COUPLED VIBRATIONS OF A SLENDER BEAM IN A CENTRIFUGAL FORCE FIELD

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The differential equations for the coupled bending and torsional vibrations of a slender beam in a centrifugal force field are obtained. A method based on the Rayleigh's quotient is used to obtain an upper bound for the fundamental frequency, and the accuracy of this upper bound is investigated by considering the application of the method to uncoupled vibration.

Notation

x = radial co-ordinate

Y = tangential co-ordinate

z = axial co-ordinate

v = deflection of elastic axis in Y direction

e = axis of minor moment of inertia I min

v' = deflection perpendicular to axis e

l = length of the blade in radial direction

m = mass per unit length of the beam

E = modulus of elasticity

I = moment of inertia of the cross-section about the axis perpendicular to the plane of bending

G = shear modulus of rigidity

J = a constant depending upon the cross-section of the beam such that GJ is the torsional rigidity

δ = distance between the elastic axis and the centroidal axis

θ = rotation of the beam cross-section about the elastic axis

S = area of cross-section of the beam

I_θ = mass moment of inertia about the elastic axis per unit length

t = time

M = moment of centrifugal force at the point x

ξ = dimensionless variable $\xi = x/l$

ω = frequency of vibration

Introduction

The analysis presented in this paper considers vibration of a slender beam that could represent a turbine blade of simple geometry. The beam is attached to a disc of radius r_0 as indicated in Fig. 1 and the disc rotates with angular velocity Ω. The shear centre of each cross-section does not coincide with the centre of gravity, consequently the torsional and bending oscillations are 'coupled'.

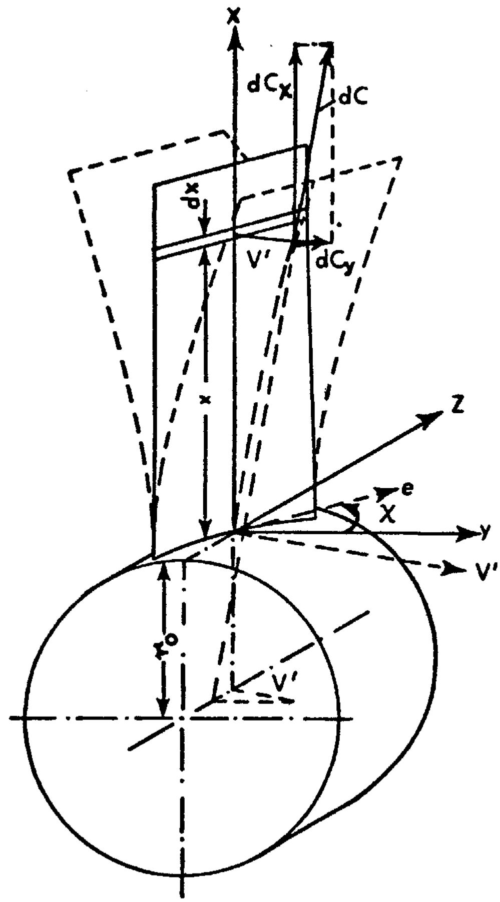

FIG. 1. Cantilever beam attached to a circular disc.

For the analysis it is assumed that S, I and J are constants. This, however, is not an essential assumption. If S, I and J are functions of x, the amount of computing work is greater than for S, I and J independent of x.

THE DIFFERENTIAL EQUATIONS

The differential equation for the deflected form of the neutral axis of a bar according to the elementary theory of bending is

$$EI\frac{\partial^4 v}{\partial x^4} = w \qquad \text{(1)}$$

where w is the intensity of the distributed load.

If the load is distributed along the centroidal axis, the given load can be replaced by the same load distributed along the shear-centre axis, and a torque of intensity $w\,.\,\delta$ distributed along the same axis.

Let the x-axis coincide with the shear-centre axis. Since the torsion is not uniform, the relation between the variable torque T and the angle of twist θ is given by (Timoshenko 1955)

$$T = GJ\frac{\partial\theta}{\partial x} - C_1\frac{\partial^3\theta}{\partial x^3}$$

where GJ is the torsional rigidity for uniform torsion and C_1 is the warping rigidity. Differentiation of this equation with respect to x gives

$$GJ\frac{\partial^2\theta}{\partial x^2} - C_1\frac{\partial^4\theta}{\theta x^4} = w\,.\,\delta \quad \quad (2)$$

with the positive torque taken as shown in Fig. 2.

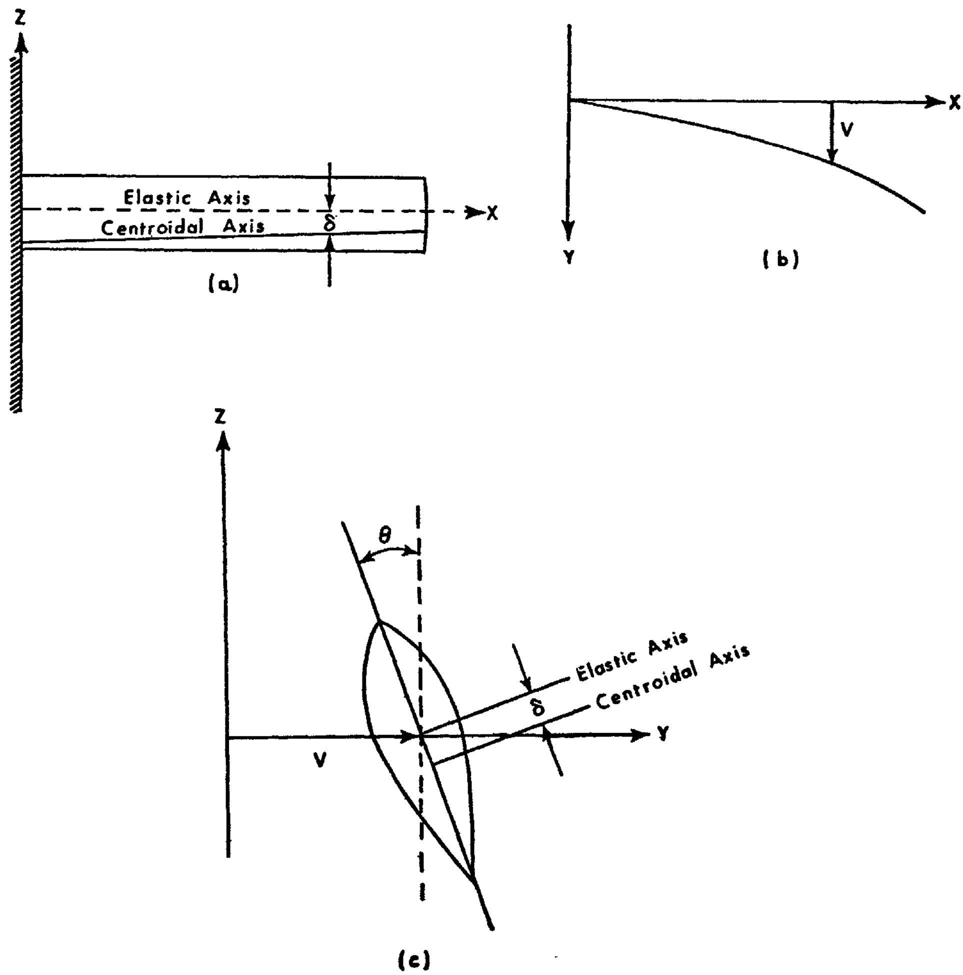

FIG. 2. Deflection and rotation of a cantilever beam from and about elastic axis.

For a vibrating bar the intensity of the inertia force is

$$-m\frac{\partial^2}{\partial t^2}(v+\delta\,.\,\theta)$$

and the intensity of the inertia moment about the x-axis is

$$-I_\theta\frac{\partial^2\theta}{\partial t^2}$$

The following differential equations for the coupled bending and torsional vibrations are obtained by replacing the statical load in eqns. (1) and (2) by the inertia forces,

$$EI\frac{\partial^4 v}{\partial x^4} = -m\frac{\partial^2}{\partial t^2}(v+\delta\,.\,\theta) \qquad \ldots \quad \ldots \quad \ldots \quad (3a)$$

$$GJ\frac{\partial^2\theta}{\partial x^2}-C_1\frac{\partial^4\theta}{\partial x^4} = m\,.\,\delta\frac{\partial^2}{\partial t^2}(v+\delta\,.\,\theta)+I_\theta\frac{\partial^2\theta}{\partial t^2} \qquad \ldots \quad \ldots \quad (3b)$$

Referring to Fig. 1 the centrifugal force dc exerted by a mass element of the radial length $d\eta$ at $x = \eta$ is

$$dc = m\Omega^2\,(r_0+\eta)\,d\eta$$

If the deflection of the blade at time t is $v'(x, t)$, the tangential and radial components of dc are

$$dc_Y = dc\frac{v'\sin\chi}{r_0+\eta}$$

and

$$dc_x = dc\left[1-\left(\frac{v'\sin\chi}{r_0+\eta}\right)^2\right]^{\frac{1}{2}}$$

respectively, where χ is the angle between Y and e axes.

If it is assumed that $\sin\chi \ll 1$, $dc_x = dc$. The components c_Y and c_x applied at x follow by integrating from $\eta = x$ to $\eta = l$ and are

$$c_x = m\Omega^2[r_0(l-x)+\tfrac{1}{2}(l^2-x^2)] \qquad \ldots \quad \ldots \quad \ldots \quad (4)$$

$$c_Y = m\Omega^2\sin\chi\int_x^l v'(x, t)\,d\eta \qquad \ldots \quad \ldots \quad \ldots \quad (5)$$

Referring to Fig. 3 the moment dM about the minor principal axis of inertia is

$$dM = c_x dv'(x, t)-c_Y\sin\chi\,dx \qquad \ldots \quad \ldots \quad \ldots \quad (6)$$

Eqns. (4)–(6) give

$$\frac{\partial^2 M}{\partial x^2} = m\Omega^2\left[\left\{r_0(l-x)+\tfrac{1}{2}(l^2-x^2)\frac{\partial^2 v'}{\partial x^2}-(r_0+x)\frac{\partial v'}{\partial x}+\sin^2\chi v'\right\}\right]$$

If χ is a small angle and $v' = v$ this may be written as

$$\frac{\partial^2 M}{\partial x^2} = m\Omega^2\left[r_0(l-x)+\tfrac{1}{2}(l^2-x^2)\frac{\partial^2 v}{\partial x^2}-(r_0+x)\frac{\partial v}{\partial x}\right] \qquad \ldots \quad \ldots \quad (7)$$

The second derivative of the moment M with respect to x can be considered as a lateral load depending upon the centrifugal effect.

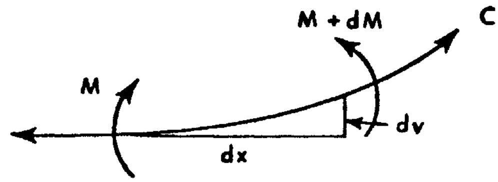

FIG. 3. Showing moment about the minor principal axis of inertia.

The R.H.S. of eqn. (3a) represents an inertia loading, and eqn. (3a) must be modified by adding the lateral load, $\frac{\partial^2 M}{\partial x^2}$, if the centrifugal force effect is to be considered and the governing equations then become

$$\left.\begin{aligned} EI\frac{\partial^4 v}{\partial x^4} &= -m\frac{\partial^2}{\partial t^2}(v+\delta\,.\,\theta)+\frac{\partial^2 M}{\partial x^2} \\ GJ\frac{\partial^2\theta}{\partial x^2}-C_1\frac{\partial^4\theta}{\partial x^4} &= m\,.\,\delta\frac{\partial^2}{\partial t^2}(v+\delta\,.\,\theta)+I_\theta\frac{\partial^2\theta}{\partial t^2} \end{aligned}\right\} \quad \ldots \quad (8)$$

where $\frac{\partial^2 M}{\partial x^2}$ is given by eqn. (7).

DETERMINATION OF NATURAL FREQUENCIES

The equations are now put in terms of a dimensionless variable $\xi = x/l$ and the substitutions,

$$\beta^2 = \frac{EI}{ml^4},\ \gamma^2 = \frac{GJ}{ml^2},\ C_2 = \frac{C_1}{ml^4},\ \text{and } I_\theta' = \frac{I_\theta}{m}.$$

are used.

Eqns. (8) become

$$\left.\begin{aligned} &\beta^2\frac{\partial^4 v}{\partial\xi^4}-\Omega^2\left[\left(\frac{r_0}{l}+\frac{1}{2}-\frac{r_0}{l}\xi-\frac{1}{2}\xi^2\right)\frac{\partial^2 v}{\partial\xi^2}-\left(\frac{r_0}{l}+\xi\right)\frac{\partial v}{\partial\xi}\right] \\ &\qquad+\frac{\partial^2 v}{\partial t^2}+\delta\frac{\partial^2\theta}{\partial t^2}=0 \\ &\gamma^2\frac{\partial^2\theta}{\partial\xi^2}-C_2\frac{\partial^4\theta}{\partial\xi^4}-\delta\frac{\partial^2 v}{\partial t^2}-(\delta^2+I_\theta')\frac{\partial^2\theta}{\partial t^2}=0 \end{aligned}\right\} \quad \ldots \quad (9)$$

The solutions of eqns. (9) are of the form:

$$\left.\begin{aligned} v(\xi, t) &= Af(\xi)e^{i\omega t} \\ \theta(\xi, t) &= B\phi(\xi)e^{i\omega t} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

where A and B are constants which are not independent; and $f(\xi)$ and $\phi(\xi)$ are functions of ξ only.

The functions $f(\xi)$ and $\phi(\xi)$ satisfy all the boundary conditions of the beam which are as follows:

$$\left.\begin{aligned} v(0) = \frac{\partial v}{\partial \xi}(0) = \theta(0) = \frac{\partial^2 \theta}{\partial \xi^2}(0) = 0 \\ \frac{\partial^2 v}{\partial \xi^2}(1) = \frac{\partial^3 v}{\partial \xi^3}(1) = \frac{\partial \theta}{\partial \xi}(1) = \frac{\partial^3 \theta}{\partial \xi^3}(1) = 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (11)$$

For an approximate determination of the fundamental frequency $f(\xi)$ is chosen as the shape function for the fundamental mode of uncoupled bending vibration and $\phi(\xi)$ as the shape function for the fundamental mode of uncoupled torsional vibration of a uniform cantilever beam. These shape functions satisfy the boundary conditions (11) and are

$$\left.\begin{aligned} f(\xi) &= \cosh \lambda\xi - \cos \lambda\xi - 0{\cdot}73410(\sinh \lambda\xi - \sin \lambda\xi) \\ \text{and} \quad \phi(\xi) &= \sin \frac{\pi}{2}\xi \end{aligned}\right\} \quad \ldots \quad \ldots \quad (12)$$

where $\lambda = 1{\cdot}87510$.

Substitution of eqns. (10) in eqns. (9) gives:

$$\left.\begin{aligned} &\left[\beta^2 \frac{d^4 f}{d\xi^4} - \Omega^2\left(\frac{r_0}{l} + \frac{1}{2} - \frac{r_0}{l}\xi - \frac{1}{2}\xi^2\right)\frac{d^2 f}{d\xi^2} + \Omega^2\left(\frac{r_0}{l} + \xi\right)\frac{df}{d\xi} - \omega^2 f\right] A - \delta\omega^2\phi B = 0 \\ &\delta\omega^2 f.A + \left[\gamma^2 \frac{d^2\phi}{d\xi^2} - C_2 \frac{d^4\phi}{d\xi^4} + (\delta^2 + I'_\theta)\omega^2\phi\right] B = 0 \end{aligned}\right\} \quad (13)$$

Equations (13) can be solved for ω^2 but the result is a function of ξ, since f and ϕ are not the exact shape function. This difficulty can be overcome (Fung 1955) by multiplying the first and the second of eqns. (13) by f and ϕ respectively, and integrating with respect to ξ from 0 to 1. The method results in the familiar Rayleigh quotient (Collatz 1960) when applied to uncoupled problems, and is an extension of Rayleigh's method to the coupled problem. The following equations are obtained:

$$\left.\begin{aligned} (a_1 - a_2 + a_3 - a_4\omega^2)A - a_5\omega^2 B = 0 \\ -a_6\omega^2 A + (a_7 + a_8 - a_9\omega^2)B = 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

where

$$a_1 = \beta^2 \int_0^1 \frac{d^4 f}{d\xi^4} f.d\xi = \beta^2 \int_0^1 \left(\frac{d^2 f}{d\xi^2}\right) d\xi$$

$$a_2 = \Omega^2 \int_0^1 \left(\frac{r_0}{l} + \frac{1}{2} - \frac{r_0}{l}\xi - \frac{1}{2}\xi^2\right)\frac{d^2 f}{d\xi^2} \cdot f \cdot d\xi$$

$$a_3 = \Omega^2 \int_0^1 \left(\frac{r_0}{l} + \xi\right)\frac{df}{d\xi} \cdot f \cdot d\xi$$

$$a_4 = \int_0^1 f^2 d\xi$$

$$a_5 = a_6 = \delta \int_0^1 \phi . f . d\xi$$

$$a_7 = -\gamma^2 \int_0^1 \frac{d^2\phi}{d\xi^2} . \phi . d\xi = \gamma^2 \int_0^1 \left(\frac{d\phi}{d\xi}\right)^2 d\xi$$

$$a_8 = C_2 \int_0^1 \frac{d^4\phi}{d\xi^4} . \phi . d\xi$$

$$a_9 = (\delta^2 + I'_\theta) \int_0^1 \phi^2 d\xi$$

For a nontrivial solution A and B must not both vanish, consequently the determinant of the coefficients of eqns. (14) must be zero:

$$\begin{vmatrix} a_1 - a_2 + a_3 - a_4\omega^2 & -a_5\omega^2 \\ -a_6\omega^2 & a_7 + a_8 - a_9\omega^2 \end{vmatrix} = 0$$

i.e.

$$P\omega^4 - Q\omega^2 + R = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (15)$$

where

$$P = a_4a_9 - a_5a_6, \; Q = (a_1 - a_2 + a_3)a_9 + (a_7 + a_8)a_4$$

and

$$R = (a_1 - a_2 + a_3)(a_7 + a_8)$$

The solution of eqn. (15) is

$$\omega^2 = \frac{Q \pm \sqrt{(Q^2 - 4PR)}}{2P} \qquad \ldots \quad \ldots \quad \ldots \quad (16)$$

The right-hand side of eqn. (16) is positive since it may be shown that $Q^2 - 4PR > 0$.

The smaller of the two values of ω^2 given by eqn. (16) is an upper bound for the frequency of the fundamental mode of vibration. The larger of the two ω^2 values is an upper bound for the next higher mode of vibration.

Also, for the larger value of ω^2, from eqns. (14), A and B have opposite signs and for the smaller value of ω^2 they are of the same sign. The corresponding two configurations are shown in Fig. 4.

Numerical Example

A numerical example for the coupled torsional vibrations of a slender rotating beam is now presented. The frequencies are computed from eqn. (16), and the cross-section of the blade is taken as a semi-circle of radius a_0 and thickness t_1. The physical constants of the blade except the additional assumed value of δ are taken from Grammel (Biezeno and Grammel 1954).

$l = 4{\cdot}41$ in $\qquad E = 29{\cdot}20 \times 10^6$ lb/in^2

$S = 0{\cdot}1488$ in^2 $\qquad I = 0{\cdot}001845$ in^4

$a_0 = 0{\cdot}23885$ in $\qquad G = 11{\cdot}53 \times 10^6$ lb/in^2

$t_1 = 0{\cdot}19706$ in $\qquad J = 0{\cdot}00193$ in$^4 = \dfrac{At_1^2}{3}$

$m = 0{\cdot}00011$ lb $\qquad \delta = 0{\cdot}15$ in

$r_0 = 29{\cdot}10$ in $\qquad = 314$ sec^{-1}

$$I_\theta = 0{\cdot}14834\, m \text{ lb/in}^2 = m\left\{a_0^2 + \left(\frac{2a_0}{\pi} + 0{\cdot}15\right)^2\right\}$$

$$C_1 = 0{\cdot}00014 \times 10^6 \text{ lb/in}^4 = Ea_0{}^5 t_1 \left(\frac{\pi^3}{12} - \frac{8}{\pi}\right)$$

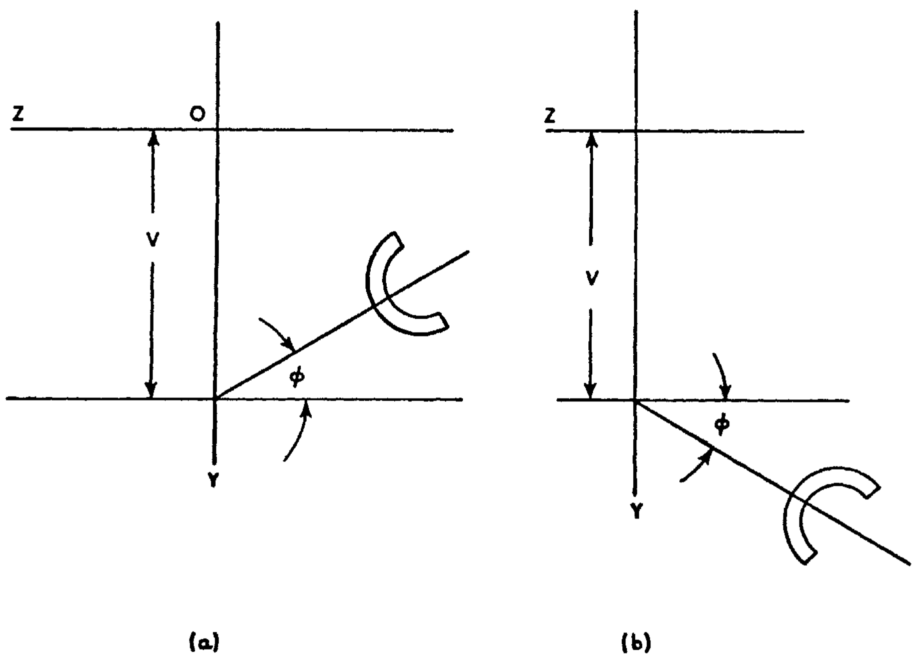

FIG. 4. Configuration corresponding to two values of ω^2.

With these values, the various parameters used in the frequency equation are as follows:

$a_1 = 16{\cdot}20711 \times 10^6 \qquad a_5 = a_6 = 0{\cdot}10151$

$a_2 = 0{\cdot}30724 \times 10^6 \qquad a_7 = 12{\cdot}75067 \times 10^6$

$a_3 = 1{\cdot}44909 \times 10^6 \qquad a_8 = 0{\cdot}01021 \times 10^6$

$a_4 = 1 \qquad a_9 = 0{\cdot}08542$

and from eqn. (16)

$$\omega_1^2 = 1{\cdot}70830 \times 10^7 \text{ sec}^{-2}$$

and

$$\omega_2^2 = 1{\cdot}72511 \times 10^8 \text{ sec}^{-2}$$

which are the upper bounds of the frequencies corresponding to the first two modes of the torsional coupled vibrations of the rotating blade.

Discussion and Concluding Remarks

For the uncoupled case (i.e. $\delta = 0$), eqns. (13) are two separate eigenvalue problems that satisfy the conditions of self-adjointness and full definiteness (Collatz 1960). The first of eqns. (13) when $\delta = 0$ is

$$\beta^2 \frac{d^4 f}{d\xi^4} - \Omega^2 \left\{ \left(\frac{r_0}{l} + \tfrac{1}{2} - \frac{r_0}{l}\xi - \tfrac{1}{2}\xi^2 \right) \frac{d^2 f}{d\xi^2} - \left(\frac{r_0}{l} + \xi \right) \frac{df}{d\xi} \right\} = \omega^2 f$$

with boundary conditions $f(0) = f'(0) = f''(1) = f'''(1) = 0$
This can be written as

$$U[f] = \omega^2 V[f]$$

where U and V are differential operators and the boundary conditions as

$$B[f] = 0.$$

A sequence of functions $f_1, f_2, f_3 \ldots$ can be obtained from an arbitrary f_0 and application of the boundary value problems

$$\begin{aligned} U[f_k] &= V[f_{k-1}] \\ B[f_k] &= 0. \end{aligned} \qquad k = 1, 2, 3 \ldots$$

If the ratio f_{k-1}/f_k tends to a constant as k increases, the sequence converges, and the function f_k converges to the form of the first eigen-function f. The quotient $U[f_k]/V[f_k]$ should provide an approximation to ω_s^2, where ω_s is first eigenvalue, but as it is still a function of ξ the numerator and denominator are each multiplied by f_k and integrated with respect to ξ from 0 to 1 to give Rayleigh's quotient which is an upper bound for ω_s^2, that is

$$\Lambda = \frac{\int_0^1 f_k U[f_k] d\xi}{\int_0^1 f_k V[f_k] d\xi} = \frac{\int_0^1 f_k V[f_{k-1}] d\xi}{\int_0^1 f_k V[f_k] d\xi}$$

The form of the integrals appearing in the above expressions for Λ may be denoted by

$$\left.\begin{aligned} b_{2k-1} &= \int_0^1 f_k V[f_{k-1}] d\xi \\ b_{2k} &= \int_0^1 f_k V[f_k] d\xi \\ b_{2k+1} &= \int_0^1 f_{k+1} V[f_k] d\xi \end{aligned}\right\} (k = 0, 1, 2, \ldots)$$

and are called Schwarz constants (Collatz 1960) and Λ is the quotient of two successive Schwarz constants

$$\Lambda = \mu_{2k} = \frac{b_{2k-1}}{b_{2k}}$$

The quotients $\mu_{2k-1} = b_{2k-2}/b_{2k-1}$, $\mu_{2k} = b_{2k-1}/b_{2k}$, $\mu_{2k+1} = b_{2k}/b_{2k+1}, \ldots$ are known as Schwarz's quotients (Collatz 1960) and the even numbered quotients are identical with Rayleigh's quotients.

Bounds for the first eigenvalue ω_s^2 can be obtained from the quotients μ_k, μ_{k+1} and l_2 where l_2 is a lower bound for the second eigenvalue and such that $l_2 > \mu_{k+1}$. The bounds are given by (Collatz 1960)

$$\mu_{k+1} - \frac{\mu_k - \mu_{k+1}}{\dfrac{l_2}{\mu_{k+1}} - 1} \leq \omega_s^2 \leq \mu_{k+1} \; (k = 1, 2, 3 \ldots)$$

For numerical computation of the quotients f_1 is taken as the shape function for free vibration of a uniform cantilever, that is

$$f_1 = \cosh 1{\cdot}8751\xi - \cos 1{\cdot}8751\xi - 0{\cdot}7341 \, (\sinh 1{\cdot}8751\xi - \sin 1{\cdot}8751\xi)$$

and is a non-zero function and satisfies all boundary conditions and possesses continuous derivatives.

Also, a function f_0 such that

$$U[f_1] = V[f_0]$$

or

$$\beta^2 \frac{d^4 f_1}{d\xi^4} - \Omega^2 \left\{ \left(\frac{r_0}{l} + \tfrac{1}{2} - \frac{r_0}{l}\xi - \tfrac{1}{2}\xi^2 \right) \frac{d^2 f_1}{d\xi^2} - \left(\frac{r_0}{l} + \xi \right) \frac{df_1}{d\xi} \right\} = f_0$$

can be readily obtained.

In this special eigenvalue problem with $V[f]$ the condition

$$\int_0^1 (f_0 V[f_1] - f_1 V[f_0]) \, d\xi = 0$$

is satisfied and according to Collatz (1960) f_0 need not satisfy any boundary conditions.

Using the same numerical values for the physical constants of the beam already considered but with $\delta = 0$ Schwarz constants and quotients are

$$b_0 = \int_0^1 f_0^2 \, d\xi = 3{\cdot}02437 \times 10^4 \qquad \mu_1 = \frac{b_0}{b_1} = 1{\cdot}74325 \times 10^7$$

$$b_1 = \int_0^1 f_0 f_1 \, d\xi = 1{\cdot}73490 \times 10^7 \qquad \mu_2 = \frac{b_1}{b_2} = 1{\cdot}73490 \times 10^7$$

$$b_2 = \int_0^1 f_1^2 \, d\xi = 1$$

and an upper bound is

$$\mu_1 \geq \mu_2 \geq \omega^2 \text{ i.e. } 1{\cdot}74325 \times 10^7 \geq 1{\cdot}73490 \times 10^7 \geq \omega^2$$

For calculating lower bound from the expression

$\mu_2 - \frac{\mu_1-\mu_2}{l_2/\mu_2-1} \leq \omega_s^2$, it is not essential to have a close lower bound l_2 for the second eigenvalue and even a rough value for l_2 is justified since changes in l_2 have little effect on the lower bound calculated from the above expression when l_2 is appreciably greater than μ_2.

Further, it is quite reasonable to assume that for flexural vibrations, the second eigenvalue for free vibration of the uniform beam is a good enough approximation for the lower bound of the second eigenvalue of the rotating cantilever beam. Hence, the calculated value for l_2 is $6{\cdot}36515\times 10^8$ sec^{-2}.

Thus, the frequency for fundamental mode of uncoupled vibration is bounded as

$$1{\cdot}73467\times 10^7 \text{ sec}^{-2} \leq \omega^2 \leq 1{\cdot}73490\times 10^7 \text{ sec}^{-2}$$

The above result shows that the method provided very narrow limits for fundamental frequency of uncoupled vibrations in the presence of a centrifugal force field.

The fundamental frequency for coupled vibrations could not be bounded at present, but the above computation indicates that Rayleigh's quotient obtained from the shape function f_1 is a very close upper bound for the first eigenvalue ω_s^2 for the uncoupled problem. Consequently, it seems reasonable to believe that the method results in a close upper bound for the fundamental frequency for the coupled problem, especially if δ is small.

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