ON INITIAL DEVELOPMENT OF AXISYMMETRIC WAVES IN FLUIDS OF FINITE DEPTH

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An initial value approach to the axisymmetric wave problem in a fluid of limited and very shallow depth is presented. An asymptotic solution of the problem has been worked out in a simple and rigorous way by the similar method developed by the author (Debnath 1969) in an earlier paper. Finally, the principal behaviour of the axisymmetric wave motions has been discussed in considerable detail and some physical arguments has been put forward in order to explain the strange character of the solution related to the concentrated pressure distribution with the forcing frequency ω.

1. INTRODUCTION

Debnath (1969) has suggested various reasons in favour of the initial value investigation into the general wave problems and explained the relation between the steady state and properly posed initial value problem in considerable detail. Usually, it is assumed that the solution of the steady state problem obtained by imposing a radiation condition at infinity is generally the limit as time $T \to \infty$ of the solution derived by considering a well-posed initial value problem. Nevertheless, there appears to be no generally rigorous proof that this is, in fact, the case and it is usually necessary to examine the limiting procedure in each individual configuration. In the same paper, the author has introduced a linearized theory of axisymmetric wave motions in a fluid of infinite depth due to local disturbances with the forcing frequency ω acting on the undisturbed free surface of the fluid. The asymptotic solution of the initial value problem has been derived in a simple and rigorous manner. Finally, the principal features of the wave motions have been described in some detail.

In this connection, reference may be made to the original works of Stoker (1957) who considered the unsteady two-dimensional surface wave problem due to a periodic concentrated pressure distribution. With due investigation, he suggested that steady state wave problem with the Sommerfeld radiation

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condition at infinity should be derivable as the limit of the corresponding initial value problem.

The corresponding axisymmetric wave problem in a fluid of limited and very shallow depth is also interesting and important on its own merit, hence it deserves a special attention. So far, no such work has been reported to the current literature at least available to the author.

The object of this discussion is to present a linearized theory of initial development of axisymmetric surface waves in an inviscid, incompressible and homogeneous fluid of limited and very shallow depth generated by an arbitrary periodic disturbance with the forcing frequency ω acting on the undisturbed free surface of the fluid. An asymptotic solution of the problem has been derived for sufficiently large times by the similar method developed by the author (Debnath 1969) in his earlier paper. Finally, the limiting behaviour of the solution has been examined in some detail and the main features of the wave motions have been predicted. This is followed by some physical arguments in favour of the strange behaviour of the solution related to the oscillating point-pressure distribution with the forcing frequency ω.

2. Mathematical Formulation of the Problem

We consider the axisymmetric wave problem in an inviscid, incompressible and homogeneous fluid of finite depth due to a periodic surface pressure distribution in the form

$$\left.\begin{aligned} P(R,T) &= P(R)\,e^{i\omega T}H(T), \quad R \leqslant A \\ &= 0 \qquad\qquad\qquad\quad , \quad R > A \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

acting on the free surface $Y = 0$ of the fluid initially at rest, where $P(R)$ is an arbitrary function of R, ω is the fixed frequency and $H(T)$ is the Heaviside unit function of time T.

We assume that X-Z plane be the undisturbed horizontal free surface and Y-axis be vertical positive upwards. We take cylindrical polar coordinates (R, Θ, Y) with cylindrical symmetry about the Y-axis such that R is equal to $\sqrt{X^2+Z^2}$ from the Y-axis and the origin of coordinate is taken on the free surface.

In view of the fact that the motion starts from rest, there exists a velocity potential $\Phi(R, Y; T)$ which is governed by the Laplace equation

$$\nabla^2\Phi \equiv \frac{\partial^2\Phi}{\partial R^2}+\frac{1}{R}\frac{\partial\Phi}{\partial R}+\frac{\partial^2\Phi}{\partial Y^2} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

$$-h \leqslant Y \leqslant 0, \; 0 \leqslant R < \infty$$

where h denotes the depth of the fluid.

The condition at the bottom surface is

$$\Phi_Y = 0 \quad \text{on } Y = -h \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.3)$$

Neglecting surface tension, the linearized free surface boundary conditions are given by

$$\left.\begin{aligned} \Phi_T+gE &= -\frac{1}{\rho}P(R,T) \\ \Phi_Y &= E_T \end{aligned}\right\} \quad \begin{aligned} Y &= 0 \\ T &> 0 \end{aligned} \qquad \begin{aligned} &(2.4) \\ &(2.5) \end{aligned}$$

where suffixes denote partial differentiation and $E = E(R, T)$ represents the vertical surface displacement, ρ the density and g the acceleration due to gravity.

The initial conditions are given by

$$\left.\begin{aligned} &\Phi = E = 0 \quad \text{at } T = 0 \\ &[\Phi_T(R, 0; T)]_{T=0} = -\frac{1}{\rho}P(R, 0) \end{aligned}\right\} \qquad (2.6)$$

We further assume that the functions $\Phi(R, Y; T)$ and $E(R, T)$ possess the Hankel transform with respect to R in the generalized sense.

Remarks: The problem formulated above can be investigated as a steady state just by omitting the initial conditions (2.6). For the steady state analysis, one has to impose Sommerfeld's radiation condition at infinity which is, in fact, essential in order to derive a solution of physical interest.

3. Solution of the Problem

It is convenient to introduce dimensionless variables r, x, y, z, a, t, η and ϕ defined by the following relations:

$$(r, x, y, z, a) = \frac{\omega^2}{g}(R, X, Y, Z, A), \quad t = \omega T, \quad \eta = \frac{P\omega^4}{\rho g^3}E,$$

and

$$\phi = \frac{P\omega^5}{\rho g^4}\Phi,$$

and we introduce a non-dimensional parameter D defined by $D = \omega^2 h/g$. Substituting these relations into the basic eqns. (2.2)–(2.6), we obtain

$$\phi_{rr}+\frac{1}{r}\phi_r+\phi_{yy} = 0 \qquad (3.1)$$

$$-D \leqslant y \leqslant 0, \; 0 \leqslant r < \infty$$

$$\phi_y = 0 \quad \text{on } y = -D \qquad (3.2)$$

$$\left.\begin{aligned} \phi_t+\eta &= -p(r, t) \\ \phi_y &= \eta_t \end{aligned}\right\} \quad \begin{aligned} y &= 0 \\ t &> 0 \end{aligned} \qquad \begin{aligned} &(3.3) \\ &(3.4) \end{aligned}$$

$$\left.\begin{aligned} &\phi = \eta = 0 && \text{at } t = 0 \\ &\phi_t = -p(r, t) && \text{at } t = 0, y = 0 \end{aligned}\right\} \qquad (3.5)$$

As in our earlier paper (Debnath and Rosenblat *in press*), we introduce the Laplace transform (Tranter 1966) $\bar{\phi}$, $\bar{\eta}$ of the functions ϕ, η respectively with respect to t and then we introduce the Hankel transform $\bar{\bar{\phi}}$, $\bar{\bar{\eta}}$ (Lighthill 1958) of the functions $\bar{\phi}$, $\bar{\eta}$ respectively with respect to r. This enables us to simplify eqns. (3.1)–(3.5) and to write down the solutions for $\bar{\bar{\phi}}$ and $\bar{\bar{\eta}}$ without any difficulty in the form

$$\bar{\bar{\phi}}(k, y; s) = -\frac{s\,\bar{\bar{p}}(k, s) \cosh k(y+D)}{(s^2+\alpha^2) \cosh kD} \qquad (3.6)$$

$$\bar{\bar{\eta}}(k, s) = -\frac{\alpha^2(k)\,\bar{\bar{p}}(k, s)}{(s^2+\alpha^2)} \qquad (3.7)$$

where α^2 is given by the relation

$$\alpha^2 = \alpha^2(k) = k \tanh kD. \qquad (3.8)$$

Making reference to the inversion formula for the Laplace transform combined with its convolution theorem and then to the inverse theorem of Hankel transformation, we readily obtain

$$\phi(r, y;\ t) = \int_0^\infty (i \cos \alpha t - \alpha \sin \alpha t - ie^{it})\, k\, \bar{p}(k) J_0(kr) \frac{\cosh k(y+D)}{(\alpha^2-1) \cosh kD}\, dk \qquad (3.9)$$

$$\eta(r, t) = \int_0^\infty (i \sin \alpha t + \alpha \cos \alpha t - \alpha e^{it})\, k\, \alpha(k)(\alpha^2-1)^{-1} \bar{p}(k) J_0(kr)\, dk. \qquad (3.10)$$

The above integrals (3.9) and (3.10) are the exact solution of the problem. But, in general, they cannot be worked out exactly. Hence asymptotic methods are utilized to evaluate them for a deeper understanding of the transient wave motions.

In the case of infinite depth of fluids, i.e. when $D \to \infty$, the integral for the velocity potential $\phi(r, y; t)$ as well as the surface elevation $\eta(r, t)$ has the following form:

$$\phi(r, y;\ t) = \int_0^\infty (i \cos \sqrt{k}t - \sqrt{k} \sin \sqrt{k}t - ie^{it}) e^{ky}\, k\, \bar{p}(k) J_0(kr)(k-1)^{-1}\, dk \qquad (3.11)$$

$$\eta(r, t) = \int_0^\infty (\sqrt{k} \cos \sqrt{k}t + i \sin \sqrt{k}t - \sqrt{k}e^{it})\, k^{3/2}\, \bar{p}(k) J_0(kr)(k-1)^{-1}\, dk. \qquad (3.12)$$

These results are exactly identical with those already obtained by Debnath (1969).

4. Asymptotic Analysis of the Solution

As already mentioned that the function $p(r)$ involved in the non-dimensional pressure distribution may be arbitrary, hence it would be sufficient for the determination of the principal behaviour of the wave motions to take the simple non-dimensional form as

$$p(r, t) = e^{it} p(r), \quad r < a$$
$$= 0 \quad , \quad r < a$$

with the prescribed form of $p(r)$ as

$$\text{(a)}\ p(r) = \delta(r)/r$$

where $\delta(r)$ is the Dirac function of distribution and

$$\text{(b)}\ p(r) = 1$$

$$\text{(c)}\ p(r) = \exp\left(-r^2/r_0^2\right)$$

$$\text{(d)}\ p(r) = J_0(\lambda r)$$

$$\text{(e)}\ p(r) = (a^2-r^2)^n, \quad n > -1.$$

We proceed to find out an asymptotic representation for the surface elevation $\eta(r, t)$ in the general form and the above particular cases (a)–(e) can be used in the end.

Let us consider the integral (3.10) for the surface elevation $\eta(r, t)$ which is given by

$$\eta(r, t) = \int_0^\infty (i \sin \alpha t + \alpha \cos \alpha t - \alpha e^{it})(\alpha^2-1)^{-1} k\, \alpha(k)\, \bar{p}(k) J_0(kr)\, dk.$$

It may be noticed that this integral has no singularities on $(0, \infty)$, therefore, its path of integration can be deformed into a path M (say) in the complex $s = k+i\mu$ plane as indicated in Fig. 1. The path M coincides with the real axis $(0, \infty)$ except that it is diverted round the zeros of the denominator of the integrand. We can then write down the integral as a sum of two integrals which are now singular at the zeros of (α^2-1).

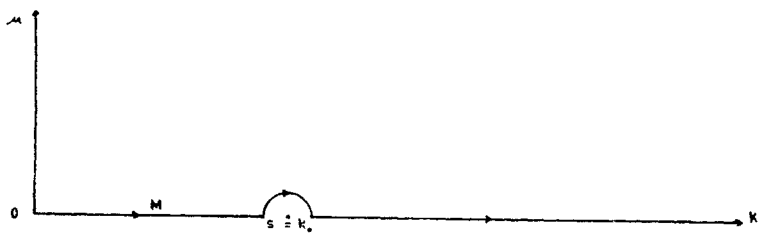

FIG. 1. The complex $s = k+i\mu$.

Hence

$$\eta(r, t) = \int_M (i \sin \alpha t + \alpha \cos \alpha t - \alpha e^{it})\, s\, \alpha(s)(\alpha^2-1)^{-1} \bar{p}(s) J_0(sr)\, ds \quad .. \quad (4.1)$$

where $\alpha = \alpha(s) = \sqrt{s \tanh sD}$ and $s = k_0$ is the only real root in $(0, \infty)$ of the equation $\alpha^2(s) = 1$.

We thus obtain

$$\eta(r, t) = (I_1 - I_2) \quad .. \quad .. \quad .. \quad .. \quad (4.2)$$

where I_1 and I_2 are given by the integrals in the form

$$I_1 = \int_M \bar{p}(s)\, s\, \alpha(s)\, (\alpha^2-1)^{-1}(i \sin \alpha t + \alpha \cos \alpha t) J_0(sr)\, ds$$

$$I_2 = e^{it} \int_M \frac{\bar{p}(s)\, s\, \alpha^2(s) J_0(sr)\, ds}{(\alpha^2-1)}$$

where $-\pi < \arg s \leqslant \pi$.

It may be noticed that the wave integrals I_1 and I_2 obtained above are very much similar in form obtained earlier (Debnath 1969), therefore, similar methods, as before, with a slight modification can be exploited to evaluate them without any difficulty.

Making reference to the method used (Debnath 1969) for the evaluation of the steady state wave integral, we readily write down the result for I_2 in the form

$$I_2 \sim -\left(\frac{2\pi}{rk_0}\right)^{\frac{1}{2}} \left[\frac{i\bar{p}(k_0)\, k_0\, \alpha^2(k_0)}{W'(k_0)}\right] e^{i\left(t-rk_0+\frac{\pi}{4}\right)} \quad \dots \quad \dots \quad (4.3)$$

where the function $W(k)$ is given by

$$W(k) = \alpha^2(k) - 1 \quad \dots \quad \dots \quad \dots \quad \dots \quad (4.4)$$

Remarks: It can be easily seen that in the case of infinite depth (i.e. $D \to \infty$), the pole $s = k_0 = 1$ and, consequently, the value of I_2 obtained above is in perfect agreement with the result of Debnath (1969).

In order to evaluate the transient integral I_1, we first replace the Bessel function $J_0(sr)$ by its standard integral formula and next from a simple rearrangement, it easily follows that

$$I_1 = \frac{1}{2\pi} \int_0^{\pi/2} (L_1 + L_2 + L_3 + L_4)\, d\theta \quad \dots \quad \dots \quad \dots \quad (4.5)$$

where the integrals L_1, L_2, L_3 and L_4 are given by

$$L_1 = \int_M \frac{s\, \alpha(s) \bar{p}(s)}{\alpha(s)+1}\, e^{-itf_-(s)}\, ds$$

$$L_2 = \int_M \frac{\bar{p}(s)\, s\, \alpha(s)}{\alpha(s)-1}\, e^{itf_-(s)}\, ds$$

$$L_3 = \int_M \frac{\bar{p}(s)\, s\, \alpha(s)}{\alpha(s)-1}\, e^{itf_+(s)}\, ds$$

$$L_4 = \int_M \frac{\bar{p}(s)\, s\, \alpha(s)}{\alpha(s)+1}\, e^{-if_+(s)t}\, ds$$

with

$$f_\pm(s) = \left(\alpha \pm \frac{r}{t} s \cos\theta\right).$$

It may be observed that these integrals are similar in nature to those already obtained earlier by Debnath (1968, 1969), Debnath and Rosenblat (*in press*). Therefore, they can be evaluated exactly in a similar way as before with a little modification. However, a few important points may be outlined here so that one can easily understand basic ideas behind the asymptotic evaluation of these integrals stated above for large values of t. Making reference to the works cited above, it is easy to show that the integrals L_3 and L_4 are $O\left(\frac{1}{t}\right)$.

And the stationary (or saddle) point and pole related to the integrals L_1 and L_2 are respectively at $s = k_1$ and $s = k_0$. In fact, $s = k_1$ is the root of the equation

$$f'_-(s) = 0.$$

A similar calculation, as before, leads us to the asymptotic answer to the integrals L_1 and L_2. Then substituting their values to the θ-integral (4.5) and making appeal to the method of stationary phase (Jeffreys and Jeffreys 1946) for large t in order to evaluate the θ-integral, it turns out that

$$I_1 \sim \begin{bmatrix} -\dfrac{\bar{p}(k_0)ik_0\,\alpha(k_0)\{\alpha(k_0)+1\}}{2W'(k_0)}\left(\dfrac{2\pi}{rk_0}\right)^{\frac{1}{2}} e^{i\{t\,\alpha(k_0)-rk_0+\frac{\pi}{4}\}}, & k_1 < k_0 \\ 0 & , \ k_1 > k_0 \end{bmatrix}$$
$$+k_1\alpha(k_1)\frac{\bar{p}(k_1)}{\{4rt\left|f''_-(k_1)\right|\}^{\frac{1}{2}}}\left[\frac{e^{i\{k_1 r-t\,\alpha(k_1)\}}}{\alpha(k_1)+1}+\frac{e^{-i\{k_1 r-t\,\alpha(k_1)\}}}{\alpha(k_1)-1}\right]+O\left(\frac{1}{t^2}\right). \quad \text{.. (4.6)}$$

Finally, combining the integrals I_1 and I_2 together, we arrive at the asymptotic representation for the surface elevation $\eta(r, t)$ in the form

$$\eta(r, t) \sim \frac{i\bar{p}(k_0)k_0\,\alpha^2(k_0)}{W'(k_0)}\left(\frac{2\pi}{rk_0}\right)^{\frac{1}{2}} e^{i\left(t-rk_0+\frac{\pi}{4}\right)}\begin{bmatrix} 1 & , \ k_1 > k_0 \\ 1-\frac{1}{2}\{\alpha(k_0)+1\}, & k_1 < k_0 \end{bmatrix}$$
$$+\frac{k_1\bar{p}(k_1)\alpha(k_1)}{\{4r\,t\,k_1\left|f''_-(k_1)\right|\}^{\frac{1}{2}}}\left[\frac{e^{i\{rk_1-t\,\alpha(k_1)\}}}{\alpha(k_1)+1}+\frac{e^{-i\{rk_1-t\,\alpha(k_1)\}}}{\alpha(k_1)-1}\right]+O\left(\frac{1}{t^2}\right) \quad (4.7)$$

Remarks: The quantity $k_1 = k_0$ renders the solution (4.7) invalid. So, a special method is needed to obtain a valid solution for $\eta(r, t)$ near the point $k_1 = k_0$. Since our range of interest is large values of t, it seems to us that the derivation of the solution at the above point is not so important here.

5. Asymptotic Solution in the case of Unlimited Depth

In the case of infinite depth of fluid, that is when the parameter $D \to \infty$, the function $\alpha^2(s)$ tends to the limit s. Hence the pole $s = k_0 = 1$ and the saddle or stationary point $s = k_1 = \left(\dfrac{t}{2r}\right)^2$ (Copson 1965) which are in perfect agreement with our previous analysis (Debnath 1969). Therefore, the above solution takes the following form:

$$\eta(r, t) \sim \begin{bmatrix} i\bar{p}(1)\left(\dfrac{2\pi}{r}\right)^{\frac{1}{2}} e^{i\left(t-r+\frac{\pi}{4}\right)}, & t \gg 2r \\ 0 & , \ t \ll 2r \end{bmatrix}$$
$$+\frac{t^3\,\bar{p}\left(\dfrac{t^2}{4r^2}\right)}{4\sqrt{2}r^4\left(\dfrac{t^2}{4r^2}-1\right)}\left\{\frac{t}{2r}\cos\left(\frac{t^2}{4r}\right)+i\sin\left(\frac{t^2}{4r}\right)\right\}+O\left(\frac{1}{t^2}\right) \quad \text{.. (5.1)}$$

This result is in perfect agreement with those obtained by Debnath (1969) in the particular cases (a)–(d) stated before. In order to compare this result (5.1) with those derived in the earlier paper, one has to substitute the value for the function $\bar{p}(k_1)$.

We would like to mention here the Hankel transform $\bar{p}(k)$ of the function $p(r)$ related to cases (a)–(e). The functions $\bar{p}(k)$ associated with the cases (a)–(e) are respectively given by

$$1,\ \frac{a}{k}J_1(ak),\ 2m^2e^{-k^2m^2},\ \left[m^2 = \left(\frac{r_0}{2}\right)^2\right];\quad \int_0^a x\,J_0(\lambda x)J_0(kx)\,dx,$$

and

$$2^n\Gamma(n+1)(ak)^{-(n+1)}a^{2(n+1)}J_{n+1}(ak),$$

where $J_n(x)$ are the Bessel function of the first kind of order n, and the finite integral is a standard one (Watson 1922; Sneddon 1951).

Thus the solution of the problem is explicitly known in both finite and infinite depth cases. And for infinite depth case, we can refer to our earlier paper (Debnath 1969). In the case of limited depth, we intend to summarize our conclusion later on.

6. Axisymmetric Wave Motions in Shallow Fluids

In the other extreme case, that is when $D \ll 1$, we can replace $\alpha^2(s) \equiv s \tanh sD$ by s^2D to the first approximation. Consequently, the integrals involved in the vertical surface elevation $\eta(r, t)$ are simpler in nature. And a procedure similar to that advanced in the papers (Debnath 1968, 1969) can be adopted to obtain the asymptotic solution for $\eta(r, t)$ in the following form:

$$\eta(r, t) \sim \begin{bmatrix} i\left(\dfrac{2\pi}{r\sqrt{D}}\right)^{\frac{1}{2}} \bar{p}\left(\dfrac{1}{\sqrt{D}}\right) e^{i\left(t-\frac{r}{\sqrt{D}}+\frac{\pi}{4}\right)}, & t \gg \dfrac{r}{\sqrt{D}} \\ 0\qquad\qquad , & t \ll \dfrac{r}{\sqrt{D}} \end{bmatrix} \quad \cdots \quad (6.1)$$

This solution for $\eta(r, t)$ suggests that the waves are advancing with the phase velocity $\sqrt{gh}$ which is independent of the wavelength and the amplitude of the waves decays like $r^{-1/2}$. Evidently, the waves are non-dispersive in character.

7. Discussion and Conclusions

From the salient and inherent features of the properly posed initial value formulation, it follows that the axisymmetric wave problem investigated above achieves a unique solution of physical interest. An asymptotic analysis of the problem enables us to derive explicitly both the steady state and transient wave solutions. Hence solution (4.7) as well as (5.1) for the surface elevation $\eta(r, t)$ describes the behaviour of wave formation at all times instead of only in the limit. And the solution for the two extreme limiting cases (i) $h \to \infty$ and (ii) $h \to 0$ has also been discussed in some detail.

The above analysis reveals that the transient component of the asymptotic solution associated with cases (b)–(e) does die out very rapidly in the limit $t \to \infty$, for fixed values of r. Therefore, the ultimate steady state is attained in the limit. In other words, the steady state is set up in the limit provided the applied pressure with the forcing frequency ω is distributed over a 'finite' circular region, however small, of the free surface of the fluid. On the other hand, classical steady state is not attained when the harmonic pressure distribution concentrated in a single point of the free surface. These conclusions are in excellent agreement with those obtained earlier in connection with the axisymmetric wave problem in a fluid of infinite depth and the two-dimensional wave problem (Debnath 1968). Although this disagreement is strange to some extent, but not really unexpected for mathematical and physical reasons which have been suggested earlier (1969) and would be stated below.

In all the above cases (b)–(e), the classical steady state solution of special interest is obtained in the form

$$\eta(r, t) \sim i\left(\frac{A}{\sqrt{r}}\right)e^{i\left(t-rk_0+\frac{\pi}{4}\right)},$$

where A is some known constant depending on the actual functional form of $\bar{p}(k)$ involved in the cases under consideration.

This solution represents the axisymmetric surface waves propagating in the medium with the amplitude decaying like $r^{-1/2}$. And in the limit $D \to \infty$, that is $h \to \infty$, this solution confirms our previous solution obtained earlier (1969).

The author (1969) has suggested some mathematical arguments to explain the strange behaviour of the asymptotic solution (5.1) for the case of the pressure distribution concentrated in a single point with the forcing frequency ω. From the physical point of view, the following argument may be put forward to resolve the strange situation. The sudden harmonic point pressure with the high frequency content generates a continuous spectrum of circular waves of all possible wavelengths and the continuous spectrum is very rich in short wavelength content indeed. Thus the sudden onset and the enrichment of short wavelength content are both needed if there is to be sufficient energy distributed among the slowly varying travelling wave trains so that more and more intense accumulations occur at fixed (large) distance r as the time t approaches to infinity. That is why the transient component of the asymptotic solution (5.1) related to the delta function pressure increases like $O(t^2)$. So the classical steady state is not attained in the limit.

On the contrary, the situation is well behaved when the pulsating pressure is distributed over a finite region of the free surface and, in particular, when the Gaussian representation of the delta function in the form

$$\delta(r) = \lim_{a \to 0} \frac{1}{a\sqrt{\pi}} e^{-r^2/a^2}$$

is used as a wave-generating mechanism and the limit $t \to \infty$ is taken first.

Therefore, in any physically realistic situation where the surface pressure is distributed over any finite region of non-zero dimension, the ultimate steady state would be attained in the limit and the attainment of the steady state is directly attributable to the double limit process $a \to 0$, $t \to \infty$ in the proper order as suggested earlier and a similar but appropriate energy argument as outlined above.

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