# INHOMOGENEOUS MINIMA OF BINARY QUADRATIC, TERNARY CUBIC AND QUATERNARY QUARTIC FORMS IN FIELDS OF FORMAL POWER SERIES*

by SATISH K. AGGARWAL, *Mathematics Department, University of Illinois, Urbana, Illinois*

(Communicated by R. P. Bambah, F.N.I.)

(*Received 9 January 1968*)

In this paper our aim is to prove the analogues of the results of Cassels (1952*a*, 1952*b*) in the field $K\{t\}$ of formal power series in an indeterminate $t$ over the finite field $K$ and equipped with the usual non-Archimedean valuation. Denote by $P_n$ the set of $n$-tuples of polynomials in $t$ over $K$. Then we prove the following theorem.

Theorem 1.—Let $K$ be a finite field in which $x^2+m = 0$ is not soluble for some integer $m$ and for which the characteristic is not equal to 2. Let $L_1$, $L_2$, $L_3$, $L_4$ be four linear forms in variables $x$, $y$, $z$, $s$ with coefficients from $K\{t\}$ and determinant $\Delta \neq 0$. Then we can find elements $x_0$, $y_0$, $z_0$, $s_0$ in $K\{t\}$ such that

$$\left| L_1^2(X)+mL_2^2(X) \right| \left| L_3^2(X)+mL_4^2(X) \right| \geq \frac{|\Delta|}{e^4}$$

for all $X = (x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}$.

Following similar methods we can also prove the following theorems.

Theorem 2.—Let $K$ be a finite field. Let $L_1$, $L_2$ be two linear forms in the variables $x$, $y$ with coefficients in $K\{t\}$ and determinant $\Delta \neq 0$. Then $x_0 \in K\{t\}$ and $y_0 \in K\{t\}$ can be found such that

$$|L_1(X)L_2(X)| \geq \frac{|\Delta|}{e^2}$$

for all $X = (x, y) \equiv (x_0, y_0) \pmod{P_2}$.

Theorem 3.—Let $K$ be a finite field in which $x^2+m$ is irreducible for some integer $m$ and for which the characteristic is not equal to 2. Let $L_1$, $L_2$, $L_3$ be three linear forms in variables $x$, $y$, $z$ with coefficients from $K\{t\}$ and determinant $\Delta \neq 0$. Then there exist $x_0$, $y_0$, $z_0$ in $K\{t\}$ such that

$$\left| L_1(X) \right| . \left| L_2^2(X)+mL_3^2(X) \right| \geq \frac{|\Delta|}{e^3}$$

for all $X = (x, y, z) \equiv (x_0, y_0, z_0) \pmod{P_3}$.

---

* This is a part of the author's dissertation approved for the Ph.D. degree of Ohio State University in December 1965 written under the guidance of Professor R. P. Bambah.

§ 1. Let $K$ be any field and $t$ be an indeterminate. Let $K[t]$ be the ring of polynomials over $K$, $K(t)$ the field of rational functions and $K\{t\}$ the field of power series in $t^{-1}$ over $K$. The elements of $K\{t\}$ are of the form

$$x = \alpha_m t^m + \alpha_{m-1} t^{m-1} + \dots \text{ (up to } -\infty).$$

In the field $K\{t\}$, we define the usual valuation $||$ as follows:

(i) $|0| = 0$.

(ii) If $x = \alpha_m t^m + \alpha_{m-1} t^{m-1} + \dots, \alpha_m \neq 0$ then $|x| = e^m$.

We also define

$$||x|| = |\alpha_{-1} t^{-1} + \alpha_{-2} t^{-2} + \dots|.$$

Write $P_n = (K[t])^n = \{X = (x_1, \dots, x_n) ; x_i \in K[t]\}$ and $R_n = (K\{t\})^n = \{X = (x_1, \dots, x_n) ; x_i \in K\{t\}\}$. For $X = (x_1, \dots, x_n) \in R_n$, we define

$$|X| = \max(|x_1|, \dots, |x_n|).$$

It is well known that the fields $K\{t\}$, $K(t)$ and the integral domain $K[t]$ are similar in their relative structure to the real field, the rational field and the ring of rational integers respectively. $||x||$ corresponds to the fractional part of the real number $x$, $P_n$ corresponds to $n$-dimensional fundamental lattice and $R_n$ corresponds to $n$-dimensional Euclidean space.

§ 2. Let $L_1, L_2, \dots, L_{r+2s}$ be linear forms in $r+2s = n$ variables such that $L_1, \dots, L_r$ have real coefficients and $L_{r+1}, \dots, L_{r+s}$ have complex coefficients and $L_{r+s+1}, \dots, L_{r+2s}$ are complex conjugates to $L_{r+1}, \dots, L_{r+s}$ respectively. Let $\Delta \neq 0$ be the determinant of these forms. In the cases $r = 2, s = 0$; $r = 1, s = 1$ and $r = 0, s = 2$, Davenport (1950*a*, 1950*b*, 1951) has proved the existence of positive constants $C_{r,s}$ and reals $\xi_1, \dots, \xi_n$ such that

$$|L_1 L_2 \dots L_n| \geq \frac{|\Delta|}{C_{r,s}}$$

for all $(x_1, \dots, x_n) \equiv (\xi_1, \dots, \xi_n) \pmod 1$. He proved the theorems with $C_{2,0} = 128$, $C_{1,1} = 8 \times 10^{13}$ and $C_{0,2} = 10^{132}$.

Cassels (1952*a*, 1952*b*) gave an alternative proof with $C_{3,0} = 45{\cdot}2$, $C_{1,1} = 420$ and $C_{0,2} = 5300$. The best values of these $C$'s, however, are not known.

The above-mentioned results have their obvious analogues in connection with the field of formal power series.

Following the techniques of Davenport, Armitage (1957*a*) proved the analogues for finite $K$ in the case of quadratic and cubic forms, but limiting himself only to the cases, where the product $L_1 L_2 \dots L_n$ does not represent zero non-trivially.

Following the methods of Cassels (1952*a*, 1952*b*) in the real case, the author proved the following theorems in his dissertation.

*Theorem* 1.—Let $K$ be a finite field. Let $L_1$, $L_2$ be two linear forms in variables $x$, $y$ with coefficients in $K\{t\}$ and determinant $\Delta \neq 0$. Then there exists $(x_0, y_0) \in R_2$ such that

$$|L_1(X) \, . \, L_2(X)| \geq \frac{|\Delta|}{e^2}$$

for all $X = (x, y) \equiv (x_0, y_0) \pmod{P_2}$.

*Theorem* 2.—Let $K$ be a finite field in which $x^2+m$ is irreducible over $K$ for some integer $m$ and characteristic of $K$ is not equal to 2. Let $L_1$, $L_2$, $L_3$ be three linear forms in variables $x$, $y$, $z$ with coefficients from $K\{t\}$ and determinant $\Delta \neq 0$. Then there exists $(x_0, y_0, z_0) \in R_3$ such that

$$\left|L_1(X)\left(L_2^2(X)+mL_3^2(X)\right)\right| \geq \frac{|\Delta|}{e^3}$$

for all $X = (x, y, z) \equiv (x_0, y_0, z_0) \pmod{P_3}$*.

*Theorem* 3.—Let $K$ be a finite field in which $x^2+m = 0$ is not soluble for some integer $m$ and characteristic of $K$ is not equal to 2. Let $L_1$, $L_2$, $L_3$, $L_4$ be four linear forms in variables $x$, $y$, $z$, $s$ with coefficients from $K\{t\}$ and determinant $\Delta \neq 0$. Then there exists $(x_0, y_0, z_0, s_0) \in R_4$ such that

$$\left|L_1^2(X)+mL_2^2(X)\right| . \left|L_3^2(X)+mL_4^2(X)\right| \geq \frac{|\Delta|}{e^4}$$

for all $X = (x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}$*.

Remark: The results of all these theorems are best possible.

The three theorems mentioned above may be proved by similar techniques. Since Armitage (1957*a*) proved parts of Theorems 1 and 2, we shall give the details of Theorem 3 only. However, whether we can choose $x_0$, $y_0$, $z_0$, $s_0$ in $K(t)$, when the product has coefficients in $K(t)$, remains open. It may be remarked that Armitage solved the problem for quadratic and cubic cases, when the product does not represent zero non-trivially.

§ 3. Before proving the actual theorem, we need a few preliminaries.

Since we are given $x^2+m = 0$ is not soluble in $K$, it is also not soluble in $K\{t\}$. Adjoin $\alpha$, the root of $x^2+m = 0$ (in the algebraic closure of $K\{t\}$) to the field $K\{t\}$ and we get the field $K\{t\}(\alpha)$ (to be denoted by $K^{(m)}\{t\}$ hereafter). The elements of $K^{(m)}\{t\}$ are of the form $z = x+\alpha y$ where $x, y \in K\{t\}$. We define a function

$f$: $K^{(m)}\{t\} \rightarrow$ the set of non-negative reals by

$$f(z) = f(x+\alpha y) = \max(|x|, |y|)$$

It can be checked that $f$ is a valuation in $K^{(m)}\{t\}$. Since the restriction of $f$ to the field $K\{t\}$ is the original valuation, without ambiguity we shall write

$$f(z) = |z|.$$

* In my dissertation the theorems are stated for $m = 1$. I am grateful to Professor Bateman for pointing out that the same proof works for general $m$.

*Definition.*—Let $z = x+\alpha y \in K^{(m)}\{t\}$. Then we define

$$\|z\| = \max(\|x\|, \|y\|).$$

*Definition.*—We shall call the elements $z_1 = x_1+\alpha y_1$ and $z_2 = x_1-\alpha y_1$ 'conjugates' of each other and we shall write $z_1 = \tilde{z}_2$ or $z_2 = \tilde{z}_1$

REMARK: If $z_1 = \tilde{z}_2$, then $|z_1| = |z_2|$.

Suppose the given linear forms are

$$L_j(X) = a_jx+b_jy+c_jz+d_js \quad (j = 1, 2, 3, 4).$$

Write

$$L_1' = L_1+\alpha L_2, \quad L_2' = L_1-\alpha L_2,$$

$$L_3' = L_3+\alpha L_4, \quad L_4' = L_3-\alpha L_4, \quad (\alpha^2 = -m).$$

Let $\Delta'$ be the determinant of $L_1'$, $L_2'$, $L_3'$, $L_4'$. It can be checked very easily that $\Delta' = -4m\Delta$. Thus $\Delta' \in K\{t\}$.

Since $x^2+m = 0$ is not soluble in the field $K$ and the characteristic of the field is not 2,

$$|\Delta'| = |-4m\Delta| = |\Delta|.$$

Let $M_1$, $M_2$, $M_3$, $M_4$ be forms in variables $u$, $v$, $w$, $r$ such that the coefficients of $M_j$ are cofactors of the corresponding coefficients of $L_j'$ $(j = 1, 2, 3, 4)$ respectively. Then

$$L_1'M_1+L_2'M_2+L_3'M_3+L_4'M_4 = \Delta'(xu+yv+zw+sr). \quad .. \quad (3.1)$$

It can be checked that $M_1 = \tilde{M}_2$ and $M_3 = \tilde{M}_4$ and det $(M_1, M_2, M_3, M_4) = \Delta'^3$.

Now we prove a few lemmas.

*Lemma* 1.—Let $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$ be a finite or infinite sequence from the field $K^{(m)}\{t\}$ such that

$$|\lambda_{n+1}| \geq e|\lambda_n|, \quad \lambda_0 \neq 0.$$

Let $\mu_0, \mu_1, \ldots$ be arbitrary given elements of $K\{t\}$.

Then there exists $\xi \in K^{(m)}\{t\}$ such that

$$||\lambda_n\xi+\tilde{\lambda}_n\tilde{\xi}+\mu_n|| = e^{-1} \quad (n = 0, 1, 2, \ldots).$$

PROOF: First we note the following. Suppose

$$\lambda_n\xi = \sigma+\alpha\tau.$$

Then

$$\tilde{\lambda}_n\tilde{\xi} = \sigma-\alpha\tau$$

and

$$\lambda_n\xi+\tilde{\lambda}_n\tilde{\xi} = 2\sigma \in K\{t\}.$$

So that for $\lambda_n\xi+\tilde{\lambda}_n\tilde{\xi}$, we have to fix our attention on '$\sigma$' of $\lambda_n\xi$ only.

Without loss of generality, we can assume that the degree of $\mu_n$ is at most $-1$. Suppose

$$\lambda_n = a_{j_n}^{(n)}t^{j_n}+a_{j_n-1}^{(n)}t^{j_n-1}+\ldots+\alpha\{b_{j_n}^{(n)}t^{j_n}+b_{j_n-1}^{(n)}t^{j_n-1}+\ldots\}$$

where $a$'s and $b$'s belong to $K$ and at least one of $a_{j_n}^{(n)}$ and $b_{j_n}^{(n)}$ is not zero. Let

$$\mu_n = r_{-1}^{(n)} t^{-1} + r_{-2}^{(n)} t^{-2} + \ldots \ .$$

Take

$$\xi = \alpha_{-j_0-1} t^{-j_0-1} + \alpha_{-j_0-2} t^{-j_0-2} + \ldots + \alpha \{\beta_{-j_0-1} t^{-j_0-1} + \beta_{-j_0-2} t^{-j_0-2} + \ldots\}$$

where $\alpha$'s and $\beta$'s belong to $K$. We shall prove that with suitable choice of $\alpha$'s and $\beta$'s, we have the required result. We choose these by induction. Coefficient of $t^{-1}$ in $\lambda_0 \xi + \tilde{\lambda}_0 \tilde{\xi} + \mu_0 = 2\{a_{j_0}^{(0)} \alpha_{-j_0-1} - m b_{j_0}^{(0)} \beta_{-j_0-1}\} + r_{-1}^{(0)}$.

Since at least one of $a_{j_0}^{(0)}$, $b_{j_0}^{(0)}$ is not zero, we can choose $\alpha_{-j_0-1}, \beta_{-j_0-1} \in K$ such that coefficient of $t^{-1}$ in $\lambda_0 \xi + \tilde{\lambda}_0 \tilde{\xi} + \mu_0$ is not zero. Suppose we have selected already

$$\alpha_{-(j_0+1)}, \ldots, \alpha_{-(j_{k-1}+1)}, \beta_{-(j_0+1)}, \ldots, \beta_{-(j_{k-1}+1)}.$$

Since $j_k \geq j_{k-1}+1$, choose $\alpha_{-(j_{k-1}+2)}, \ldots, \alpha_{-j_k}, \beta_{-(j_{k-1}+2)}, \ldots, \beta_{-j_k}$, arbitrarily and $\alpha_{-(j_k+1)}, \beta_{-(j_k+1)}$ such that $t^{-1}$ has a non-zero coefficient in

$$\lambda_k \xi + \tilde{\lambda}_k \tilde{\xi} + \mu_k$$

$$= 2\Big\{\Big(\alpha_{-(j_k+1)} a_{j_k}^{(k)} + \ldots + \alpha_{-(j_0+1)} a_{j_0}^{(k)}\Big) - m\Big(\beta_{-(j_k+1)} b_{j_k}^{(k)} + \ldots + \beta_{-(j_0+1)} b_{j_0}^{(k)}\Big)\Big\} + r_{-1}^{(k)}.$$

This is possible, since $\big(a_{j_k}^{(k)}, b_{j_k}^{(k)}\big) \neq (0, 0)$.

Thus the induction is complete and lemma is proved.

In the subsequent discussion $K$ stands for a finite field.

*Lemma* 2.—Let $L_1(U), \ldots, L_n(U)$ be $n$ linear forms over $K\{t\}$ in $n$ variables $u_1, \ldots, u_n$ with determinant $\Delta \neq 0$. Then for every constant $C > 0$, there exists only a finite number of elements $U$ of $P_n$ satisfying

$$\max(|L_1(U)|, \ldots, |L_n(U)|) \leq C.$$

PROOF: We define a new function

$$F(U) = \max(|L_1(U)|, \ldots, |L_n(U)|)$$

and by a result of Mahler (1941), we get

$$F(U) \geq \gamma |U|$$

for a constant $\gamma > 0$ and for all $U \in R_n$ and our lemma follows immediately from this.

Since $K$ is a finite field, $K\{t\}$ is locally compact and we have

*Lemma* 3.—Every infinite bounded sequence in $K\{t\}$ (in the sense of valuation) has a convergent subsequence.

*Lemma* 4.—Let $p$, $q$ be integers such that

$$e^{2p+2q} \geq e^{-3} |\Delta|^3.$$

Then there exists a non-zero vector $U = (u, v, w, r) \in P_4$ such that

$$|M_1(U)| = |M_2(U)| \leq e^p, \ |M_3(U)| = |M_4(U)| \leq e^q \qquad \ldots \quad (3.2)$$

Further the inequalities in (3.2) have only a finite number of solutions from $P_4$.

PROOF: Since $M_1 = \tilde{M}_2$ and $M_3 = \tilde{M}_4$, let

$$M_1 = N_1+\alpha N_2,\ M_2 = N_1-\alpha N_2$$

$$M_3 = N_3+\alpha N_4,\ M_4 = N_3-\alpha N_4$$

where $N_1$, $N_2$, $N_3$, $N_4$ are linear forms with coefficients from $K\{t\}$. Then we can see

$$0 \neq |\det(M_1, M_2, M_3, M_4)| = |\det(N_1, N_2, N_3, N_4)| = |\Delta|^3.$$

Since $e^{2p+2q} \geq e^{-3}|\Delta|^3$ by a theorem of Mahler (*see* Armitage 1957*b*), there exists a non-zero $U = (u, v, w, r) \in P_4$ satisfying

$$|N_1(U)| \leq e^p,\ |N_2(U)| \leq e^p,\ |N_3(U)| \leq e^q,\ |N_4(U)| \leq e^q \quad \ldots \quad (3.3)$$

But, by definition, for any $U \in P_4$,

$$|M_1(U)| = |M_2(U)| = \max(|N_1(U)|, |N_2(U)|)$$

and

$$|M_3(U)| = |M_4(U)| = \max(|N_3(U)|, |N_4(U)|).$$

So non-zero $U \in P_4$, which satisfies (3.3), must also satisfy (3.2) and this proves the first part.

To prove that (3.2) has only a finite number of solutions $U \in P_4$, we first observe that any solution of (3.2) is also a solution of (3.3). But, by Lemma 2, (3.3) has only a finite number of solutions $U = (u, v, w, r) \in P_4$. This proves the lemma.

REMARK: Let $K$ be a field (not necessarily finite), in which $x^2+m = 0$ is not soluble. Let $L_1$, $L_2$, $L_3$, $L_4$ be four linear forms in variables $(x, y, z, s) = X$ with determinant $\Delta \neq 0$. Then Lemma 4 incidentally proves that there exists a non-zero vector $X \in P_4$, satisfying

$$|L_1^2(X)+mL_2^2(X)|\,|L_3^2(X)+mL_4^2(X)| \leq e^{-3}|\Delta|.$$

Now we distinguish the following three cases:

(i) $M_1 \neq 0 \neq M_2$, $M_3 \neq 0 \neq M_4$ for any non-zero $U \in P_4$.

(ii) One and only one of the pairs $(M_1, M_2)$, $(M_3, M_4)$, say $(M_3, M_4)$ assumes the value $(0, 0)$ for some non-zero vector $U \in P_4$.

(iii) $M_1 = 0 = M_2$ for some non-zero $U \in P_4$ and $M_3 = 0 = M_4$ for another non-zero vector $U \in P_4$.

## § 4. *Proof of Theorem 3 in Case I*

First we shall prove a few more lemmas.

*Lemma* 5.—Let $M \geq 1$ be any given integer. Then there exists a finite sequence of values $\alpha_n$, $\beta_n = \tilde{\alpha}_n$, $\gamma_n$, $\delta_n = \tilde{\gamma}_n (0 \leq n \leq N)$ of $M_1$, $M_2$, $M_3$, $M_4$

respectively corresponding to a sequence $\mathfrak{J}_M$ of elements $(u_n, v_n, w_n, r_n) \in P_4$ with gcd $(u_n, v_n, w_n, r_n) = 1$ and satisfying the following properties:

(i) $|\alpha_n^2 \gamma_n^2| \leq e^{-2} |\Delta|^3$.

(ii) $|\alpha_n| \leq e^{-1} |\alpha_{n-1}|$, $(n \geq 1)$.

(iii) $|\alpha_{n-1}^2 \gamma_n^2| \leq |\Delta|^3$, $(n \geq 1)$.

(iv) $|\gamma_0| \leq e^{-M}$, $|\alpha_0| \geq e^M$.

(v) $|\alpha_N| \leq e^{-M}$.

(vi) $|\gamma_n| \geq |\gamma_{n-1}|$, $(n \geq 1)$.

Proof: Let $|\Delta| = e^d$. Consider the solutions of

$$\left.\begin{aligned} |M_1(u, v, w, r)| = |M_2(u, v, w, r)| &\leq \max \left(e^{M-1}, e^{M+\left[\frac{3d}{2}\right]-2}\right) \\ |M_3(u, v, w, r)| = |M_4(u, v, w, r)| &\leq e^{-M+1} \end{aligned}\right\} \quad \ldots \quad (4.1)$$

By Lemma 4, the above inequalities have a non-zero solution $U = (u, v, w, r) \in P_4$ and the number of such solutions is finite. Consider

$$\min |M_3(u, v, w, r)|$$

as $(u, v, w, r)$ runs over non-zero solutions from $P_4$ of inequalities (4.1). Since the number of these solutions is finite, this minimum is attained and by hypothesis, it is not zero. Let this minimum be $e^{-T}$. Then $T \geq M-1$.
Consider the solutions of

$$\left.\begin{aligned} |M_1(u, v, w, r)| = |M_2(u, v, w, r)| &\leq e^{T+\left[\frac{3d}{2}\right]} \\ |M_3(u, v, w, r)| = |M_4(u, v, w, r)| &\leq e^{-T-1} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (4.2)$$

By Lemma 4, the inequalities (4.2) have at least one non-zero solution $U = (u, v, w, r) \in P_4$. Further the number of such solutions is finite. Hence min $|M_3(u, v, w, r)|$ taken over non-zero vectors in $P_4$ satisfying (4.2) is attained, say at $(u_0, v_0, w_0, r_0)$. Obviously, the gcd $(u_0, v_0, w_0, r_0) = 1$.

Let $\alpha_0 = M_1(u_0, v_0, w_0, r_0)$, $\beta_0 = M_2(u_0, v_0, w_0, r_0)$,

$\gamma_0 = M_3(u_0, v_0, w_0, r_0)$ and $\delta_0 = M_4(u_0, v_0, w_0, r_0)$. Then

$|\gamma_0| = |\delta_0| \leq e^{-T-1} \leq e^{-M}$ (since $T \geq M-1$).

Also for every non-zero $U = (u, v, w, r) \in P_4$ satisfying (4.2), we must have $|M_1(U)| > e^{M-1}$. Otherwise, we have $|M_1(U)| \leq e^{M-1}$ and then this $U$ satisfies (4.1), which contradicts the minimal nature of $T$. Hence, in particular $|\alpha_0| \geq e^M$.
Since $(u_0, v_0, w_0, r_0)$ satisfies (4.2), $(u_0, v_0, w_0, r_0)$ also satisfies condition (i) of Lemma 5. Suppose there exists non-zero $U = (u, v, w, r) \in P_4$ satisfying $|M_1(U)| \leq e^{-1}|\alpha_0|$. Then we claim $|M_3(U)| \geq |\gamma_0|$. For suppose we have $|M_3(U)| < |\gamma_0|$. Then this non-zero $U$ also satisfies (4.2), and this contradicts the minimal nature of $|\gamma_0|$.

Now we shall construct by induction a sequence $\mathfrak{J}_M$ satisfying the conditions of Lemma 5. $(u_0, v_0, w_0, r_0)$ satisfies the conditions of the lemma. Suppose $(u_0, v_0, w_0, r_0), \ldots, (u_{n-1}, v_{n-1}, w_{n-1}, r_{n-1})$ have been constructed. Let

$$\alpha_i = M_1(u_i, v_i, w_i, r_i), \beta_i = M_2(u_i, v_i, w_i, r_i)$$

$$\gamma_i = M_3(u_i, v_i, w_i, r_i) \text{ and } \delta_i = M_4(u_i, v_i, w_i, r_i) \text{ for all } i.$$

Consider the solutions of

$$\left.\begin{aligned} |M_1(u, v, w, r)| = |M_2(u, v, w, r)| \leq e^{-1}|\alpha_{n-1}|, \\ |M_3(u, v, w, r)| = |M_4(u, v, w, r)| \leq |\alpha_{n-1}|^{-1}e^{\left[\frac{3d}{2}\right]}. \end{aligned}\right\} \quad .. \quad (4.3)$$

By Lemma 4, there exists at least one non-zero solution $U = (u, v, w, r) \in P_4$ satisfying (4.3) and there are only a finite number of such solutions. As before, min $|M_3(U)|$, where minimum is being taken over all non-zero $U \in P_4$ satisfying (4.3) is attained, say at $(u_n, v_n, w_n, r_n)$. Conditions (i), (ii), (iii) of the lemma are obviously satisfied. It must satisfy also condition (vi), for otherwise, it would have been selected at the previous stage. Since $|\alpha_n| \leq e^{-1}|\alpha_{n-1}|$ for all $n$, we get integer $N'$ such that

$$|\alpha_n| \leq e^{-M}$$

for all $n \geq N'$. We stop at such an $N'$.
REMARK: 1. Clearly $\mathfrak{J}_M$ is not unique.
2. If $M_1 > M_2$, every $\mathfrak{J}_{M_1}$ is an $\mathfrak{J}_{M_2}$ also.

*Lemma* 6.—There exists

$$(x_0^{(M)}, \quad y_0^{(M)}, \quad z_0^{(M)}, \quad s_0^{(M)}) \in R_4,$$

such that

$$|xu_n + yv_n + zw_n + sr_n| \geq e^{-1}$$

for all $\qquad (x, y, z, s) \equiv (x_0^{(M)}, \quad y_0^{(M)}, \quad z_0^{(M)}, \quad s_0^{(M)}) \pmod{P_4}$

and for all $n (0 \leq n \leq N)$.

PROOF: Consider

$$\lambda_n = \frac{\alpha_{N-n}}{\Delta'}, \quad \mu_n = 0 \quad (n = 0, 1, \ldots, N).$$

Owing to condition (ii) of Lemma 5,

$$|\lambda_n| \geq e|\lambda_{n-1}| \quad (n = 1, \ldots, N).$$

We can, therefore, apply Lemma 1 and get $\xi \in K^{(m)}\{t\}$, such that

$$\left|\left|\frac{\alpha_n\xi + \tilde{\alpha}_n\tilde{\xi}}{\Delta'}\right|\right| = \left|\left|\frac{\alpha_n\xi + \beta_n\tilde{\xi}}{\Delta'}\right|\right| = e^{-1} (n = 0, 1, \ldots, N).$$

Let $\xi = \eta+\alpha\zeta;\ \eta,\ \zeta \in K\{t\}$. Define

$(x_0^{(M)},\ y_0^{(M)},\ z_0^{(M)},\ s_0^{(M)}) \in R_4$ by the equations

$$L_1(x_0^{(M)},\ y_0^{(M)},\ z_0^{(M)},\ s_0^{(M)}) = \eta,\quad L_2(x_0^{(M)},\ y_0^{(M)},\ z_0^{(M)},\ s_0^{(M)}) = \zeta,$$

$$L_3(x_0^{(M)},\ y_0^{(M)},\ z_0^{(M)},\ s_0^{(M)}) = 0 = L_4(x_0^{(M)},\ y_0^{(M)},\ z_0^{(M)},\ s_0^{(M)}).$$

Then $(x_0^{(M)},\ y_0^{(M)},\ z_0^{(M)},\ s_0^{(M)})$ also satisfies

$$L_1'(x_0^{(M)},\ y_0^{(M)},\ z_0^{(M)},\ s_0^{(M)}) = \xi,\quad L_2'(x_0^{(M)},\ y_0^{(M)},\ z_0^{(M)},\ s_0^{(M)}) = \tilde{\xi}$$

$$L_3'(x_0^{(M)},\ y_0^{(M)},\ z_0^{(M)},\ s_0^{(M)}) = 0 = L_4'(x_0^{(M)},\ y_0^{(M)},\ z_0^{(M)},\ s_0^{(M)}).$$

Substituting $x = x_0^{(M)},\quad y = y_0^{(M)},\quad z = z_0^{(M)},\quad s = s_0^{(M)}$, $u = u_n$, $v = v_n$, $w = w_n$ and $r = r_n$ in (3.1), we get

$$x_0^{(M)}u_n+y_0^{(M)}v_n+z_0^{(M)}w_n+s_0^{(M)}r_n = \frac{\xi\alpha_n+\tilde{\xi}\tilde{\alpha}_n}{\Delta'}.$$

Hence

$$\left\| x_0^{(M)}u_n+y_0^{(M)}v_n+z_0^{(M)}w_n+s_0^{(M)}r_n \right\|$$

$$= \left\| \frac{\xi\alpha_n+\tilde{\xi}\tilde{\alpha}_n}{\Delta'} \right\| = e^{-1} (0 \le n \le N).$$

Further if $(x, y, z, s) \equiv (x_0^{(M)},\ y_0^{(M)},\ z_0^{(M)},\ s_0^{(M)}) \pmod{P_4}$, we have

$$(xu_n+yv_n+zw_n+sr_n)-(x_0^{(M)}u_n+y_0^{(M)}v_n+z_0^{(M)}w_n+s_0^{(M)}r_n)$$

$$= (x-x_0^{(M)})u_n+(y-y_0^{(M)})v_n+(z-z_0^{(M)})w_n+(s-s_0^{(M)})r_n$$

and the right hand side is an element of $K[t]$. Consequently

$$\| xu_n+yv_n+zw_n+sr_n \| = \left\| x_0^{(M)}u_n+y_0^{(M)}v_n+z_0^{(M)}w_n+s_0^{(M)}r_n \right\| = e^{-1} (0 \le n \le N).$$

Hence

$$| xu_n+yv_n+zw_n+sr_n | \ge \| xu_n+yv_n+zw_n+sr_n \| = e^{-1} (0 \le n \le N).$$

This proves the lemma.

*Lemma* 7.—There exists a doubly infinite sequence of polynomials $(u_n, v_n, w_n, r_n)$ $(-\infty < n < \infty)$ with gcd $(u_n, v_n, w_n, r_n) = 1$, such that conditions (i), (ii), (iii) and (vi) of Lemma 5 hold and

$$\lim_{n\to\infty} | M_1(u_n, v_n, w_n, r_n) | = 0 = \lim_{n\to-\infty} | M_3(u_n, v_n, w_n, r_n) |.$$

Also

$$\lim_{n\to\infty} | M_3(u_n, v_n, w_n, r_n) | = \infty = \lim_{n\to-\infty} | M_1(u_n, v_n, w_n, r_n) |.$$

Also there exists $(x_0, y_0, z_0, s_0) \in R_4$ such that for all $(x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}$, the condition

$$| xu_n+yv_n+zw_n+sr_n | \ge e^{-1} (-\infty < n < \infty)$$

is satisfied.

PROOF: Let $L$ be any given integer. First we assert that there are only a finite number of pairs of non-zero elements $(u, v, w, r), (u', v', w', r') \in P_4$ satisfying

(a) $|M_1(u, v, w, r)|^2 |M_3(u', v', w', r')|^2 \leq |\Delta|^3$,

(b) $|M_1(u, v, w, r)|^2 |M_3(u, v, w, r)|^2 \leq e^{-2} |\Delta|^3$,

(c) $|M_1(u', v', w', r')| \leq e^L \leq e^{-1} |M_1(u, v, w, r)|$.

From (a) and (c) we get

$$|M_3(u', v', w', r')|^2 \leq e^{-2L-2} |\Delta|^3.$$

Therefore

$$|M_3(u', v', w', r')| \leq e^{-L-1+[3d/2]}.$$

But for $(u', v', w', r') \in P_4$,

$$|M_1(u', v', w', r')| = |M_2(u', v', w', r')|$$

and

$$|M_3(u', v', w', r')| = |M_4(u', v', w', r')|.$$

So we have

$$|M_1(u', v', w', r')| = |M_2(u', v', w', r')| \leq e^L \text{ [from (c)]}$$

and

$$|M_3(u', v', w', r')| = |M_4(u', v', w', r')| \leq e^{-L-1+[3d/2]}.$$

By Lemma 4, there exist only a finite number of $(u', v', w', r') \in P_4$ satisfying the above.

Further given non-zero $(u', v', w', r') \in P_4$, we assert there are only a finite number of non-zero $(u, v, w, r) \in P_4$. Since by assumption, $M_3(u', v', w', r') \neq 0$, from (a) we get

$$|M_1(u, v, w, r)|^2 \leq |\Delta|^3 . \; |M_3(u', v', w', r')|^{-2}.$$

From (b) we get,

$$|M_3(u, v, w, r)|^2 \leq e^{-2} |\Delta|^3 |M_1(u, v, w, r)|^{-2}$$
$$\leq e^{-4-2L} |\Delta|^3.$$

Now by using Lemma 2, we can see that there are only a finite number of $(u, v, w, r) \in P_4$ satisfying these.

Now suppose $M \geq 1$ is a given integer. Construct a sequence $\mathfrak{S}_M$ satisfying the conditions of Lemma 5. By conditions (iv), (v) and (ii) of Lemma 5, there exists a unique integer $N_1$ such that $N \geq N_1 \geq 1$ and

$$|\alpha_{N_1}| \leq 1 \leq e^{-1} |\alpha_{N_1-1}|.$$

Let $\mathfrak{S}'_M$ be the sequence of $(u'_n, v'_n, w'_n, r'_n)$ defined by

$$(u'_n, v'_n, w'_n, r'_n) = (u_{n+N_1}, v_{n+N_1}, w_{n+N_1}, r_{n+N_1}) \text{ for}$$
$$-N_1 \leq n \leq N_2 = N - N_1.$$

Let $\alpha'_n, \beta'_n, \gamma'_n, \delta'_n$ be the values assumed by $M_1$, $M_2$, $M_3$, $M_4$ respectively at $(u'_n, v'_n, w'_n, r'_n)$. Then conditions (i), (ii), (iii) and (vi) of Lemma 6 hold for

$$(u'_n, v'_n, w'_n, r'_n) \text{ and}$$

$$|\alpha'_0| \leq 1 \leq e^{-1}|\alpha'_{-1}|, \; |\alpha'_{N_2}| \leq e^{-M}, \; |\alpha'_{-N_1}| \geq e^M \quad \ldots \quad \ldots \quad (4.4)$$

We now construct $\mathfrak{I}$ by a diagonal process. By the observation already made, there are only a finite number of possibilities for

$$(u'_0, v'_0, w'_0, r'_0), \quad (u'_{-1}, v'_{-1}, w'_{-1}, r'_{-1})$$

and so at least one pair of these must occur infinitely often, as $M$ runs through the set of natural numbers. We fix one such possible pair, say $(\bar{u}_0, \bar{v}_0, \bar{w}_0, \bar{r}_0)$, $(\bar{u}_{-1}, \bar{v}_{-1}, \bar{w}_{-1}, \bar{r}_{-1})$. We now consider only those $\mathfrak{I}'_M$'s which contain these.

For these $M$, conditions (4.4) imply that for all $M$ large enough $N_2 \geq 1$, for otherwise $M_1(\bar{u}_0, \bar{v}_0, \bar{w}_0, \bar{r}_0) = 0$, which contradicts the hypothesis.

Denote by $\bar{\alpha}_n, \bar{\beta}_n, \bar{\gamma}_n, \bar{\delta}_n$ the values assumed by $M_1$, $M_2$, $M_3$, $M_4$ respectively at $(\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{r}_n)$. Now

$$|M_1(u'_1, v'_1, w'_1, r'_1)| \leq e^{-1}|\bar{\alpha}_0| \leq e^{-1}|M_1(\bar{u}_0, \bar{v}_0, \bar{w}_0, \bar{r}_0)|$$

for each $\mathfrak{I}'_M$.

Again, there are only a finite number of possible choices for $(u'_1, v'_1, w'_1, r'_1) \in P_4$. We pick out $(\bar{u}_1, \bar{v}_1, \bar{w}_1, \bar{r}_1) \in P_4$ which occurs infinitely often. Let $\mathfrak{S}_2$ be the infinite subsequence of $\mathfrak{S}_1$, consisting of those $\mathfrak{I}'_M$'s to which $(\bar{u}_1, \bar{v}_1, \bar{w}_1, \bar{r}_1)$ belong. We now consider only these $\mathfrak{I}'_M$'s. For these $\mathfrak{I}'_M$'s, if $M$ is large enough, $N_1 \geq 2$.

Consider

$$|M_1(\bar{u}_{-1}, \bar{v}_{-1}, \bar{w}_{-1}, \bar{r}_{-1})| \leq |\bar{\alpha}_{-1}| \leq e^{-1}|M_1(u'_{-2}, v'_{-2}, w'_{-2}, r'_{-2})|$$

for all those $\mathfrak{I}'_M$'s. As before, there are only a finite number of choices for $(u'_{-2}, v'_{-2}, w'_{-2}, r'_{-2}) \in P_4$. We pick out $(\bar{u}_{-2}, \bar{v}_{-2}, \bar{w}_{-2}, \bar{r}_{-2})$, which occurs infinitely often. Let $\mathfrak{S}_3$ be the infinite subsequence of $\mathfrak{S}_2$ composed of those $\mathfrak{I}'_M$'s to which $(\bar{u}_{-2}, \bar{v}_{-2}, \bar{w}_{-2}, \bar{r}_{-2})$ belongs and we consider only these $\mathfrak{I}'_M$'s. For these $\mathfrak{I}'_M$'s, $N_2 \geq 2$ if $M$ is large enough.

Continuing this process, we get a doubly infinite sequence $(\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{r}_n)$ $(-\infty < n < \infty)$, which satisfies conditions (i), (ii), (iii) and (vi) of Lemma 5.

Obviously, by construction

$$\lim_{n \to \infty} |\bar{\alpha}_n| = 0, \quad \lim_{n \to -\infty} |\bar{\alpha}_n| = \infty.$$

Due to condition (i) of this lemma, we must have

$$\lim_{n \to -\infty} |\bar{\gamma}_n| = 0.$$

Also $\lim_{n\to\infty} |\bar{\gamma}_n| = \infty$, for otherwise, there exists an integer $k$ such that

$$|\bar{\gamma}_n| \leq e^k$$

for all $n$. Then

$$|M_1(u, v, w, r)| = |M_2(u, v, w, r)| \leq 1$$

$$|M_3(u, v, w, r)| = |M_4(u, v, w, r)| \leq e^k$$

has infinitely many solutions $(u, v, w, r) \in P_4$, which by using Lemma 2, we can see, is not true. Thus this sequence has all the required properties.

Since there is no danger of confusion, we now suppress the bar on $(u_n, v_n, w_n, r_n)$ and $\alpha_n, \beta_n, \gamma_n, \delta_n$.

During our construction of double sequence, we have used the subsequences $\mathfrak{D}_1, \mathfrak{D}_2, \ldots$ of the sequences $\mathfrak{J}'_M$'s which have the properties

(i) $\mathfrak{D}_{t+1}$ is a subsequence of $\mathfrak{D}_t$

(ii) Given any finite set of $(u_j, v_j, w_j, r_j)$'s, this is a subset of $\mathfrak{D}_j$ for $j$ large.

Let $\mathfrak{J}^*_j$ be the first member of $\mathfrak{D}_j$. Then the sequences $\mathfrak{J}^*_j$ has the following properties:

(1) $\mathfrak{J}^*_j = \mathfrak{J}_{m_j}$ for some $m_j$; $m_j \to \infty$ as $j \to \infty$.

(2) Each finite set of $(u_j, v_j, w_j, r_j)$ is a subset of $\mathfrak{J}^*_j$ for all $j$ large enough.

By Lemma 6, there exists a sequence

$$\{(x_0^{(m_j)}, \ y_0^{(m_j)}, \ z_0^{(m_j)}, \ s_0^{(m_j)}) \,;\, j = 1, 2, \ldots\}$$

and we may assume that

$$|x_0^{(m_j)}| < 1, \quad |y_0^{(m_j)}| < 1, \quad |z_0^{(m_j)}| < 1, \quad |s_0^{(m_j)}| < 1$$

for all $j$.

By using Lemma 3, we can see that

$$\{(x_0^{(m_j)}, \ y_0^{(m_j)}, \ z_0^{(m_j)}, \ s_0^{(m_j)}), j = 1, 2, \ldots\}$$

has a convergent subsequence tending to a limit point $(x_0, y_0, z_0, s_0)$ (say). Without loss of generality, assume this itself is convergent. This choice of $(x_0, y_0, z_0, s_0)$ does what is required. For take $(x, y, z, s) \equiv (x_0, y_0, z_0, s_0)$ (mod $P_4$). Let $(u_n, v_n, w_n, r_n)$ be any member of this double sequence. For all $j$ large enough, $(u_n, v_n, w_n, r_n) \in \mathfrak{J}^*_j = \mathfrak{J}_k$ (say),
and

$$\begin{aligned}
&|xu_n + yv_n + zw_n + sr_n| \\
&= \big| (x - x_0 + x_0^{(k)})u_n + (y - y_0 + y_0^{(k)})v_n + (z - z_0 + z_0^{(k)})w_n + (s - s_0 + s_0^{(k)})r_n + \\
&\qquad (x_0 - x_0^{(k)})u_n + (y_0 - y_0^{(k)})v_n + (z_0 - z_0^{(k)})w_n + (s_0 - s_0^{(k)})r_n \big|.
\end{aligned}$$

Since

$$\left(x-x_0+x_0^{(k)},\quad y-y_0+y_0^{(k)},\quad z-z_0+z_0^{(k)},\quad s-s_0+s_0^{(k)}\right) \equiv \left(x_0^{(k)},\quad y_0^{(k)},\quad z_0^{(k)},\quad s_0^{(k)}\right) \pmod{P_4},$$

$$\left|\left(x-x_0+x_0^{(k)}\right)u_n+\left(y-y_0+y_0^{(k)}\right)v_n+\left(z-z_0+z_0^{(k)}\right)w_n+\left(s-s_0+s_0^{(k)}\right)r_n\right| \geq e^{-1} \text{ for all } n.$$

If $j \to \infty$, $k \to \infty$; therefore, if $j$ is large enough

$$\left|\left(x_0-x_0^{(k)}\right)u_n+\left(y_0-y_0^{(k)}\right)v_n+\left(z_0-z_0^{(k)}\right)w_n+\left(s_0-s_0^{(k)}\right)r_n\right| < e^{-1}$$

and hence

$$|xu_n+yv_n+zw_n+sr_n| \geq e^{-1} \text{ for all } n.$$

This completes the proof of Lemma 7.

*Proof of Theorem 3 in Case I.*—Let $(x_0, y_0, z_0, s_0)$ and the doubly infinite sequence $(u_n, v_n, w_n, r_n)$ $(-\infty < n < \infty)$ be given by Lemma 7. Take any $(x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}$. Suppose $a$, $b = \bar{a}$, $c$, $d = \bar{c}$ are the values assumed by $L_1'$, $L_2'$, $L_3'$, $L_4'$ respectively at $(x, y, z, s)$.

First we assert neither of $a$, $b$, $c$, $d$ is zero. For suppose $a = 0 = b$. Then from (3.1) we get for any $n$,

$$xu_n+yv_n+zw_n+sr_n = \frac{c\gamma_n+d\delta_n}{\Delta'}.$$

So

$$|xu_n+yv_n+zw_n+sr_n| = \left|\frac{c\gamma_n+d\delta_n}{\Delta'}\right| \leq \left|\frac{c\gamma_n}{\Delta}\right|.$$

(Since $|c\gamma_n| = |d\delta_n|$ and $|\Delta'| = |\Delta|$.) Making $n \to -\infty$, we get

$$|xu_n+yv_n+zw_n+sr_n| \to 0$$

and this contradicts the fact that $|xu_n+yv_n+zw_n+sr_n| \geq e^{-1}$ for all $n$. Hence $a \neq 0 \neq b$. Similarly $c \neq 0 \neq d$.

Since $|\gamma_n| \to \infty$ as $n \to \infty$ and $|\gamma_n| \to 0$ as $n \to -\infty$ and $|\gamma_{n+1}| \geq |\gamma_n|$, we can choose an integer $n$
such that

$$|\gamma_n|^2 \leq |\Delta|^{\frac{3}{2}}\left|\frac{a}{c}\right| \leq |\gamma_{n+1}|^2. \qquad \ldots \quad \ldots \quad \ldots \quad (4.5)$$

From condition (iii) of Lemma 7, we get

$$|\alpha_n|^2 \leq |\Delta|^3|\gamma_{n+1}|^{-2} \leq \left|\frac{c}{a}\right||\Delta|^{\frac{3}{2}} \qquad \ldots \quad \ldots \quad \ldots \quad (4.6)$$

Put

$$p = |ab||\alpha_n\beta_n| = |a|^2|\alpha_n|^2, \quad q = |cd||\gamma_n\delta_n| = |c|^2|\gamma_n|^2$$

and

$$k^2 = |abcd||\Delta|^3 = |ac|^2|\Delta|^3.$$

Then from (4.5) and (4.6), we get

$$p = |a|^2||\alpha_n|^2 \leq |ac||\Delta|^{\frac{3}{2}} = k,$$

$$q = |c|^2|\gamma_n|^2 \leq |ac|||\Delta|^{\frac{3}{2}} = k.$$

From (3.1), we get

$$|\Delta'(xu_n+yv_n+zw_n+sr_n)| = |a\alpha_n+b\beta_n+c\gamma_n+d\delta_n| \leq \max(|a\alpha_n|, |c\gamma_n|)$$

i.e.
$$|\Delta(xu_n+yv_n+zw_n+sr_n)|^2 \leq \max(|a\alpha_n|^2, |c\gamma_n|^2) \leq k.$$

But from Lemma 7, we get

$$|xu_n+yv_n+zw_n+sr_n)| \geq e^{-1}$$

$$|\Delta(xu_n+yv_n+zw_n+sr_n)|^2 \geq e^{-2}|\Delta|^2.$$

Hence

$$k \geq e^{-2}|\Delta|^2$$

or
$$|abcd|^{\frac{1}{2}}|\Delta|^{\frac{3}{2}} \geq e^{-2}|\Delta|^2$$

or
$$|abcd| \geq |\Delta|/e^4.$$

This proves the theorem in this case.

## § 5. *Proof of Theorem 3 in Case II*

Here we prove the following lemmas.

*Lemma 5'.—Let $M > 0$ be a given integer.* Then there exists a finite sequence of values $\alpha_n$, $\beta_n = \bar{\alpha}_n$, $\gamma_n$, $\delta_n = \bar{\gamma}_n$ $(0 \leq n \leq N)$ of $M_1$, $M_2$, $M_3$, $M_4$ respectively corresponding to a sequence $\mathfrak{S}_M$ of polynomials $(u_n, v_n, w_n, r_n)$ $(0 \leq n \leq N)$ with gcd $(u_n, v_n, w_n, r_n) = 1$ satisfying conditions (i), (ii), (iii), (v) and (vi) of Lemma 5 and

(iv'): $\gamma_0 = 0, \gamma_n \neq 0$ for $0 < n \leq N$.

PROOF: By hypothesis there exists $(\alpha, \beta, \gamma, \delta) \in P_4$, $(\alpha, \beta, \gamma, \delta) \neq 0$ and with gcd $(\alpha, \beta, \gamma, \delta) = 1$, such that

$$M_3(\alpha, \beta, \gamma, \delta) = 0 = M_4(\alpha, \beta, \gamma, \delta).$$

Consider the solutions of

$$\left.\begin{aligned} |M_1(u, v, w, r)| = |M_2(u, v, w, r| \leq |M_1(\alpha, \beta, \gamma, \delta)| \\ |M_3(u, v, w, r)| = |M_4(u, v, w, r)| = 0 \leq 1. \end{aligned}\right\} \quad .. \quad (5.1)$$

By hypothesis, (5.1) has at least one non-zero solution $(\alpha, \beta, \gamma, \delta) \in P_4$. Using Lemma 2, we can see that (5.1) has only a finite number of solutions $U = (u, v, w, r) \in P_4$. Consider

$$\min |M_1(u, v, w, r)|,$$

the minimum being taken over all non-zero $(u, v, w, r) \in P_4$ with gcd $(r, v, w, r) = 1$ satisfying (5.1). As there are only a finite number of solutions, minimum is attained, say at $(u_0, v_0, w_0, r_0)$. Then $M_1(u_0, v_0, w_0, r_0) \neq 0$.

Now suppose there exists non-zero $(u, v, w, r) \in P_4$ with gcd $(u, v, w, r) = 1$, such that

$$|M_1(u, v, w, r)| \leq e^{-1} |M_1(u_0, v_0, w_0, r_0)|.$$

Then $M_3(u, v, w, r) \neq 0$, for otherwise, this $(u, v, w, r)$ contradicts the choice of $(u_0, v_0, w_0, r_0)$. In particular, if there exists $(u_1, v_1, w_1, r_1)$ satisfying condition (ii) of Lemma 5, then $\gamma_1 \neq 0$.

With this choice of $(u_0, v_0, w_0, r_0)$, we proceed as in Lemma 5 and determine the sequence $\mathfrak{S}_M$ satisfying conditions (i), (ii), (iii), (v) and (vi) of Lemma 5.

Since $|\gamma_n| \geq |\gamma_{n-1}|$ and $\gamma_1 \neq 0$, we get condition (iv′) of Lemma 5.

*Lemma* 6′.—For the sequence obtained in Lemma 5′, there exists

$$\left(x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}\right) \in R_4$$

such that

$$|xu_n + yv_n + zw_n + sr_n| \geq e^{-1}$$

for all

$$(x, y, z, s) \equiv \left(x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}\right) (\mathrm{mod}\ P_4)$$

and for all $n$ $(0 \leq n \leq N)$.

The proof of Lemma 6′ is the same as that of Lemma 6.

*Lemma* 7′.—There exists an infinite sequence $\mathfrak{S}$ of

$$(u_n, v_n, w_n, r_n) \in P_4 \ (0 \leq n < \infty) \text{ with gcd } (u_n, v_n, w_n, r_n) = 1$$

satisfying conditions (i), (ii), (iii) and (vi) of Lemma 5, and

$$\lim_{n\to\infty} |\gamma_n| = \infty, \quad \lim_{n\to\infty} |\alpha_n| = 0$$

and

$$\gamma_0 = 0, \gamma_n \neq 0 \ \text{ for } \ 0 < n < \infty.$$

Also there exists $(x_0, y_0, z_0, s_0) \in R_4$ such that for all $(x, y, z, s) \equiv (x_0, y_0, z_0, s_0)$ (mod $P_4$) the condition

$$|xu_n + yv_n + zw_n + sr_n| \geq e^{-1} \quad (0 \leq n < \infty)$$

is satisfied.

PROOF: Let $(u, v, w, r) \in P_4$ be a fixed element of $P_4$. Let $L$ be any integer such that

$$e^L \leq e^{-1}|M_1(u, v, w, r)|.$$

Then we claim there are only a finite number of non-zero $(u', v', w', r') \in P_4$ with gcd $(u', v', w', r') = 1$ satisfying

$$(a) \ |M_1(u, v, w, r)|^2 |M_3(u', v', w', r')|^2 \leq |\Delta|^3,$$

$$(b) \ |M_1(u', v', w', r')| \leq e^L.$$

The proof of this is exactly the same as in Lemma 7.

Now choose $(u_0, v_0, w_0, r_0)$ as in Lemma 5′. We note here that $(u_0, v_0, w_0, r_0)$ as chosen in Lemma 5′ is independent of $M$ and we use same $(u_0, v_0, w_0, r_0)$ for all $\mathfrak{S}_M$.

For each $\mathfrak{S}_M$, the corresponding $(u_1, v_1, w_1, r_1)$ satisfies

$$|M_3(u_1, v_1, w_1, r_1)|^2 |M_1(u_0, v_0, w_0, r_0)|^2 \leq |\Delta|^3$$

and

$$|M_1(u_1, v_1, w_1, r_1)| \leq e^{-1}|\alpha_0|.$$

By the above observation, there are only a finite number of $(u_1, v_1, w_1, r_1) \in P_4$ satisfying above. So we pick out $(\bar{u}_1, \bar{v}_1, \bar{w}_1, \bar{r}_1) \in P_4$ which occurs infinitely often. Let $\mathfrak{S}_1$ be the infinite subsequence of those $\mathfrak{J}_M$'s to which $(\bar{u}_1, \bar{v}_1, \bar{w}_1, \bar{r}_1)$ belong. Now we consider only these $\mathfrak{J}_M$'s.

Then if $M$ is large and $\mathfrak{J}_M \in \mathfrak{S}_1$, then $\mathfrak{J}_M$ contains more than three terms. Now for each $\mathfrak{J}_M \in \mathfrak{S}_1$,

$$|M_1(u_2, v_2, w_2, r_2)| \leq e^{-1}|M_1(\bar{u}_1, \bar{v}_1, \bar{w}_1, \bar{r}_1)|$$

and

$$|M_3(u_2, v_2, w_2, r_2)|^2|M_1(\bar{u}_1, \bar{v}_1, \bar{w}_1, \bar{r}_1)|^2 \leq |\Delta|^3.$$

As before, there are only a finite number of choices for $(u_2, v_2, w_2, r_2)$. So pick $(\bar{u}_2, \bar{v}_2, \bar{w}_2, \bar{r}_2) \in P_4$ which occurs in infinitely many $\mathfrak{J}_M$. Let $\mathfrak{S}_2$ be the infinite subsequence of $\mathfrak{S}_1$ of those $\mathfrak{J}_M$'s to which $(\bar{u}_2, \bar{v}_2, \bar{w}_2, \bar{r}_2)$ belong.

Continuing this process, we have an infinite sequence $(\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{r}_n)$ $(0 \leq n < \infty)$ which satisfies conditions (i), (ii), (iii), (vi) of Lemma 5 and further $M_3(\bar{u}_0, \bar{v}_0, \bar{w}_0, \bar{r}_0) = 0$, $M_3(\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{r}_n) \neq 0$ for $n \neq 0$.
Also by construction

$$\lim_{n\to\infty} |M_1(\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{r}_n)| = 0.$$

As in Lemma 7, we can see that

$$\lim_{n\to\infty} |M_3(\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{r}_n)| = \infty.$$

In the construction of $(\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{r}_n)$'s, we have used subsequences $\mathfrak{S}_1$, $\mathfrak{S}_2, \ldots$ which have the same properties as in Lemma 7.

Following the same method as in Lemma 7, we can get $(x_0, y_0, z_0, s_0)$ with the required properties. In the rest of the proof, we suppress the bar.

*Proof of Theorem 3 in Case II.*—Let $(u_n, v_n, w_n, r_n)$ $(0 \leq n < \infty)$ and $(x_0, y_0, z_0, s_0)$ be given by Lemma 7′. Suppose $a$, $b = \tilde{a}$, $c$, $d = \tilde{c}$ are the values assumed by $L_1'$, $L_2'$, $L_3'$, $L_4'$ respectively for some $(x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}$.

First we assert neither of $a$, $b$, $c$, $d$ is zero. For suppose $a = 0 = b$. Then from (3.1), we get for every $n (0 \leq n < \infty)$,

$$xu_n + yv_n + zw_n + sr_n = \frac{c\gamma_n + d\delta_n}{\Delta'}.$$

So
$xu_0 + yv_0 + zw_0 + sr_0 = 0$ (since $\gamma_0 = 0 = \delta_0$), and this contradicts Lemma 7′.

Now suppose $c = 0 = d$. Then from (3.1) we get that for every $n$,

$$xu_n + yv_n + zw_n + sr_n = \frac{a\alpha_n + b\beta_n}{\Delta'}$$

or $$|xu_n+yv_n+zw_n+sr_n| = \left|\frac{a\alpha_n+b\beta_n}{\Delta'}\right| \leq \frac{|a\alpha_n|}{|\Delta|}.$$

Making $n \to \infty$, we get
$$\lim_{n\to\infty} |xu_n+yv_n+zw_n+sr_n| = 0.$$

This contradicts Lemma 7′ again.

The proof of Theorem 3 in this case is exactly the same as in earlier case.

§ 6. *Proof of Theorem 3 in Case III.*— $M_1 = 0 = M_2$ for some non-zero $U \in P_4$ and $M_3 = 0 = M_4$ for some other non-zero $U \in P_4$.

Then we must have one of the following:

(*a*) For each $M \geq 1$ there exists an $\mathfrak{S}_M$ with the properties of Lemma 5′; or

(*b*) there exists a finite sequence of values $\alpha_n$, $\beta_n = \tilde{\alpha}_n$, $\gamma_n$, $\delta_n = \tilde{\gamma}_n$ of $M_1$, $M_2$, $M_3$, $M_4$ respectively corresponding to $(u_n, v_n, w_n, r_n) \in P_4$ with gcd $(u_n, v_n, w_n, r_n) = 1$ satisfying conditions (i), (ii), (iii) and (vi) of Lemma 5 and also the following:

(iv′) $\gamma_0 = 0 = \delta_0$, $\gamma_n \neq 0$ for $0 < n \leq N$.

(v′) $\alpha_N = 0 = \beta_N$, $\alpha_n \neq 0$ for $0 \leq n < N$.

The proof of this obviously follows from that of Lemma 5′. If we have (*a*), the rest of the proof is the same as in case II. If we have (*b*), we prove the following lemma.

*Lemma* 6″.—For the finite sequence obtained above in (*b*), there exists $(x_0, y_0, z_0, s_0) \in R_4$ such that

$$|xu_n+yv_n+zw_n+sr_n| \geq e^{-1} \quad \ldots \quad \ldots \quad \ldots \quad (6.1)$$

for all $(x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}$ and all $n(0 \leq n \leq N)$.

PROOF: Since $\alpha_N = 0$, $\gamma_N \neq 0$. So we can choose $\theta = \sigma+\alpha\tau$; $\sigma, \tau \in K\{t\}$ such that

$$\| \gamma_N\theta+\tilde{\gamma}_N\tilde{\theta} \| = e^{-1}.$$

Owing to condition (ii) of Lemma 5, we can apply Lemma 1 with

$$A_n = \alpha_{N-n},\ \mu_n = \gamma_{N-n}\theta+\delta_{N-n}\tilde{\theta}(1 \leq n \leq N) \text{ and we get}$$
$$\xi = \eta+\alpha\zeta \in K^{(m)}\{t\};\ \eta, \zeta \in K\{t\},$$

such that

$$\| \alpha_n\xi+\tilde{\alpha}_n\tilde{\xi}+\gamma_n\theta+\delta_n\tilde{\theta} \| = e^{-1}(0 \leq n < N).$$

Since $\alpha_N = 0$, due to (6.1) we have

$$\| \alpha_N\xi+\beta_N\tilde{\xi}+\gamma_N\theta+\delta_N\tilde{\theta} \| = \| \gamma_N\theta+\tilde{\gamma}_N\tilde{\theta} \| = e^{-1}$$

and

$$\| \alpha_n\xi+\beta_n\tilde{\xi}+\gamma_n\theta+\tilde{\gamma}_n\tilde{\theta} \| = e^{-1}(0 \leq n \leq N). \quad \ldots \quad \ldots \quad (6.2)$$

Let $x_0, y_0, z_0, s_0$ be the solution of

$$L_1(x, y, z, s) = \eta\Delta', \quad L_2(x, y, z, s) = \zeta\Delta',$$
$$L_3(x, y, z, s) = \sigma\Delta', \quad L_4(x, y, z, s) = \tau\Delta'.$$

This $(x_0, y_0, z_0, s_0)$ also satisfies

$$L'_1(x, y, z, s) = \xi \Delta', \quad L'_2(x, y, z, s) = \tilde{\xi} \Delta',$$

$$L'_3(x, y, z, s) = \theta \Delta', \quad L'_4(x, y, z, s) = \tilde{\theta} \Delta'.$$

From (3.1) we get

$$x_0 u_n + y_0 v_n + z_0 w_n + s_0 r_n = \xi \alpha_n + \tilde{\xi} \beta_n + \theta \gamma_n + \tilde{\theta} \delta_n.$$

Then from (6.2) we get

$$||x_0 u_n + y_0 v_n + z_0 w_n + s_0 r_n|| = ||\xi \alpha_n + \tilde{\xi} \beta_n + \theta \gamma_n + \tilde{\theta} \delta_n|| = e^{-1}$$
$$(0 \le n \le N).$$

Take $(x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}$.
Then

$$|xu_n + yv_n + zw_n + sr_n| \ge ||xu_n + yv_n + zw_n + sr_n||$$
$$= ||x_0 u_n + y_0 v_n + z_0 w_n + s_0 r_n|| = e^{-1} \ (0 \le n \le N).$$

*Proof of Theorem 3.*—Let $(x_0, y_0, z_0, s_0) \in R_4$ be given by Lemma 6″. Take

$$(x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}.$$

Let $a$, $b$, $= \tilde{a}$, $c$, $d = \tilde{c}$ be the values taken by

$$L'_1, L'_2, L'_3, L'_4$$

respectively at $(x, y, z, s)$. As in case II, we can suppose $a \ne 0 \ne b$. Now suppose $c = 0 = d$. From (3.1) we get for every $n$

$$xu_n + yv_n + zw_n + sr_n = \frac{a\alpha_n + b\beta_n}{\Delta'}.$$

So

$$xu_N + yv_N + zw_N + sr_N = 0.$$

This contradicts Lemma 6″.

Now as in (4.5), if we can choose an integer $n (1 \le n \le N)$ such that

$$|\gamma_{n-1}|^2 \le |\Delta|^{\frac{1}{2}} \left|\frac{a}{c}\right| \le |\gamma_n|^2,$$

the proof is similar as in case I. If not, we have

$$|\gamma_N|^2 < |\Delta|^{\frac{1}{2}} \left|\frac{a}{c}\right|.$$

But from (3.1) and using $\alpha_N = 0 = \beta_N$, we get

$$|xu_N + yv_N + zw_N + sr_N| = \frac{|c\gamma_N + d\delta_N|}{|\Delta'|} \le \frac{|c\gamma_N|}{|\Delta|}.$$

So

$$|xu_N + yv_N + zw_N + sr_N|^2 \le \frac{|c^2||\gamma_N|^2}{|\Delta|^2} \le \frac{|ac|}{|\Delta|^{\frac{1}{2}}}.$$

But by Lemma 6″, we get

$$|xu_N+yv_N+zw_N+sr_N| \geq e^{-1}.$$

Hence

$$\frac{|ac|}{|\Delta|^{\frac{1}{2}}} \geq e^{-2} \text{ or } |a^2c^2| \geq \frac{|\Delta|}{e^4} \text{ or } |abcd| \geq \frac{|\Delta|}{e^4}.$$

This proves the theorem in this case.


## Acknowledgement

The author takes this opportunity to thank Professor R. P. Bambah for his kind guidance.


## References


Armitage, J. V. (1957*a*). Euclid's algorithm in certain algebraic function fields. *Proc. Lond. math. Soc.*, **7**, 498–509.

——— (1957*b*). The product of *n* linear forms in a field of series. *Mathematika*, **4**, 132–137.

Cassels, J. W. S. (1952*a*). The inhomogeneous minimum of binary quadratic, ternary cubic and quaternary quartic forms. *Proc. Camb. Phil. Soc.*, **48**, 72-86.

——— (1952*b*). Addendum to the paper mentioned above. *Proc. Camb. Phil. Soc.*, **48**, 519-520.

Davenport, H. (1950*a*). Euclid's algorithm in cubic fields of negative discriminant. *Acta math.*, **84**, 159–179.

——— (1950*b*). Euclid's algorithm in certain quartic fields. *Trans. Am. math. Soc.*, **68**, 508–532.

——— (1951). Indefinite binary quadratic forms and Euclid's algorithm in real quadratic fields. *Proc. Lond. math. Soc.*, **53**, 65–82.

Mahler, K. (1941). An analogue to Minkowski's geometry of numbers in a field of series. *Ann. math.*, **42**, 488–522.