RADIAL OSCILLATIONS OF COMPOSITE POLYTROPES—PART II

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Small radial oscillations of four composite polytropic models with interfaces $\xi_i = 0{\cdot}2094$, $0{\cdot}3915$, $0{\cdot}6160$ and $0{\cdot}8250$, having indices 1·5 in the core and 3 in the envelope, have been considered. The anharmonic pulsations of these models have also been considered.

1. INTRODUCTION

We consider in this paper the small adiabatic radial oscillations of the composite polytropes having indices 1·5 in the core and 3 in the envelope. Four such models with interfaces at $\xi_i = 0{\cdot}2094$, $0{\cdot}3915$, $0{\cdot}6160$ and $0{\cdot}8250$ have been obtained in the Part I [Manmohan Singh (1969)]. The radial oscillations of these models have been worked out for the first four modes taking $\alpha = 0{\cdot}6$, $0{\cdot}5$ and $0{\cdot}4$. We have also considered their anharmonic pulsations for the case $\alpha = 0{\cdot}6$.

2. SMALL RADIAL OSCILLATIONS

Using Emden variables ξ and ϕ, the equations of pulsation for the core ($n = 1{\cdot}5$) may be written as

$$\frac{d^2\eta}{d\xi^2}+\frac{1}{\xi}\left(4+\frac{5}{2}\frac{\xi}{\phi}\frac{d\phi}{d\xi}\right)\frac{d\eta}{d\xi}+\left(\frac{\omega^2}{\phi}+\frac{5}{2}\frac{\alpha}{\xi\phi}\frac{d\phi}{d\xi}\right)\eta = 0 \quad \ldots \quad \ldots \quad (1)$$

where

$$\frac{\omega^2}{A^4} = \frac{R^2\sigma^2\rho_c}{\gamma P_c} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

and other symbols have the same meanings as given by Prasad and Gurm (1961).

For the envelope ($n = 3$) the equations of pulsation in terms of variables ξ and θ are

$$\frac{d^2\eta}{d\xi^2}+\frac{4}{\xi}\left(1+\frac{\xi}{\theta}\frac{d\theta}{d\xi}\right)\frac{d\eta}{d\xi}+\left(\frac{\Omega^2}{\theta}+\frac{4\alpha}{\xi\theta}\frac{d\theta}{d\xi}\right)\eta = 0 \quad \ldots \quad \ldots \quad (3)$$

Here

$$\Omega^2 = \frac{R^2\sigma^2\rho_c}{\gamma P_c}\frac{\theta_i}{\phi_i}$$

$$= \frac{\omega^2}{A^4}\frac{\theta_i}{\phi_i} \quad \ldots \quad (4)$$

where θ_1 and ϕ_1 are the values of θ and ϕ at the interface.

Eqn. (1) has singularities at the centre ($\xi = 0$) and also at the surface. But as the solution of this equation does not extend to the surface, we only require a solution finite at $\xi = 0$. So we develop a solution in series near the centre which is non-singular at the centre $\xi = 0$.

With the help of well-known expansion [B.A.M. Tables (1932)] for ϕ we have

$$\phi = A^4\left\{1-\frac{A^2\xi^2}{6}+\frac{A^4\xi^4}{80}-\frac{A^6\xi^6}{1440}+\ldots\right\} \quad \ldots \quad (5)$$

The solution in series for the centre near $\xi = 0$ is assumed to be

$$\eta = c_0+c_2\xi^2+c_4\xi^4+c_6\xi^6+\ldots \quad \ldots \quad (6)$$

On substitution in eqn. (1) we can find the coefficients c_2, c_4, c_6. . . . by equating the coefficients of various powers of ξ.

The eqn. (3) for the envelope has regular singularities at $\xi = 0$ and 1. A solution in series started from one extremity usually becomes divergent at the other. For the envelope we require a solution finite at $\xi = 1$ and not extending to the centre. To obtain it we put $\xi = 1-t$ in eqn. (3). We have

$$\frac{d^2\eta}{dt^2}-\frac{4.}{1-t}\left(1-\frac{1-t}{\theta}\frac{d\theta}{dt}\right)\frac{d\eta}{dt}+\left(\frac{\Omega^2}{\theta}-\frac{4\alpha}{1-t}\frac{1}{\theta}\frac{d\theta}{dt}\right)\eta = 0 \quad \ldots \quad (7)$$

To envelope a series solution we have taken θ as follows:

$$\theta = \omega_n\left\{t+t^2+t^3+t^4+\left(1-\frac{1}{20}\omega_n^2\right)t^5+\ldots\right\} \quad \ldots \quad (8)$$

where

$$\omega_n = -\left(\frac{d\theta}{d\xi}\right)_{\xi=1} \quad \ldots \quad (9)$$

and the series assumed is

$$\eta = 1+a_1t+a_2t^2+a_3t^3+\ldots \quad \ldots \quad (10)$$

On substituting (8) and (10) in (7) we can find the coefficients a_1, a_2, a_3, . . . by equating to zero the coefficients of various power of t.

The solutions of eqns. (1) and (7), satisfying boundary conditions at the two extremities as well as the interface, have been found out in the same way as by Prasad (1953).

The values of ω^2/A^4 for the first four modes for three values of α (= 0·6, 0·5 and 0·4) are given for the four models in Table I.

TABLE I

Values of ω^2/A^4

Model with interface at	α	zero-th mode	1st mode	2nd mode	3rd mode
$\xi = 0{\cdot}2094$	0·6	6·023504	11·226409	19·105066	29·279197
	0·5	5·306420	10·560777	18·494241	28·689014
	0·4	4·481545	9·916510	17·899210	28·108813
$\xi = 0{\cdot}3915$	0·6	5·554554	11·919778	21·318248	33·279639
	0·5	4·744743	11·402158	20·829913	32·793357
	0·4	3·884831	10·903206	20·348435	32·309936
$\xi = 0{\cdot}6160$	0·6	4·396333	14·232968	25·697994	39·415017
	0·5	3·693355	13·807355	25·247873	38·919717
	0·4	2·978025	13·382857	24·797359	38·428582
$\xi = 0{\cdot}8250$	0·6	3·498776	15·719071	29·998478	46·498867
	0·5	2·931580	15·239002	29·561448	46·048814
	0·4	2·358007	14·759121	29·124757	45·599013

3. Anharmonic Pulsations

For small radial oscillations the square of the amplitude is neglected in the equation of motion. This gives a symmetrical sine curve for the radial velocity of pulsating stars. But observation shows that the 'Cepheid Variables' have velocity curves, which are far from being represented by a simple sine curve. They usually show a steep rise to a maximum and then a slower decline to minimum. This necessitates the consideration of the second order terms.

The equations of anharmonic pulsations (Prasad 1949*b*) for the composite polytrope may be written as

$$\frac{d^2q}{d^2}+\beta_m q_m = \frac{A^4}{\omega_1^2 I_m}\left\{\sum_{jj} D_{jjm}q_j^2+2\sum_{jk} D_{jkm}q_jq_k\right\} \qquad (11)$$

where

$$r = R\xi \qquad (12)$$

$$\tau = \sigma_1 t \qquad (13)$$

$$\beta_m = \frac{\sigma_m^2}{\sigma_1^2} \qquad (14)$$

$$I_m = \int_0^{\xi_i} \phi^{3/2}\xi^4\eta_m^2\, d\xi+\frac{\phi_i^{3/2}}{\theta_i^3}\int_{\xi_i}^1 \theta^3\xi^4\eta_m^2\, d\xi \qquad (15)$$

and

$$D_{jkm} = -\frac{1}{2}\left(3-\frac{4}{\gamma}\right)(3\gamma+1)\left\{\frac{5}{2}\int_0^{\xi_i} \phi^{3/2}\phi'\xi^3\eta_j\eta_k\eta_m\,d\xi\right.$$

$$\left.+4\frac{\phi_i^{5/2}}{\phi_i^4}\int_{\xi_i}^1 \theta^3\theta'\xi^3\eta_j\eta_k\eta_m\,d\xi\right\}$$

$$+\frac{3\gamma-1}{2}\left\{\int_0^{\xi_i} \phi^{5/2}\xi^4\left(\eta_j\eta_k'\eta_m'+\eta_j'\eta_k\eta_m'+\eta_j'\eta_k'\eta_m\right)d\xi\right.$$

$$\left.+\frac{\phi_i^{5/2}}{\theta_i^4}\int_{\xi_i}^1 \theta^4\xi^4\left(\eta_j\eta_k'\eta_m'+\eta_j'\eta_k\eta_m'+\eta_j'\eta_k'\eta_m\right)d\xi\right\}$$

$$+\frac{\gamma-1}{2}\left\{\int_0^{\xi_i} \phi^{5/2}\xi^5\eta_j\eta_k\eta_m\,d\xi+\frac{\phi_i^{5/2}}{\theta_i^4}\int_{\xi_i}^1 \theta^4\xi^5\eta_j'\eta_k'\eta_m'\,d\xi\right\} \quad .. \quad (16)$$

where j, k, m are the positive integral values starting with unity, q's are unknown functions of time, η's are the relative amplitudes normalized to unity at the surface of the star and $2\pi/\sigma_m$ are the periods for small oscillations, ξ_i, θ_i and ϕ_i are the values of ξ, θ and ϕ at the interface.

I's and D's are evaluated by numerical integration. This gives the following differential equations for the first three modes:

(i) $\xi_i = 0{\cdot}2094$

$$\left.\begin{aligned}
\frac{d^2q_1}{d\tau^2}+q_1 &= 0{\cdot}952650\ q_1^2+0{\cdot}593014\ q_1q_2+0{\cdot}069846\ q_1q_3\\
\frac{d^2q_2}{d\tau^2}+1{\cdot}863767\ q_2 &= 4{\cdot}982256\ q_1^2+7{\cdot}873256\ q_1q_2+2{\cdot}463878\ q_1q_3\\
\frac{d^2q_3}{d\tau^2}+3{\cdot}171753\ q_3 &= 5{\cdot}144444\ q_1^2+21{\cdot}600000\ q_1q_2+19{\cdot}511111\ q_1q_3
\end{aligned}\right\} \quad .. \quad (17)$$

(ii) $\xi_i = 0{\cdot}3915$

$$\left.\begin{aligned}
\frac{d^2q_1}{d\tau^2}+q_1 &= 0{\cdot}874390\ q_1^2+0{\cdot}370095\ q_1q_2+0{\cdot}005027\ q_1q_3\\
\frac{d^2q_2}{d\tau^2}+2{\cdot}145947\ q_2 &= 6{\cdot}514288\ q_1^2+9{\cdot}596708\ q_1q_2+2{\cdot}338361\ q_1q_3\\
\frac{d^2q_3}{d\tau^2}+3{\cdot}837979\ q_3 &= 0{\cdot}809235\ q_1^2+21{\cdot}908740\ q_1q_2+21{\cdot}827020\ q_1q_3
\end{aligned}\right\} \quad .. \quad (18)$$

(iii) $\xi_i = 0{\cdot}6160$

$$\left.\begin{aligned}
\frac{d^2q_1}{d\tau^2}+q_1 &= 0{\cdot}906197\ q_1^2+0{\cdot}264662\ q_1q_2-0{\cdot}029630\ q_1q_3\\
\frac{d^2q_2}{d\tau^2}+3{\cdot}237463\ q_2 &= 11{\cdot}073776\ q_1^2+17{\cdot}342202\ q_1q_2+3{\cdot}174770\ q_1q_3\\
\frac{d^2q_3}{d\tau^2}+5{\cdot}845325\ q_3 &= -11{\cdot}054628\ q_1^2+28{\cdot}307066\ q_1q_2+33{\cdot}9788648\ q_1q_3
\end{aligned}\right\} \quad (19)$$

(iv) $\xi_i = 0{\cdot}8250$

$$\left.\begin{aligned}
\frac{d^2q_1}{d\tau^2}+q_1 &= 0{\cdot}706618\ q_1^2+0{\cdot}319456\ q_1q_2-0{\cdot}014567\ q_1q_3\\
\frac{d^2q_2}{d\tau^2}+4{\cdot}492734\ q_2 &= 7{\cdot}379650\ q_1^2+18{\cdot}095526\ q_1q_2+0{\cdot}180129\ q_1q_3\\
\frac{d^2q_3}{d\tau^2}+8{\cdot}573992\ q_3 &= -4{\cdot}044655\ q_1^2+21{\cdot}650820\ q_1q_2+38{\cdot}715066\ q_1q_3
\end{aligned}\right\} \quad (20)$$

The method of the solution of these equations is that given by Prasad (1949*a*). The solutions of eqns. (17), (18), (19) and (20) have been started by taking $a_1 = 0{\cdot}06$. The radial velocity curves for the four cases are shown in Fig. 1.

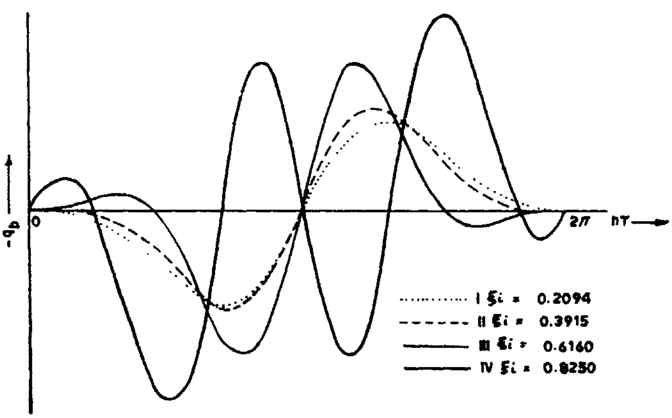

FIG. 1. The radial velocity curves for the composite polytropes.

4. CONCLUSIONS

From the results of § 2 we conclude that the composite model with interface $\xi_i = 0{\cdot}2094$ is quite near to the standard model in the pulsational behaviour, the composite model with interface $\xi_i = 0{\cdot}3915$ has drifted away from the standard model and the composite model with interface $\xi_i = 0{\cdot}6160$ is very near to the complete polytrope $n = 2$. The last composite model with interface $\xi_i = 0{\cdot}8250$ has shifted closer to the complete polytrope $n = 1{\cdot}5$.

As for the anharmonic pulsations, we see from Fig. 1 that radial velocity curves of model with interface at $\xi_i = 0{\cdot}2094$ and $0{\cdot}3915$ lie very near to each other and they have skewness near that of the standard model. In the last two models with interface at $\xi_i = 0{\cdot}6160$ and $0{\cdot}8250$ the radial velocity curves show humps and so we regard these two models unsuitable as far as their anharmonic pulsations are concerned.

The comparison of the pulsation characteristics of these models with some other models has been given in another paper by the author (1968).

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