NOTE ON RESPONSES IN A PIEZOELECTRIC CRYSTAL WITH DIVIDED ELECTRODES

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The present note is an attempt to obtain the electrical responses in a piezoelectric crystal plate with divided electrodes with given electrical input employing the method of Laplace transform.

1. Introduction

The piezoelectric problems, particularly those concerning the electromechanical responses in a piezoelectric transducer, have long been treated from the standpoint of equivalent circuit theory, vide Mason (1948, 1950). It is only in recent years that such investigations are carried out using the principles of mechanics of continuous media and electromagnetism and the pioneering efforts in this regard have been made by Redwood (1961*a*, *b*). It is pertinent in this connection to refer here to similar attempts made by Sinha (1963, 1965, 1967*a*, *b*), Giri (1966, 1967) and Das (1967, 1969) on the determination of responses in piezoelectric crystals. The present note is an analogous attempt on a problem as yet unsolved and it seeks to work out the responses in a piezoelectric crystal with divided electrodes, some time-dependent inputs being prescribed. The analysis presented here proceeds on the lines of the mechanism of a crystal provided with electrodes as considered by Van der Veen (1956). It has been found that the Laplace transform serves as an effective tool for the solution of the problem.

2. Problem

We take a bar whose centre is chosen as the origin of coordinates (Fig. 1). The three axes X, Y, Z are in the direction of thickness ($= 2a$), the length ($= 2b$) and the width ($= 2c$). A first set of electrodes $x = \pm a$, $-b \leqslant y < 0$ and $-c \leqslant z \leqslant c$ is connected to the terminals J_1 and J_1' while a second set of electrodes with coordinates $x = \pm a$, $0 < y \leqslant b$ and $-c \leqslant z \leqslant c$ is connected to a second set of terminals J_2 and J_2'.

A longitudinal disturbance is set in piezoelectrically by applying a time-dependent voltage $V_1(t)$ across the terminals where

$$V_1(t) = \begin{cases} 0, & t \leqslant 0 \\ V_0 \dfrac{t}{t_0}, & 0 < t < t_0 \\ V_0, & t > t_0 \end{cases} \qquad \qquad (1)$$

and the current induced I_1 is given by

$$I_1 = H(t)$$

$H(t)$ being Heaviside's steep function defined as

$$\left.\begin{aligned} H(t) &= 1, \quad t > 0 \\ &= 0, \quad t = 0 \end{aligned}\right\} \qquad \text{(1}a\text{)}$$

Our object here is to find out the electrical responses represented by the voltage V_2 (say) and the current I_2 (say) across the second set of terminals arising out of the mechanical coupling between the two halves of the bar.

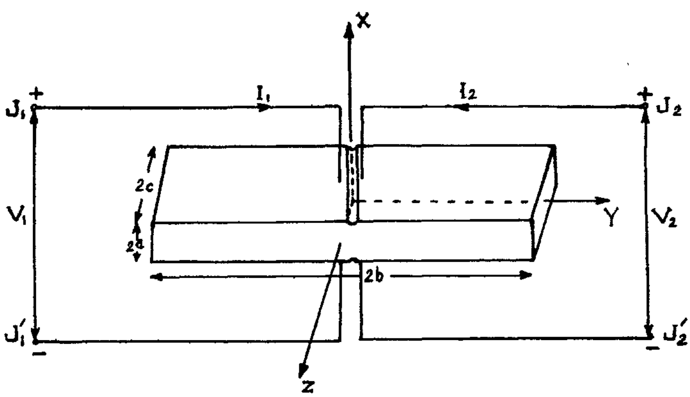

FIG. 1. Piezoelectric crystal with divided electrodes.

3. Fundamental Equations and Boundary Conditions

The fundamental equations of the problem are evidently those which represent the interacting fields. Thus the equations are the constitutive relations of the piezoelectric material, equation of motion and the boundary conditions. The piezoelectric equations, as in Mason (1950), are given by

$$S = sT + dE \qquad \text{(2}a\text{)}$$

$$D = dT + \epsilon E \qquad \text{(2}b\text{)}$$

where $S = \partial\eta/\partial y$, η being the displacement in the direction of length; T, normal stress in the length direction; E, electric field strength in the thickness direction; D, electric displacement in the thickness direction; s, electric compliance; d, piezoelectric strain constant; ϵ, permittivity.

The equation of motion is

$$\frac{\partial T}{\partial y} = \rho \frac{\partial^2 \eta}{\partial t^2} \qquad (3)$$

ρ being the density of the bar.

Since the electrodes are equipotential surfaces

$$\frac{\partial E}{\partial y} = 0 \quad \text{for } y \gtrless 0 \qquad (4)$$

From the continuity of η and T in the plane $y = 0$, we get

$$\left.\begin{aligned}(\eta)_1 &= (\eta)_2 \\ (T)_1 &= (T)_2\end{aligned}\right\} \text{ on } y = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (5)$$

where the suffixes indicate the respective electrodes. A third set of boundary conditions is obtained if, following Van der Veen (1956), we take into account the influence of various causes of damping of a vibrating bar by assuming a fictitious acoustical medium which exerts on the boundary surfaces ($y = \pm b$) a stress

$$(T)_{\pm b} = \mp\alpha\left(\frac{\partial\eta}{\partial t}\right)_{\pm b} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

α being a positive constant.

Again eliminating T from eqns. ($2a$) and ($2b$), integration over the first set of electrodes J_1, J_1' leads to the total charge Q on the electrodes J_1, J_1' given by

$$Q = \frac{d}{s}\int_{-c}^{c} dz \int_{-b}^{0} S\,dy + \left(\epsilon - \frac{d^2}{s}\right)\int_{-c}^{c} dz \int_{-b}^{0} E\,dy$$

$$= \phi\{\eta_{y=-0} - \eta_{y=-b}\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

where $\phi = 2cd/s$.

The second integral is $\frac{1}{2}C_p V_1$ where C_p is given by

$$\tfrac{1}{2}C_p = \left(\epsilon - \frac{d^2}{s}\right)\frac{cb}{a}$$

Differentiating eqn. (7) with respect to time we get the current through the terminals J_1, J_1' where

$$I_1 = \phi\frac{d}{dt}\{\eta_{-0} - \eta_{-b}\} + \frac{d}{dt}\left(\tfrac{1}{2}C_p V_1\right) \qquad \ldots \quad \ldots \quad \ldots \quad (8)$$

The second half of the crystal gives in the same way,

$$I_2 = \phi\frac{d}{dt}\{\eta_b - \eta_{+0}\} + \frac{d}{dt}\left(\tfrac{1}{2}C_p V_2\right) \qquad \ldots \quad \ldots \quad \ldots \quad (9)$$

Equations (4) to (8) constitute the boundary conditions of the problem.

4. Solution of the Problem

To solve the problem we take Laplace transform of eqns. (3)–(9) where all the quantities with bars indicate the transform of the corresponding quantity, for example,

$$\bar{\eta} = \int_0^\infty \eta e^{-pt}\,dt \quad [\mathrm{Re}\,(p) > 0]$$

Therefore, all equations mentioned above yield a second set of equations as follows:

$$\frac{\partial \bar{T}}{\partial y} = \rho p^2 \bar{\eta} \quad \text{(10)}$$

$$\frac{\partial \bar{\eta}}{\partial y} = 0 \qquad \text{for } y \gtrless 0 \quad \text{(11)}$$

$$(\bar{\eta})_1 = (\bar{\eta})_2 \qquad \text{for } y = 0 \quad \text{(12)}$$

$$(\bar{T})_1 = (\bar{T})_2 \qquad \text{for } y = 0 \quad \text{(12a)}$$

$$(\bar{T})_{\pm b} = \mp \alpha p (\bar{\eta})_{\pm b} \quad \text{(13)}$$

or

$$\left(\frac{\partial \bar{\eta}}{\partial y}\right)_{-b} - \alpha s p (\bar{\eta})_{-b} = \frac{d}{2a} \bar{V}_1 \quad \text{(13a)}$$

$$\left(\frac{\partial \bar{\eta}}{\partial y}\right)_{b} + \alpha s p (\bar{\eta})_{b} = \frac{d}{2a} \bar{V}_2 \quad \text{(13b)}$$

$$\bar{I}_1 = \phi p \{\bar{\eta}_{-0} - \bar{\eta}_{-b}\} + \tfrac{1}{2} p C_p \bar{V}_1 \quad \text{(14)}$$

and

$$\bar{I}_2 = \phi p \{\bar{\eta}_{b} - \bar{\eta}_{+0}\} + \tfrac{1}{2} p C_p \bar{V}_2 \quad \text{(15)}$$

Solving eqn. (10) we get

$$\left.\begin{aligned} \bar{\eta} &= A e^{\frac{py}{v}} + B e^{-\frac{py}{v}} \quad \text{for } y < 0 \\ &= C e^{\frac{py}{v}} + D e^{-\frac{py}{v}} \quad \text{for } y > 0 \end{aligned}\right\} \quad \text{(16)}$$

where $\rho = 1/v^2$.

The boundary conditions (12) to (13b) yield respectively

$$A + B = C + D \quad \text{(17a)}$$

$$A - B - C + D = \frac{dv}{2ap}(\bar{V}_1 - \bar{V}_2) \quad \text{(17b)}$$

$$A e^{-\frac{pb}{v}} K_1 - B e^{\frac{pb}{v}} K_2 - \frac{d\bar{V}_1}{2ap} = 0 \quad \text{(17c)}$$

$$C e^{\frac{pb}{v}} K_2 - D e^{-\frac{pb}{v}} K_1 - \frac{d\bar{V}_2}{2ap} = 0 \quad \text{(17d)}$$

where

$$K_1 = 1 - \alpha v s, \quad K_2 = 1 + \alpha v s$$

From eqns. (17*a*)–(17*d*) using eqn. (14) A, B, C and D can be obtained as follows:

$$
\left.
\begin{aligned}
A &= \lambda_1 \left\{ K_2 M e^{\frac{pb}{v}} + \frac{dv\bar{V}_1}{2ap}\left(1-e^{\frac{pb}{v}}\right) \right\} \\
B &= \lambda_1 \left\{ K_1 M e^{-\frac{pb}{v}} - \frac{dv\bar{V}_1}{2ap}\left(1-e^{-\frac{pb}{v}}\right) \right\} \\
C &= L\lambda_2 \Bigg[M \left\{ \left(1-K_1 e^{-\frac{pb}{v}}\right)\left(K_2 e^{\frac{pb}{v}}+K_1 e^{-\frac{pb}{v}}\right) - \left(K_1 e^{-\frac{pb}{v}} - K_2 e^{\frac{pb}{v}}\right) \right\} \\
&\quad + \frac{dv}{2ap}\bar{V}_1 \left\{ \left(1-K_1 e^{-\frac{pb}{v}}\right)\left(e^{-\frac{pb}{v}} - e^{\frac{pb}{v}}\right) - \left(e^{-\frac{pb}{v}} + e^{\frac{pb}{v}} - 2\right) \right\} \Bigg] \\
&\quad - L\frac{dv}{2ap}\bar{V}_1 \\
D &= L\lambda_2 \Bigg[M \left\{ \left(K_1 e^{-\frac{pb}{v}} - K_2 e^{\frac{pb}{v}}\right) - \left(K_2 e^{\frac{pb}{v}} - 1\right)\left(K_2 e^{\frac{pb}{v}} + K_1 e^{-\frac{pb}{v}}\right) \right\} \\
&\quad + \frac{dv}{2ap}\bar{V}_1 \left\{ e^{-\frac{pb}{v}} + e^{\frac{pb}{v}} - 2 - \left(K_2 e^{\frac{pb}{v}} - 1\right)\left(e^{-\frac{pb}{v}} - e^{\frac{pb}{v}}\right) \right\} \Bigg] + L\frac{dv}{2ap}V_1
\end{aligned}
\right\} \quad \cdots \quad (18)
$$

where

$$
\left.
\begin{aligned}
L^{-1} &= 2 - K_1 e^{-\frac{pb}{v}} - K_2 e^{\frac{pb}{v}} \\
\lambda_1^{-1} &= e^{-\frac{pb}{v}} K_1 \left(1-e^{\frac{pb}{v}}\right) + K_2 e^{\frac{pb}{v}}\left(1-e^{-\frac{pb}{v}}\right) \\
\lambda_2^{-1} &= K_1\left(e^{-\frac{pb}{v}} - 1\right) + K_2\left(e^{\frac{pb}{v}} - 1\right)
\end{aligned}
\right\} \quad \cdots \quad \cdots \quad (19)
$$

and

$$M = \frac{\bar{I}_1}{\phi p} - \tfrac{1}{2} C_p \frac{\bar{V}_1}{\phi}$$

and from (1) and (1*a*)

$$\bar{V}_1 = \frac{V_0}{p^2 t_0}\left(1-e^{-pt_0}\right) \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (20a)$$

and

$$\bar{I}_1 = \frac{1}{p} \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (20b)$$

knowing the values of C and D, $\bar{V}_2$ and $\bar{I}_2$ and hence V_2 and I_2 can be computed. Here we only compute V_2.

From eqn. (17*d*)

$$\bar{V}_2 = \frac{2ap}{dv}\left[C e^{\frac{pb}{v}} K_2 - D e^{-\frac{pb}{v}} K_1\right]$$

Substituting the relevant values of C and D we get $\bar{V}_2(p)$. This inversion is very complicated. Therefore, proceeding as in Redwood (1961*a*) we suppose

t small, i.e. p large, such that higher powers of $1/p$ can be neglected and we get on simplification

$$\bar{V}_2 = \frac{2a}{dv}\frac{\bar{I}_1}{\phi}\left[\frac{K_2}{2(2-K_2)} - \frac{K_2}{K_1}\right]$$

$$- \frac{2ap}{dv}\bar{V}_1\left[\frac{K_2}{2-K_2}\left\{\frac{dv}{2ap}\left(\frac{2}{K_2}+1\right)+\frac{C_p}{\phi}\right\}\right.$$

$$\left.- \frac{K_1}{K_2}\left(\frac{dv}{2ap}+\tfrac{1}{2}C_pK_2\right)\right]$$

$$= \frac{2a}{v\,d\phi p}\left[\frac{K_2}{2(2-K_2)} - \frac{K_1}{K_2}\right]$$

$$- \frac{2a}{dv}\frac{V_0}{pt_0}(1-e^{-pt_0})\left[\left\{\left(\frac{2}{K_2}+1\right)\frac{K_2}{2-K_2} - \frac{K_1}{K_2}\right\}\frac{dv}{2ap}\right.$$

$$\left.+\left\{\frac{K_2}{(2-K_2)\phi} - \frac{K_1}{2}\right\}C_p\right]$$

$$= \frac{2a}{dv}\frac{1}{\phi p}\left[\frac{K_2}{2(2-K_2)} - \frac{K_1}{K_2}\right]$$

$$- \frac{2a}{dv}\frac{V_0}{t_0}\left[\frac{dv}{2a}\left\{\left(\frac{2}{K_2}+1\right)\frac{K_2}{2-K_2} - \frac{K_1}{K_2}\right\}\frac{(1-e^{-pt_0})}{p^2}\right.$$

$$\left.+C_p\left\{\frac{K_2}{(2-K_2)\phi_2} - \frac{K_1}{2}\right\}\frac{(1-e^{-pt_0})}{p}\right]$$

Inverting this we have, vide Churchill (1958),

$$V_2(t) = \frac{2a}{dv\phi}\left[\frac{K_1}{2(2-K_2)} - \frac{K_1}{K_2}\right]H(t)$$

$$- \frac{2a}{dv}\frac{V_0}{t_0}\left[\frac{dv}{2a}\left\{\left(\frac{2}{K_2}+1\right)\frac{K_2}{2-K_2} - \frac{K_1}{K_2}\right\}t_0t\right.$$

$$\left.+C_p\left\{\frac{K_2}{(2-K_2)\phi} - \frac{K_2}{2}\right\}\right] \quad \text{for } 0 < t \leqslant t_0$$

$$= \frac{2a}{dv\phi}\left[\frac{K_1}{2(2-K_2)} - \frac{K_1}{K_2}\right]H(t)$$

$$- \frac{2a}{dv}\frac{V_0}{t_0}\left[\frac{dv}{2a}\left\{\left(\frac{2}{K_2}+1\right)\frac{K_2}{2-K_2} - \frac{K_1}{K_2}\right\}\right]t_0 \quad \text{for } t > 0$$

Thus we find that the electrical voltage response has different characteristics for different ranges of time, namely part of it is step in character throughout while another part of it is linear in $0 < t \leqslant t_0$ and this latter part continues to be constant beyond time t_0.

This expression $V_2(t)$ enables us to calculate $I_2(t)$ from eqn. (15).

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