SOME SPECIAL CASES OF SPHERICALLY SYMMETRIC GRAVITATIONAL COLLAPSE WITH REFERENCE TO NEUTRINO EMISSION

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Misner's field equations (1965) for spherically symmetric gravitational collapse with escaping neutrinos have been considered in some particular cases. It is shown that spherically symmetric gravitational collapse with neutrino emission is not possible if the congruence associated with the matter field satisfies any two of the following properties: (i) it is geodetic, (ii) it is expansion-free and (iii) it is shear-free. Also the field does not permit neutrino emission if the null congruence associated with neutrino emission is expansion-free. A case has been worked out with geodetic matter congruence where the possibility of neutrino emission is demonstrated. Similarly it is shown that neutrino emission is possible in the case of shear-free matter congruence.

1. INTRODUCTION

It has been suggested by Colgate and White (Misner 1965) and also by Chiu (1964) that in case of gravitational collapse of supernovae and quasi-stellar radio sources the emission of neutrinos is possible. Misner (1965) formulated the field equations for a spherically symmetric case of gravitational collapse with escaping neutrinos. It has been assumed that the neutrinos after emission travel in the radial direction. In the absence of neutrino emission the field reduces to that obtained by Misner and Sharp (Misner 1965). The internal field of collapsing matter is continued with the field of pure radiation such as the one obtained by Vaidya (1953). In this paper we discuss some of the special features of the field equations of spherical gravitational collapse with reference to the possibility of neutrino emission. It is shown that spherically symmetric gravitational collapse with neutrino emission is not possible if the congruence associated with the matter field satisfies any two of the following properties: (i) it is geodetic, (ii) it is expansion-free and (iii) it is shear-free. Also the field does not permit neutrino emission if the null congruence associated with neutrino is expansion-free. A case has been worked out with geodetic matter congruence where the possibility of neutrino emission is demonstrated. Similarly it is shown that neutrino emission is possible in the case of shear-free matter congruence.

2. The Field Equations, Kinematical Quantities and Optical Parameters

We give below the system of field equations and define the various symbols involved. The meaning of the symbols is explained subsequently.

The field equations are as follows (Misner 1965):

$$A \equiv (nu^a)_{;\,a} = 0 \qquad (2.1)$$

$$B \equiv -nc + u^a T^b_{a;\,b} = 0 \qquad (2.2)$$

$$D^a_b \equiv R^a_b - \tfrac{1}{2}\delta^a_b R + 8\pi(T^a_b + N^a_b) = 0 \qquad (2.3)$$

where

$$E \equiv u^a u_a + 1 = 0 \qquad (2.4)$$

$$T^a_b = (\epsilon + p)u^a u_b + \delta^a_b p \qquad (2.5)$$

$$N^a_b = qk^a k_b \qquad (2.6)$$

and

$$F \equiv k^a k_a = 0. \qquad (2.7)$$

Next we consider the following kinematical quantities associated with u^a:

$$v_I \equiv u^a_{;\,a} \qquad (2.8)$$

$$\omega^a \equiv \tfrac{1}{2}\eta^{abcd} u_b u_{c;\,d} \qquad (2.9)$$

$$-\omega_{ab} \equiv \tfrac{1}{2}(u_{a;\,b} - u_{b;\,a}) + \tfrac{1}{2}(\dot{u}_a u_b - \dot{u}_b u_a) \qquad (2.10)$$

$$\sigma_{ab} \equiv \tfrac{1}{2}(u_{a;\,b} + u_{b;\,a}) + \tfrac{1}{2}(\dot{u}_a u_b + \dot{u}_b u_a) - \frac{v_I}{3}(g_{ab} + u_a u_b) \qquad (2.11)$$

where

$$\dot{u}_a \equiv u_{a;\,b} u^b \qquad (2.12)$$

and η^{abcd} is the pseudo tensor density with $\eta^{1234} = \dfrac{1}{\sqrt{-g}}$.

Also the 'optical parameters' are

$$v_{II} \equiv \tfrac{1}{2}k^a_{;\,a} \qquad (2.13)$$

$$\omega \equiv \{\tfrac{1}{4}(k_{a;\,b} - k_{b;\,a})k^{a;\,b}\}^{\frac{1}{2}} \qquad (2.14)$$

$$\Sigma \equiv \tfrac{1}{2}k_{a;\,b}\bar{l}^a \bar{l}^b \qquad (2.15)$$

$$\Omega \equiv \tfrac{1}{2}k_{a;\,b}\bar{l}^a m^b \qquad (2.16)$$

where

$$G^a \equiv k^a_{;\,b} k^b = 0 \qquad (2.17)$$

$$l^a l_a = 0 \qquad (2.18)$$

$$\bar{l}^a \bar{l}_a = 0 \qquad (2.19)$$

$$m^a m_a = 0 \qquad (2.20)$$

$$l^a k_a = 0 \qquad (2.21)$$

$$\bar{l}^a k_a = 0 \qquad (2.22)$$

$$l^a m_a = 0 \qquad (2.23)$$

$$\bar{l}^a m_a = 0 \qquad (2.24)$$

$$l^a \bar{l}_a - 1 = 0 \qquad (2.25)$$

$$k^a m_a - 1 = 0 \qquad (2.26)$$

and

$$H_{ab} \equiv g_{ab} - (l_a \bar{l}_b + l_b \bar{l}_a) - (k_a m_b + k_b m_a) = 0. \qquad (2.27)$$

Equation (2.1) is the 'equation of continuity' for matter, eqn. (2.2) correlates the cooling rate of unit volume of matter with the rate of decrease of internal energy due to neutrino emission and eqn. (2.3) represents the space-time structure in terms of total stress-energy. Here n is the baryon number density and c the cooling rate of unit amount of matter. The assumption of perfect fluid distribution form of the stress-energy tensor T_{ab} for matter is inherent in (2.5) while (2.6) gives the 'geometrical optics' form of the stress-energy tensor N_{ab} for neutrinos. Here ϵ, p and q are respectively the matter density, pressure and neutrino energy density.

The kinematical quantities (Witten 1962) for an observer following one of the curves of the time-like congruence and using Fermi-propagated axes are described as follows: v_I, ω_{ab}, σ_{ab} are respectively the velocity of expansion, rotation and shear of the neighbouring cloud of particles. The vector ω^a is the angular velocity in the infinitesimal rest-space of the observer. The 'optical parameters' (Witten 1962, p. 58) v_{II}, ω, Σ and Ω are called the expansion, twist, shear and rotation of the null congruence with respect to the observer. A semi-colon (;) preceding a suffix indicates covariant derivative. Here we have 37 equations:

$$1(A)+1(B)+10(D^a_b)+1(E)+1(F)+4(G^a)+9\{\text{eqns. } (2.18)\text{–}(2.26)\}+10(H_{ab})$$

in 35 unknowns:

$$1(n)+4(u^a)+1(c)+3(\epsilon, p, q)+16(l^a, \bar{l}^a, k^a, m^a)+10(g_{ab}).$$

3. The Case of Spherical Symmetry

We consider the spherically symmetric metric

$$ds^2 = -e^{2\alpha} dt^2 + e^{\beta} dr^2 + \gamma^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \qquad (3.1)$$

where α, β, γ are functions of r and t. The coordinates r, θ, ϕ and t will be denoted by x^1, x^2, x^3 and x^4 respectively.

$$u^a = 0 \quad \text{for } a = 1, 2, 3; \quad u^4 = e^{-\alpha}. \qquad (3.2)$$

Next we choose the orthonormal tetrad given by

$$\{\lambda^a_{b|}\} = \text{diag}\{\sqrt{g^{11}}, \sqrt{g^{22}}, \sqrt{g^{33}}, \sqrt{-g^{44}}\}$$

and pseudo-orthonormal tetrad (l^a, $\bar{l}^a$, k^a, m^a) is defined as

$$\left.\begin{aligned} l^a &= \frac{1}{\sqrt{2}}(\lambda^a_{2|}+i\lambda^a_{3|}), \quad \bar{l}^a = \frac{1}{\sqrt{2}}(\lambda^a_{2|}-i\lambda^a_{3|}) \\ k^a &= \frac{1}{\sqrt{2}}(\lambda^a_{1|}+\lambda^a_{4|}), \quad m^a = \frac{1}{\sqrt{2}}(\lambda^a_{1|}-\lambda^a_{4|}) \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (3.3)$$

where a suffix followed by a vertical stroke indicates the particular vector. Equation (2.27) can equivalently be written as

$$H_{ab} \equiv g_{ab} - \sum_{\nu=1,2,3} \lambda_{\nu|a}\lambda_{\nu|b} + \lambda_{4|a}\lambda_{4|b} = 0.$$

For the metric (3.1) the non-vanishing components of (l^a, $\bar{l}^a$, k^a, m^a) are

$$\left.\begin{aligned} l_2 &= l_3/i \sin\theta = \bar{l}_2 = -\bar{l}_3/i \sin\theta = \frac{\gamma}{\sqrt{2}} \\ k_1 &= m_1 = \frac{e^{\beta/2}}{\sqrt{2}}, \quad k_4 = -m_4 = -\frac{e^{\alpha}}{\sqrt{2}} \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (3.4)$$

and due to (3.2) and (3.4) eqns. (2.1) and (2.2) reduce to

$$n = F(r)\frac{e^{-\beta/2}}{\gamma^2} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.5)$$

$$c = e^{-\alpha}(\epsilon+p)\frac{n_{,4}}{n} - e^{-\alpha}\frac{\epsilon_{,4}}{n} \qquad \ldots \quad \ldots \quad \ldots \quad (3.6)$$

where $F(r)$ is an arbitrary function and $n_{,4} \equiv \dfrac{\partial n}{\partial t}$.

The non-vanishing components of D_{ab} in (2.3) are

$$D_{14} \equiv -8\pi\frac{q}{2} + 2e^{-\alpha-\beta/2}\left(\frac{\gamma_{14}}{\gamma} - \frac{\gamma_1}{\gamma}\frac{\beta_4}{2} - \frac{\gamma_4}{\gamma}\alpha_1\right) = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (3.7)$$

$$D^1_1 \equiv -8\pi\left(p+\frac{q}{2}\right) - e^{-2\alpha}\left(\frac{2\gamma_{44}}{\gamma} - \frac{2\gamma_4}{\gamma}\alpha_4 + \frac{\gamma_4^2}{\gamma^2}\right) + e^{-\beta}\left(\frac{\gamma_1^2}{\gamma^2} + \frac{2\gamma_1}{\gamma}\alpha_1\right) - \frac{1}{\gamma^2} = 0 \qquad \ldots \quad (3.8)$$

$$D^2_2 = D^3_3 \equiv -8\pi p - e^{-2\alpha}\left(\frac{\gamma_{44}}{\gamma} - \frac{\gamma_4}{\gamma}\alpha_4 + \frac{\gamma_4}{\gamma}\frac{\beta_4}{2} + \frac{\beta_{44}}{2} + \frac{\beta_4^2}{4} - \alpha_4\frac{\beta_4}{2}\right)$$
$$-e^{-\beta}\left(-\frac{\gamma_{11}}{\gamma} + \frac{\gamma_1}{\gamma}\frac{\beta_1}{2} - \frac{\gamma_1}{\gamma}\alpha_1 + \alpha_1\frac{\beta_1}{2} - \alpha_1^2 - \alpha_{11}\right) = 0 \quad \ldots \quad (3.9)$$

$$D^4_4 \equiv 8\pi\left(\epsilon+\frac{q}{2}\right) - e^{-2\alpha}\left(\frac{\gamma_4^2}{\gamma^2} + \frac{\gamma_4}{\gamma}\beta_4\right) - e^{-\beta}\left(-\frac{2\gamma_{11}}{\gamma} - \frac{\gamma_1^2}{\gamma^2} + \frac{\gamma_1}{\gamma}\beta_1\right) - \frac{1}{\gamma^2} = 0 \qquad \ldots \quad (3.10)$$

where

$$\gamma_1 \equiv \frac{\partial\gamma}{\partial r}, \quad \gamma_4 \equiv \frac{\partial\gamma}{\partial t}, \text{ etc.}$$

Equation (2.17) reduces to

$$e^{\beta/2}\frac{\beta_4}{2} + e^{\alpha}\alpha_1 = 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.11)$$

Here we observe that for spherical symmetry ω^a, ω_{ab} and Σ vanish identically, whereas ω and Ω vanish due to (3.11). Consequently, the congruences associated with u^a and k^a are necessarily normal. The non-vanishing kinematical quantities and parameters are given by

$$v_I = e^{-\alpha}\left(\frac{\beta_4}{2}+\frac{2\gamma_4}{\gamma}\right) \quad \ldots \quad (3.12)$$

$$\sigma_{11} = -\frac{2e^{\beta}}{\gamma^2}\sigma_{22} = -\frac{2e^{\beta}}{\gamma^2}\frac{\sigma_{33}}{\sin^2\theta} = \frac{e^{\beta-\alpha}}{3}\left(\beta_4-\frac{2\gamma_4}{\gamma}\right) \quad \ldots \quad (3.13)$$

$$\dot{u}_1 = \alpha_1, \quad \dot{u}_2 = \dot{u}_3 = \dot{u}_4 = 0 \quad \ldots \quad (3.14)$$

$$v_{II} = \frac{1}{\sqrt{2}}\left(e^{-\beta/2}\frac{\gamma_1}{\gamma}+e^{-\alpha}\frac{\gamma_4}{\gamma}\right). \quad \ldots \quad (3.15)$$

Thus we have seven equations, viz. (3.5) to (3.11) in eight unknowns, viz. n, c, ϵ, p, q, α, β, γ, along with v_I, v_{II}, σ_{11}, $\dot{u}_1$. In the next section we study the consequences when some of the quantities v_I, v_{II}, σ_{11} and $\dot{u}_1$ vanish.

4. Special Cases

The eqns. (3.7), (3.8), (3.9) imply

$$-2e^{-\alpha-\beta/2}\left(\frac{\gamma_{14}}{\gamma}-\frac{\gamma_1}{\gamma}\frac{\beta_4}{2}-\frac{\gamma_4}{\gamma}\alpha_1\right)+e^{-2\alpha}\left(-\frac{\gamma_{44}}{\gamma}+\frac{\gamma_4}{\gamma}\alpha_4+\frac{\gamma_4}{\gamma}\frac{\beta_4}{2}-\frac{\gamma_4^2}{\gamma^2}+\frac{\beta_{44}}{2}+\frac{\beta_4^2}{4}-\alpha_4\frac{\beta_4}{3}\right)$$
$$+e^{-\beta}\left(-\frac{\gamma_{11}}{\gamma}+\frac{\gamma_1}{\gamma}\frac{\beta_1}{2}+\frac{\gamma_1}{\gamma}\alpha_1+\frac{\gamma_1^2}{\gamma^2}+\alpha_1\frac{\beta_1}{2}-\alpha_1^2-\alpha_{11}\right)-\frac{1}{\gamma^2}=0. \quad (4.1)$$

Now we consider the following cases:

Case (i)—$v_I = \sigma_{11} = 0$.

Equations (3.12) and (3.13) give

$$\frac{\beta_4}{2}+\frac{2\gamma_4}{\gamma}=0$$

$$\beta_4-\frac{2\gamma_4}{\gamma}=0$$

which imply

$$\beta_4 = \gamma_4 = 0$$

and hence from (3.7) we get

$$q = 0.$$

Since $e^{-\alpha}\gamma_4$ is the velocity of the fluid surface, $\gamma_4 = 0$ implies no collapse while $q = 0$ implies absence of neutrinos. Hence expansion-free and shear-free (or rigid) congruences associated with matter field imply no spherical gravitational collapse with escaping neutrinos.

Case (ii)—$\dot{u}_1 = \sigma_{11} = 0$.

Equations (3.14), (3.13), (3.11) and (3.7) imply

$$\alpha_1 = \beta_4 = \gamma_4 = q = 0.$$

Hence geodetic shear-free congruences associated with matter field do not permit spherical gravitational collapse with escaping neutrinos.

Combining cases (i) and (ii) with $\omega^a \equiv 0$ we have:

Spherical gravitational collapse with escaping neutrinos does not admit a space-time which is the direct product of a time-like line and a 3-space (Witten 1962, p. 58).

Case (iii)—$\dot{u}_1 = v_I = 0$.

Equations (3.14), (3.12), (3.11) and (3.7) imply

$$\alpha_1 = \beta_4 = \gamma_4 = q = 0.$$

Hence geodetic expansion-free congruences associated with matter field imply no spherical gravitational collapse with escaping neutrinos.

Case (iv)—$v_{II} = 0$.

Equation (3.15) gives

$$e^{-\beta/2}\gamma_1 + e^{-\alpha}\gamma_4 = 0. \qquad (4.2)$$

From (4.1), (4.2) and (3.9) we get

$$e^{-2\alpha}\left(\frac{\beta_{44}}{2} - \alpha_4\frac{\beta_4}{2}\right) + e^{-\beta}\left(\alpha_1\frac{\beta_1}{2} - \beta_{11}\right) = \frac{1}{\gamma^2} \qquad (4.3)$$

and

$$-8\pi p = e^{-2\alpha}\left(\frac{\gamma_{44}}{\gamma} - \frac{\gamma_4}{\gamma}\alpha_4\right) + e^{-\beta}\left(-\frac{\gamma_{11}}{\gamma} + \frac{\gamma_1}{\gamma}\frac{\beta_1}{2}\right) + \frac{1}{\gamma^2}. \qquad (4.4)$$

Equations (4.2), (3.7) and (3.10) give

$$8\pi\epsilon = e^{-2\alpha}\left(\frac{\gamma_{44}}{\gamma} - \frac{\gamma_4}{\gamma}\alpha_4\right) + e^{-\beta}\left(-\frac{\gamma_{11}}{\gamma} + \frac{\gamma_1}{\gamma}\frac{\beta_1}{2}\right) + \frac{1}{\gamma^2} \qquad (4.5)$$

which along with (4.4) implies

$$\epsilon = -p.$$

Since ϵ and p both should be positive we have

$$\epsilon = p = 0;$$

then field equations (3.7) and (3.8) give

$$-2e^{-\alpha-\beta/2}\left(\frac{\gamma_{14}}{\gamma} - \frac{\gamma_1}{\gamma}\frac{\beta_4}{2} - \frac{\gamma_4}{\gamma}\alpha_1\right) = e^{-2\alpha}\left(\frac{2\gamma_{44}}{\gamma} - \frac{2\gamma_4}{\gamma}\alpha_4\right) - 2e^{-\beta}\frac{\gamma_1}{\gamma}\alpha_1 + \frac{1}{\gamma^2}$$

which by virtue of (4.2) reduces to

$$\frac{1}{\gamma^2} = 0.$$

This is not compatible with the metric. Hence geodetic expansion-free null congruences associated with neutrino emission imply no spherical gravitational collapse with escaping neutrinos.

Case (*v*)—$\dot{u}_1 = 0$.

Equation (3.14) gives

$$\alpha_1 = 0$$

and (3.11) gives

$$\beta_4 = 0.$$

Then (3.1) can be written as

$$ds^2 = -f(t)\,dt^2 + g(r)\,dr^2 + \gamma^2(r,t)(d\theta^2 + \sin^2\theta\,d\phi^2)$$

which can be transformed to

$$ds^2 = -dt^2 + dr^2 + \gamma^2(d\theta^2 + \sin^2\theta\,d\phi^2). \quad \ldots \quad \ldots \quad (4.6)$$

For this metric we have

$$u^1 = u^2 = u^3 = 0, \quad u^4 = 1$$

$$l_1 = l_4 = \bar{l}_1 = \bar{l}_4 = k_2 = k_3 = m_2 = m_3 = 0$$

$$l_2 = \frac{l_3}{i\sin\theta} = \bar{l}_2 = -\frac{\bar{l}_3}{i\sin\theta} = \frac{\gamma}{\sqrt{2}}, \quad k_1 = -k_4 = m_1 = m_4 = \frac{1}{\sqrt{2}}$$

$$n = \frac{F(r)}{\gamma^2}, \quad c = \frac{1}{4\pi F}\{\gamma_{14}(\gamma_1 + \gamma_4) + \gamma(\gamma_{114} + \gamma_{144})\}$$

$$2\pi q = \frac{\gamma_{14}}{\gamma} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.6a)$$

$$-8\pi\left(p + \frac{q}{2}\right) = \frac{2\gamma_{44}}{\gamma} + \frac{\gamma_4^2}{\gamma^2} - \frac{\gamma_1^2}{\gamma^2} + \frac{1}{\gamma^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.6b)$$

$$-8\pi p = \frac{\gamma_{44}}{\gamma} - \frac{\gamma_{11}}{\gamma} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.6c)$$

$$8\pi\left(\epsilon + \frac{q}{2}\right) = -\frac{2\gamma_{11}}{\gamma} - \frac{\gamma_1^2}{\gamma^2} + \frac{\gamma_4^2}{\gamma^2} + \frac{1}{\gamma^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.6d)$$

$$v_I = 2\gamma_4/\gamma$$

$$\sigma_{11} = -2\sigma_{22}/\gamma^2 = -2\sigma_{33}/\gamma^2\sin^2\theta = -2\gamma_4/3\gamma$$

$$v_{II} = \frac{1}{\sqrt{2}}\left(\frac{\gamma_1}{\gamma} + \frac{\gamma_4}{\gamma}\right)$$

and (4.1) reduces to

$$\gamma(\gamma_{11} + 2\gamma_{14} + \gamma_{44}) + \gamma_4^2 - \gamma_1^2 + 1 = 0. \quad \ldots \quad \ldots \quad (4.7)$$

One solution of (4.7) for which neutrino emission is possible is

$$\gamma = f(lr + mt)$$

where l, m are constants and function f satisfies the equation

$$(l+m)^2 ff'' + (m^2 - l^2)f'^2 + 1 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4.8)$$

where

$$f' = \frac{df}{du}, \quad u = lr + mt.$$

For this value of γ, q, p and ϵ are given by

$$2\pi q = lmf''/f$$

$$8\pi p = (l^2-m^2)f''/f$$

$$8\pi\epsilon = -2l^2\frac{f''}{f}-2lm\frac{f''}{f}+(m^2-l^2)\frac{f'^2}{f^2}+\frac{1}{f^2}.$$

If $p = 0$, which implies a distribution of discrete particles, we have from (4.6c)

$$\gamma_{44}-\gamma_{11} = 0$$

for which the most general solution is

$$\gamma = f(r+t)+g(r-t).$$

Equation (4.7) reduces to

$$4f''(f+g)-4f'\dot{g}+1 = 0 \quad \quad .. \quad .. \quad .. \quad (4.9)$$

where

$$f' \equiv \frac{df}{d(r+t)}, \quad \dot{g} \equiv \frac{dg}{d(r+t)}.$$

Differentiating (4.9) with respect to $(r-t)$, we get

$$f''\dot{g}-f'\ddot{g} = 0$$

or

$$\frac{f''}{f'} = \frac{\ddot{g}}{\dot{g}} = k \text{ (constant)}.$$

Then

$$f = \frac{e^{\{k(r+t)+k_1\}}}{k}+k_2, \quad g = \frac{e^{\{k(r-t)+k_3\}}}{k}+k_4.$$

Equation (4.9) becomes

$$4e^{2\{k(r+t)+k_1\}}+4(k_2+k_4)e^{\{k(r+t)+k_1\}}+1 = 0$$

which implies $\exp\{k(r+t)+k_1\} = $ constant.
The above equation on further examination leads to the conclusion that discrete particle distribution with geodetic congruence associated with matter is not consistent with spherical gravitational collapse with escaping neutrinos.

In particular, when $g = 0$, ϵ and q can be calculated directly from (4.6*a*, *d*). In this case it is found that q turns out to be negative.

Case (vi)—$\sigma_{11} = 0$.
Equation (3.13) gives

$$\beta_4 = \frac{2\gamma_4}{\gamma}$$

which on integration gives

$$e^{\beta/2} = g(r)\gamma(r, t).$$

Now (3.1) becomes

$$ds^2 = -e^{2\alpha}\,dt^2+\gamma^2\{g^2\,dr^2+d\theta^2+\sin^2\theta\,d\phi^2\}$$

which can be written as

$$ds^2 = -e^{2\alpha}\,dt^2+\gamma^2\{dr^2+d\theta^2+\sin^2\theta\,d\phi^2\}. \qquad (4.10)$$

For this metric we have

$$u^1 = u^2 = u^3 = 0, \quad u^4 = e^{-\alpha}$$

$$l_1 = l_4 = \bar{l}_1 = \bar{l}_4 = k_2 = k_3 = m_2 = m_3 = 0$$

$$l_2 = \frac{l_3}{i\sin\theta} = \bar{l}_2 = -\frac{\bar{l}_3}{i\sin\theta} = k_1 = m_1 = \frac{\gamma}{\sqrt{2}}$$

$$k_4 = -m_4 = -\frac{e^{\alpha}}{\sqrt{2}}$$

$$n = \frac{F(r)}{\gamma^3}$$

$$c = \frac{e^{-\alpha}}{8\pi F}\Big\{\gamma_4\Big(-1+4e^{-2\alpha}\gamma\gamma_{44}-6e^{-2\alpha}\alpha_4\gamma\gamma_4+3e^{-\alpha}\gamma_{14}-2e^{-3\alpha}\gamma\gamma_4^2+2e^{-2\alpha}\gamma_4^2-\frac{3\gamma_{11}}{\gamma}+\frac{4\gamma_1^2}{\gamma^2}-2e^{-2\alpha}\gamma_1\gamma_4+2e^{-2\alpha}\gamma\gamma_{14}\Big)+2\gamma_{114}-\frac{2\gamma_1\gamma_{14}}{\gamma}+2e^{-\alpha}\gamma\gamma_{144}-2e^{-\alpha}\gamma_1\gamma_{44}\Big\}$$

$$-8\pi\frac{q}{2} = \frac{2e^{-\alpha}}{\gamma}\left(-\frac{\gamma_{14}}{\gamma}+\frac{\gamma_1\gamma_4}{\gamma^2}-e^{-\alpha}\frac{\gamma_4^2}{\gamma}\right)$$

$$-8\pi\left(p+\frac{q}{2}\right) = e^{-2\alpha}\left(\frac{2\gamma_{44}}{\gamma}+\frac{\gamma_4^2}{\gamma^2}-\frac{2\gamma_4}{\gamma}\alpha_4\right)-\frac{1}{\gamma^2}\left(\frac{\gamma_1^2}{\gamma^2}-2\frac{\gamma_1\gamma_4}{\gamma}e^{-\alpha}\right)+\frac{1}{\gamma^2}$$

$$-8\pi p = e^{-2\alpha}\left(\frac{2\gamma_{44}}{\gamma}+\frac{\gamma_4^2}{\gamma^2}-\frac{2\gamma_4}{\gamma}\alpha_4\right)-\frac{1}{\gamma^2}\left(-\frac{\gamma_{11}}{\gamma}+\frac{\gamma_1^2}{\gamma^2}+e^{-\alpha}\gamma_{14}\right)$$

$$8\pi\left(\epsilon+\frac{q}{2}\right) = 3e^{-2\alpha}\frac{\gamma_4^2}{\gamma^2}+\frac{1}{\gamma^2}\left(-\frac{2\gamma_{11}}{\gamma}+\frac{\gamma_1^2}{\gamma^2}\right)+\frac{1}{\gamma^2}$$

$$v_I = 3e^{-\alpha}\gamma_4/\gamma$$

$$v_{II} = \frac{1}{\sqrt{2}}\left(\frac{\gamma_1}{\gamma^2}+e^{-\alpha}\frac{\gamma_4}{\gamma}\right)$$

and (3.11) and (4.1) become

$$\gamma_4+e^{\alpha}\alpha_1 = 0 \qquad (4.11)$$

$$e^{-\alpha}\gamma_{14}+2e^{-2\alpha}\gamma_4^2+\frac{\gamma_{11}}{\gamma}-\frac{2\gamma_1^2}{\gamma^2}+1 = 0. \qquad (4.12)$$

One solution of (4.11) is

$$\gamma = -F'f, \quad e^{\alpha} = F\dot{f}$$

where

$$F \equiv F(r), \quad f \equiv f(t), \quad F' \equiv \frac{dF}{dr}, \quad \dot{f} \equiv \frac{df}{dt}.$$

Equation (4.12) reduces to

$$\frac{F'''}{F'}-\frac{2F''^2}{F'^2}-\frac{F''}{F}+\frac{2F'^2}{F^2}+1=0$$

and

$$2\pi q=\frac{1}{F^2f^2}$$

$$8\pi p=-\frac{1}{F^2f^2}+\frac{1}{F'^2f^2}\left(\frac{F'''}{F'}-\frac{F''^2}{F'^2}\right)+\frac{F''}{f^2FF'^2}$$

$$8\pi\epsilon=\frac{1}{F^2f^2}+\frac{1}{F'^2f^2}\left(-\frac{2F'''}{F'}+\frac{F''^2}{F'^2}\right)+\frac{1}{f^2F'^2}$$

which demonstrates the possibility of neutrino emission in this case.

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