2 = (dx^0)^2 - (dx^1)^2 + 2\beta dx^0\, dx^2 - \gamma^2 (dx^2)^2 - \alpha^2 (dx^3)^2 \qquad \qquad (19)$$

where α, β and γ are functions of the variable x^1 only. Now the Einstein's field equations with a cosmological term are satisfied by the metric provided one chooses the functions α, β and γ in the following manner (Wright 1965)

$$\alpha_1^2 = B^2(\beta^2+\gamma^2) \qquad (20)$$

$$\beta_1^2 = \frac{A^2}{B^2}\left(\frac{\alpha_1^2}{\alpha^2}\right) \qquad (21)$$

$$\alpha_1^2 = D-\lambda\alpha^2-\tfrac{1}{2}A^2 \log \alpha \qquad (22)$$

$$\mu = \frac{1}{2}\frac{A^2}{\alpha^2} - \frac{2\alpha_{11}}{\alpha} \qquad (23)$$

where A, B and D are constants and λ is the cosmological constant. It is found, then, that the conform tensor for the gravitational field (19) is type I. It should be noted here that the density of the incoherent matter is a function of the variable x^1.

However, for a constant α (say $\alpha = 1$) the field equations are again satisfied. The solution in this case is (Gödel 1949)

$$\alpha = 1, \quad \beta = \exp(x^1), \quad \gamma^2 = \tfrac{1}{2}\exp(2x^1) \qquad (24)$$

and

$$\mu = \tfrac{1}{2}A^2 = \text{constant}. \qquad (25)$$

The Weyl tensor in this case is type I degenerate.

Thus even in the case of the simplest type of the energy momentum tensor, i.e. that of non-interacting dust particles, different types of gravitational fields are possible.

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