ESCAPE OF PHOTONS FROM A ROTATING STAR

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In this note the motion of light rays in the stationary gravitational field of a rotating star is discussed. It has been shown that rotation drastically modifies the behaviour of the ray. However, for stars with small angular velocity the escape of ray is similar to a non-rotating case.

1. Introduction

Recently, interest in the problem of gravitational collapse has been revived by the discovery of quasi-stellar radio sources and the possibility of ascribing their enormous energy output to gravitational origin. Under continued gravitational collapse the energy of the star is given away in the form of electromagnetic radiation. It is interesting to study the effect of the gravitational field of the collapsing matter on the escaping radiation. The effect of the gravitational field of a stationary spherically symmetric body on escaping radiation has been extensively discussed by Hagihara (1931), Darwin (1959, 1961) and Synge (1966). In this note we wish to discuss the escape of photons from a spherically symmetric massive body which is rotating about an axis passing through the centre of the body.

The rigorous solution of Einstein's field equations corresponding to the above problem is, unfortunately, not known and, therefore, we make use of an approximation method. We assume that the angular velocity of the rotating body is small and consider the effect of rotation as a small perturbation away from the static Schwarzschild metric. A general perturbation away from the Schwarzschild metric will give rise to various modes of gravitational radiation (Brill and Hartle 1964). Here we confine ourselves to the case when resulting metric is stationary.

2. The Metric and the Path of the Rays

Let the axis of rotation of the body be taken as z-axis of the coordinate system. Then we may take the line element of the external field of the rotating star as (Kerr 1963)

$$ds^2 = \left(1-\frac{2m}{r}\right)^{-1} dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2)-\left(1-\frac{2m}{r}\right) dt^2 + \frac{2ma}{r^3} r^2 \sin^2\theta\, d\phi \left\{dt+\left(1-\frac{2m}{r}\right)^{-1} dr\right\} \quad \ldots \quad (1)$$

Here m is the mass and a is a constant such that ma is the angular momentum of the body. The form of the line element is suggested from the linearized theory. It is seen that (1) differs from Schwarzschild metric only in the addition of the last term. (The metric given in eqn. (1) is, however, an approximation of Kerr's metric.)

It is convenient to introduce the dimensionless coordinates

$$\rho = \frac{r}{2m}, \quad \tau = \frac{t}{2m}$$

so that eqn. (1) reads

$$ds^2 = 4m^2[(1-\rho^{-1})^{-1}\,d\rho^2+\rho^2(d\theta^2+\sin^2\theta\,d\phi^2)-(1-\rho^{-1})\,d\tau^2 + K\rho^{-1}\sin^2\theta\,d\phi\{d\tau+(1-\rho^{-1})^{-1}d\rho\}] \quad \ldots \quad (2)$$

where $K = a/m$ is a characteristic constant of the rotating body and for a given body of spherical shape depends on its angular velocity. The above equation shows that for all stars without rotation the behaviour of the ray of light may be discussed in a single argument because m appears as a common factor if the last term were absent. Thus we conclude that in case of rotating stars the path of light rays will be determined by their mass as well as angular velocity.

In order to discuss the path of a given light ray we choose a coordinate system in which the ray is emitted initially in the plane $\theta = \pi/2$. Now it has been shown in the Appendix that it will continue to move in the same plane in the approximation we are using here. Then we obtain the equations of motion of the null geodesic as

$$\left.\begin{aligned} &\frac{d\phi}{du} = A\rho^{-2}, \quad \frac{d\tau}{du} = AB(1-\rho^{-1})^{-1} \\ &(1-\rho^{-1})^{-1}\left(\frac{d\rho}{du}\right)^2+\rho^2\left(\frac{d\phi}{du}\right)^2-(1-\rho^{-1})\left(\frac{d\tau}{du}\right)^2 \\ &+K\rho^{-1}\frac{d\phi}{du}\left\{\frac{d\tau}{du}+(1-\rho^{-1})^{-1}\frac{d\rho}{du}\right\} = 0 \end{aligned}\right\} \quad \ldots \quad (3)$$

where A and B are constants depending upon the initial conditions and u is an affine parameter along the ray. The constant B gives the inverse of the distance of the light path from the point of origin of the coordinate system (Bergmann 1960). In order to avoid Schwarzschild singularity we have to discuss the above differential equations for $\rho_0 \leqslant \rho < \infty$ where $1 < \rho_0 = \frac{r_0}{2m}$. Suppose now that we follow a particular ray emanating from the surface of the star. Then, from (3), the differential equation of the ray is obtained as

$$\left(\frac{d\rho}{d\phi}+\tfrac{1}{2}K\rho^{-1}\right)^2 = \rho^4\left[B^2-\frac{\rho-1}{\rho^3}-\frac{KB}{\rho^3}+\frac{K^2}{4\rho^6}\right] \quad \ldots \quad (4)$$

Thus for a star of given gravitational intensity the path of the ray will be given by the above equation.

3. Discussion and Conclusion

To discuss the escape of rays in general we draw curves with equation

$$B^2 - \frac{K}{\rho^3}B - \frac{\rho-1}{\rho^3} + \frac{K^2}{4\rho^6} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (5)$$

with B and ρ as coordinates for different values of K. A point in this diagram represents a point on a ray. In Fig. 1 we have drawn curves for $K = 0{\cdot}5$, 1, 1·5, 2. For the sake of comparison we have also drawn the corresponding

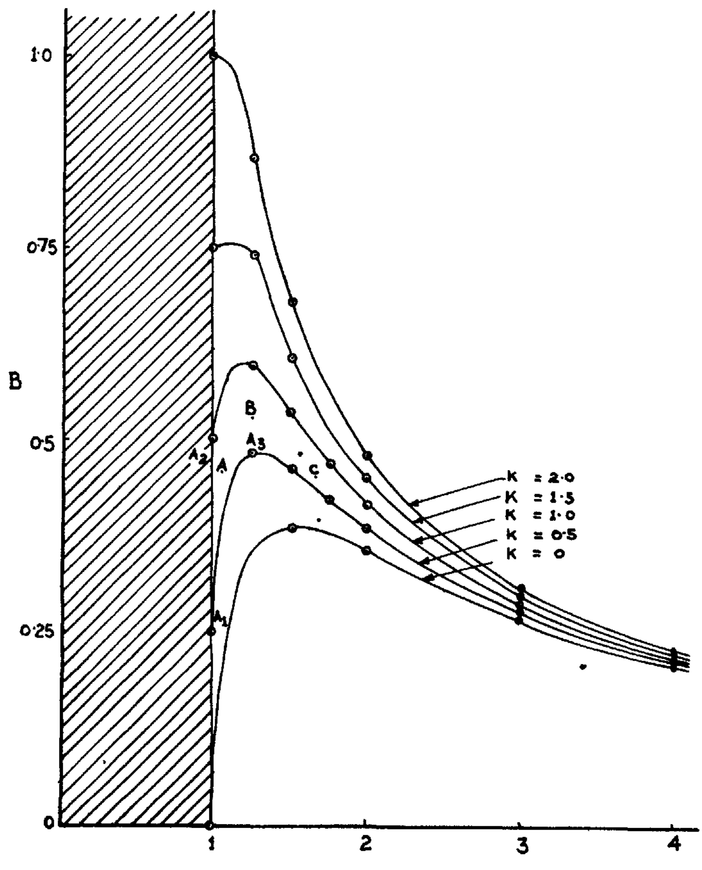

Fig. 1. Escape of photons from a rotating star (curve with $K = 0$ corresponds to a non-rotating star and curves for $K = 0{\cdot}5$, 1·0, 1·5 and 2·0 correspond to stars with increasing angular velocity).

curve for a non-rotating star. Since the right-hand side of eqn. (4) cannot be negative, no representative point can lie below the curve for a given value of

K. Further since ρ lies in the range $1 < \rho \leqslant \infty$ no point can lie to the left of the line $\rho = 1$ (shown shaded in Fig. 1).

The interesting point is that for small values of K the curves have maxima, similar to the case of non-rotating stars, which shift towards left for increasing values of K. But for $K > 1{\cdot}5$ (approx.) there are no maxima. This leads to the conclusion that as long as the angular velocity of rotation is small, the escape of the ray from a rotating star is similar to that of a stationary star. Thus, three typical points, say A, B and C, may be seen in the diagram for a particular curve, say for $K = 0{\cdot}5$, having a maxima. In each case, as we move along an outgoing ray, the representative point moves horizontally, starting to the right and continuing unless it runs into the curve. For those points, e.g. A, for which the horizontal line runs into the curve the ray has an apse. But if the representative point lies above or to the right of the maximum (e.g. B or C), it may be moved horizontally to infinity without meeting the curve. It is clear from the diagram that the curvilinear triangle A_1 A_2 A_3 is the domain of recapture.

In case of stars with large angular velocity we have a large value of K and corresponding curves have no maxima. It means that none of the rays has an apse and a ray once emitted from the surface of the star escapes to infinity.

The above considerations apply to all rays, whether incoming or outgoing. If we wish to discuss the behaviour of an incoming ray, the representative point will move horizontally from right to left. All the previous arguments equally apply to this case as well.

Thus, we see that the escape of a ray emitted from the surface of a star is sufficiently modified due to rotation. But there are certain limitations to the conclusions which we have drawn here. Firstly, we have used an approximate form of Kerr's exact metric (1963) retaining the first power of the constant a. In doing so we intended the metric to be valid to first order in the angular momentum parameter. But to this order we obtain the Schwarzschild metric plus a term in $d\phi dt$. Although many authors have used this procedure (Brill and Cohen 1966, and Landau and Lifschitz 1962) but there is no reasonable justification for adding a term proportional to $d\phi dt$. The conclusions drawn here are, therefore, not rigorous.* Secondly, the behaviour of the rays in only equatorial plane is discussed.† Thirdly, the constant K in eqn. (2) will generally have lower values than the one we have assumed here.

Nevertheless, the conclusions drawn in this note are interesting. It is expected that more exact treatment of this problem will lead to similar conclusions.

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* Note added in the proof: For the full discussion of Kerr metric, *see* Boyer, R. H., and Lindquist, R. W. *J. Math. Phys.* *8*, 265 (1967).

† For a complete discussion all the rays should be discussed.

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Appendix

The equation of motion of the ray is obtained from the geodesic equation

$$\frac{d^2x^\mu}{du^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{du}\frac{dx^\beta}{du}=0$$

When the above equation is specialized for the metric (2) for $x^2=\theta$, $x^3=\phi$ and $x^4=t$ we obtain the respective equations for the plane $\theta=\pi/2$

$$\frac{d^2\theta}{du^2}=0 \qquad (A1)$$

$$\frac{d^2\phi}{du^2}+\frac{2}{\rho}\frac{d\rho}{du}\frac{d\phi}{du}+\frac{a}{2m\rho^2}\left(\frac{d\phi}{du}\right)^2$$

$$-\frac{a}{4m\rho^5}\left(\frac{d\tau}{du}\right)^2+\left\{\frac{1}{2\rho}(-1-\rho^{-1})^{-2}\right\}\frac{a}{2m\rho^4}\left(\frac{d\rho}{du}\right)^2$$

$$+\frac{a}{2m\rho^4}\left\{\frac{(1-\rho^{-1})^{-1}}{\rho}-1\right\}\frac{d\tau}{du}\frac{d\rho}{du}-\frac{a^2}{4m^2\rho^5}\frac{d\tau}{du}\frac{d\phi}{du}$$

$$+\frac{a^2}{4m^2\rho^5}(1-\rho^{-1})^{-1}\frac{d\rho}{du}\frac{d\phi}{du}=0 \qquad (A2)$$

$$(1-\rho^{-1})\frac{d^2\tau}{du^2}+\frac{1}{\rho^2}\frac{d\tau}{du}\frac{d\rho}{du}-\frac{a}{2m\rho^2}\frac{d\rho}{du}\frac{d\phi}{du}$$

$$+\frac{a^2}{8m^2\rho^6}\left(\frac{d\tau}{du}\right)^2+(1-\rho^{-1})^{-2}\frac{a^2}{4m^2\rho^5}\left(1-\frac{1}{2\rho}\right)\left(\frac{d\rho}{du}\right)^2$$

$$-\frac{a^2}{4m^2\rho^3}\left(\frac{d\phi}{du}\right)^2+\frac{a^2}{4m^2\rho^5}\left\{1-\frac{(1-\rho^{-1})^{-1}}{\rho}\right\}\frac{d\tau}{du}\frac{d\rho}{du}$$

$$+\frac{a^3}{8m^3\rho^6}\frac{d\tau}{du}\frac{d\phi}{du}+\frac{a^3}{8m^3\rho^6}(1-\rho^{-1})^{-1}\frac{d\rho}{du}\frac{d\phi}{du}=0 \quad \ldots \quad \ldots \quad (A3)$$

Equation ($A1$) shows that a ray initially moving in the plane $\theta = \pi/2$ will continue to move in the same plane. The other two equations cannot be integrated as such and we, therefore, make use of the approximation referred to earlier. We expand different terms in decreasing powers of ρ and retain terms of the order $O(\rho^{-3})$. Thus, we obtain

$$\frac{d^2\phi}{du^2}+\frac{2}{\rho}\frac{d\rho}{du}\frac{d\phi}{du}+\frac{a}{2m\rho^2}\left(\frac{d\phi}{du}\right)^2=0 \quad \ldots \quad \ldots \quad \ldots \quad (A4)$$

and

$$(1-\rho^{-1})\frac{d^2\tau}{du^2}+\frac{1}{\rho^2}\frac{d\tau}{du}\frac{d\rho}{du}-\frac{a}{2m\rho^2}\frac{d\rho}{du}\frac{d\phi}{du}=0 \quad \ldots \quad \ldots \quad (A5)$$

Equations ($A4$) and ($A5$) are completely integrable but for the presence of terms $\frac{a}{2m\rho^2}(d\phi/du)^2$ and $-\frac{a}{2m\rho^2}\frac{d\rho}{du}\frac{d\phi}{du}$. But as we have already remarked, we consider the effect of rotation as a small perturbation away from the stationary body. The corresponding equations for spherically symmetric stationary body are completely integrable and if we compute the order of magnitude of $\frac{a}{2m\rho^2}\left(\frac{d\phi}{du}\right)^2$ and $-\frac{a}{2m\rho^2}\frac{d\rho}{du}\frac{d\phi}{du}$ this is found to be much smaller quantity than we are interested in. Therefore, we are justified in neglecting even these terms. Thus we obtain eqn. (3).