ON MECHANICAL RESPONSE IN A PIEZOELECTRIC ANNULAR DISK TRANSDUCER OWING TO A STEP INPUT WITH A PRESCRIBED TEMPERATURE FIELD

by B. CHAUDHURI, *Department of Mathematics, Jadavpur University, Calcutta*

(Communicated by B. Sen, F.N.I.)

(*Received 18 April 1968*)

The methods of transform calculus are used as tools to calculate the mechanical response in a piezoelectric annular disk owing to a step voltage with a prescribed temperature field.

INTRODUCTION

It is sufficiently well known that the piezoelectric transducers play an important role in the generation and detection of ultrasonic waves, vide Mason (1950), Cady (1946). The problems of producing mechanical response by a voltage step in a piezoelectric transducer, in the form of a plate and bar, have been investigated by Redwood (1961*a*, *b*), Sinha (1962) and Giri (1966) and their discussions are confined to the plate and the bar transducers. Sinha (1965) in his recent paper has studied the mechanical response of a piezoelectric annular disk transducer owing to an impulsive voltage input. The present paper is an attempt to study the mechanical response in a piezoelectric transducer in the form of an annular disk owing to a step voltage input with a prescribed temperature field. The solution has been facilitated by the use of Laplace transforms and under suitable assumptions similar to those of Redwood (1961*a*).

PROBLEM, FUNDAMENTAL EQUATIONS AND BOUNDARY CONDITIONS

For the present case we need an annular disk transducer of piezoelectric material. Let r_1 and r_2 be the inner and outer radii of the boundary of the disk. Let V be the electrical voltage in the form of a step function. Let the temperature at the inner boundary $r = r_1$ be T_1 and the outer boundary $r = r_2$ be at zero temperature. Now our problem is to investigate the nature of the mechanical response in the above type of transducer.

To serve the purpose of our problem we first need a relation connecting the variables representing mechanical, electrical and thermal fields. We choose the mechanical (radial) displacement u as the mechanical variable, the electrical voltage V as the electrical variable and the temperature T as the thermal variable. To simplify the algebra, we assume the disk to be open-circuited. Referred to the usual coordinates, the equations of piezoelectricity as suggested by Mason (1950) have been modified for the present problem and they are given by

$$\sigma_r = C_{11}\frac{du}{dr}+C_{12}\frac{u}{r}-hD_r-\lambda_1 T \qquad (1)$$

$$\sigma_\theta = C_{12}\frac{du}{dr}+C_{11}\frac{u}{r}-hD_r-\lambda_2 T \qquad (2)$$

$$E_r = -h\left(\frac{du}{dr}-\frac{u}{r}\right)+\frac{D_r}{\epsilon}-\nu T \qquad (3)$$

where $(\sigma_r, \sigma_\theta)$ are the components of stress, D_r is the radial component of the electric displacement $\vec{D}$, E_r the radial component of the electric intensity $\vec{E}$, C_{11}, C_{12} the elastic compliances, h the piezoelectric constant, ϵ the permittivity and T the temperature.

The equation of motion in the radial direction is given by

$$\frac{\partial \sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta}{r} = \rho\frac{\partial^2 u}{\partial t^2}. \qquad (4)$$

Gauss's equation given by

$$\operatorname{div}\vec{D} = 0$$

in polar coordinates (r, θ) becomes

$$\frac{\partial D_r}{\partial r}+\frac{1}{r}\frac{\partial D_\theta}{\partial \theta} = 0.$$

Assuming the disturbances to propagate in the radial direction, we have from the last equation,

$$\frac{\partial D_r}{\partial r} = 0. \qquad (5)$$

The equation of steady heat flow to be satisfied by the temperature T is

$$\nabla^2 T = 0. \qquad (6)$$

The solution of eqn. (6), subject to the boundary conditions

$$\left.\begin{array}{l} T = T_1 \text{ on } r = r_1 \\ T = 0 \quad \text{on } r = r_2 \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

is

$$T = \frac{T_1}{\log \frac{r_2}{r_1}} \log \frac{r_2}{r}.$$

Considering eqns. (1), (2), (4), (5) and (7), we have

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\lambda_1}{C_{11}} \cdot \frac{T_1}{\log \frac{r_2}{r_1}} \cdot \frac{1}{r} + \frac{(\lambda_2 - \lambda_1)}{C_{11} r} \cdot \frac{T_1}{\log \frac{r_2}{r_1}} \log \frac{r_2}{r} = \frac{\rho}{C_{11}} \frac{\partial^2 u}{\partial t^2}. \quad (8)$$

Taking Laplace transform defined by

$$\bar{u}(r, p) = \int_0^\infty u(r, t) e^{-pt} dt, \quad p > 0$$

we have

$$\frac{d^2 \bar{u}}{dr^2} + \frac{1}{r}\frac{d\bar{u}}{dr} - \left(\frac{1}{r^2} + p^2 \alpha^2\right) \bar{u} = \left(K_2 + K_2' \log \frac{r_2}{r}\right) \cdot \frac{1}{rp} \quad \ldots \quad \ldots \quad (9)$$

where

$$\alpha = \left(\frac{\rho}{C_{11}}\right)^{\frac{1}{2}}$$

$$K_2 = -\frac{\lambda_1}{C_{11}} \cdot \frac{T_1}{\log \frac{r_2}{r_1}}$$

$$K_2' = \frac{\lambda_1 - \lambda_2}{C_{11}} \cdot \frac{T_1}{\log \frac{r_2}{r_1}}.$$

Solving eqn. (9), we get

$$\bar{u} = A I_1(\alpha p r) + B K_1(\alpha p r) + \frac{K_2' C_{11}}{\rho r p^3} \log r - \frac{(K_2 + K_2' \log r_2) C_{11}}{\rho p^3 r} \quad \ldots \quad (10)$$

where I_1 and K_1 are Bessel functions of imaginary argument and of order one, A and B are constants to be determined from boundary conditions. From eqn. (3)

$$\bar{E}_r = -h\left(\frac{d\bar{u}}{dr} - \frac{\bar{u}}{r}\right) + \frac{\bar{D}_r}{\epsilon} - \frac{\nu T_1}{p \log \frac{r_2}{r_1}} \log \frac{r_2}{r}.$$

Therefore the potential $\bar{V}$ across the disk is given by

$$\bar{V} = \int_{r_1}^{r_2} \bar{E}_r dr = -h[(\bar{u})_{r=r_2} - (\bar{u})_{r=r_1}] - \frac{\nu T_1}{p \log \frac{r_2}{r_1}} (r_2 - r_1) \log r_2$$

$$+ \frac{\nu T_1}{p \log \frac{r_2}{r_1}} (r_2 \log r_2 - r_2 - r_1 \log r_1 + r_1) + h \int_{r_1}^{r_2} \frac{\bar{u}}{r} dr \quad \ldots \quad \ldots \quad (11)$$

where we have considered the fact that the disk is open-circuited (so that $\bar{Q}$, the surface charge is zero and hence $\bar{D}_r$ related to $\bar{Q}$ is zero). Now eqns. (10) and (11) constitute the fundamental equations of the problem.

With the value of $\bar{u}$ from eqns. (10) and (11) can also be put in the following form:

$$\bar{V} = -h\Bigg[AI_1(\alpha p r_2) + BK_1(\alpha p r_2) + \frac{K_2' C_{11}}{\rho r_2 p^3} \log r_2 - \frac{(K_2 + K_2' \log r_2) C_{11}}{\rho r_2 p^3}$$

$$- AI_1(\alpha p r_1) - BK_1(\alpha p r_1) - \frac{K_2' C_{11}}{\rho r_1 p^3} \log r_1 + \frac{(K_2 + K_2' \log r_2) C_{11}}{\rho r_1 p^3} \Bigg]$$

$$- \frac{\nu T_1 (r_2 - r_1) \log r_2}{p \log \frac{r_2}{r_1}} + \frac{\nu T_1}{p \log \frac{r_2}{r_1}} (r_2 \log r_2 - r_2 - r_1 \log r_1 + r_1)$$

$$+ h\left[\int_{r_1}^{r_2} \left\{ A \frac{I_1(\alpha p r)}{r} + B \frac{K_1(\alpha p r)}{r} \right\} dr \right]$$

$$+ \frac{h K_2' C_{11}}{\rho p^3} \left(\frac{\log r_1}{r_1} - \frac{\log r_2}{r_2} \right) + \frac{(K_2' - K_2 - K_2' \log r_2) h C_{11}}{\rho p^3} \left(\frac{1}{r_1} - \frac{1}{r_2} \right). \quad \ldots \quad (12)$$

In accordance with our assumption, we have

$$V = V_0 H(t) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

where $H(t)$ is Heaviside unit function equal to unity when $t > 0$ and equal to zero when $t < 0$. We, therefore, have

$$\bar{V} = \frac{V_0}{p}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

In addition to the condition (13), we assume the boundary of the disk to be rigidly fixed, so that

$$(\bar{u})_{r=r_2} = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

Because of eqns. (14) and (15), it follows from eqns. (10) and (12) that

$$AI_1(\alpha p r_2) + BK_1(\alpha p r_2) = -\frac{L}{p^3} \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

$$Aa_{11} + Ba_{12} = -\frac{M}{p^3} + \frac{(V_0 + N)}{p} \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

where

$$a_{11} = h\left[I_1(\alpha p r_1)+\int_{r_1}^{r_2}\frac{I_1(\alpha p r)}{r}\,dr\right] \qquad (18)$$

$$a_{12} = h\left[K_1(\alpha p r_1)+\int_{r_1}^{r_2}\frac{K_1(\alpha p r)}{r}\,dr\right] \qquad (19)$$

$$L = \frac{K_2' C_{11}\log r_2}{\rho r_2} - \frac{(K_2+K_2'\log r_2)C_{11}}{\rho r_2}$$

$$M = \frac{2hK_2' C_{11}\log r_1}{\rho r_1} - \frac{h(K_2+K_2'\log r_2)C_{11}}{\rho r_1} - \frac{hK_2' C_{11}\log r_2}{\rho r_2} + \frac{h(K_2'-K_2-K_2'\log r_2)C_{11}}{\rho}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

$$N = \frac{\nu T_1(r_2-r_1)\log r_2}{\log\frac{r_2}{r_1}} - \frac{\nu T_1}{\log\frac{r_2}{r_1}}(r_2\log r_2 - r_2 - r_1\log r_1 + r_1).$$

Solving eqns. (16) and (17) we get

$$A = \frac{[MK_1(\alpha p r_2)-La_{12}]\frac{1}{p^3}-\frac{1}{p}(V_0+N)K_1(\alpha p r_2)}{[a_{12}I_1(\alpha p r_2)-a_{11}K_1(\alpha p r_2)]} \qquad (20)$$

$$B = \frac{[MI_1(\alpha p r_2)-La_{11}]\frac{1}{p^3}-\frac{1}{p}(V_0+N)I_1(\alpha p r_2)}{[a_{11}K_1(\alpha p r_2)-a_{12}I_1(\alpha p r_2)]}. \qquad (21)$$

Substituting these values of A and B in eqn. (10) and then taking the inverse transform of the resulting expression, we get the mechanical displacement u corresponding to V given by condition (13). From this, the displacement of the inner boundary $r = r_1$ can also be calculated. This displacement, from eqn. (10), is given by

$$(\bar{u})_{r=r_1} = AI_1(\alpha p r_1)+BK_1(\alpha p r_1)+\frac{W}{p^3}$$

where

$$W = \frac{K_2' C_{11}\log r_1}{\rho r_1} - \frac{(K_2+K_2'\log r_2)C_{11}}{\rho r_1}.$$

By eqns. (20) and (21)

$$(\bar{u})_{r=r_1} = \frac{1}{[a_{12}I_1(\alpha p r_2)-a_{11}K_1(\alpha p r_2)]p^3}[\{MK_1(\alpha p r_2)-La_{12}-(V_0+N)p^2K_1(\alpha p r_2)\} I_1(\alpha p r_1)-\{MI_1(\alpha p r_2)-La_{11}-(V_0+N)p^2I_1(\alpha p r_2)\}K_1(\alpha p r_1)]+\frac{W}{p^3}.$$

Inverting the Laplace transform, as given by Carslaw and Jaeger (1959), we have

$$(u)_{r=r_1} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{pt}[\{MK_1(\alpha pr_2)-La_{12}-(V_0+N)p^2K_1(\alpha pr_2)\}I\ (\alpha pr_1)-\{MI_1(\alpha pr_2)-La_{11}-(V_0+N)p^2I_1(\alpha pr_2)\}\times K_1(\alpha pr_1)]}{p^3[a_{12}I_1(\alpha pr_2)-a_{11}K_1(\alpha pr_2)]}+\frac{Wt^2}{2}. \quad (22)$$

Now putting the value of a_{11} and a_{12} from eqns. (18) and (19) it follows from eqn. (22) that

$$(u)_{r=r_1} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{-Me^{pt}[K_1(\alpha pr_2)I_1(\alpha pr_1)-I_1(\alpha pr_2)K_1(\alpha pr_1)]}{hp^3\left[I_1(\alpha pr_1)K_1(\alpha pr_2)-I_1(\alpha pr_2)K_1(\alpha pr_1)+K_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}\,dr-I_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}\,dr\right]}$$

$$+\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{(V_0+N)e^{pt}[K_1(\alpha pr_2)I_1(\alpha pr_1)-I_1(\alpha pr_2)K_1(\alpha pr_1)]}{hp\left[I_1(\alpha pr_1)K_1(\alpha pr_2)-I_1(\alpha pr_2)K_1(\alpha pr_1)+K_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}\,dr-I_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}\,dr\right]}$$

$$+\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{Le^{pt}\left[I_1(\alpha pr_1)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}\,dr-K_1(\alpha pr_1)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}\,dr\right]}{p^3\left[I_1(\alpha pr_1)K_1(\alpha pr_2)-I_1(\alpha pr_2)K_1(\alpha pr_1)+K_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}\,dr-I_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}\,dr\right]}+\frac{Wt^2}{2}$$

or

$$(u)_{r=r_1} = \frac{1}{2\pi i}(S_1+S_2+S_3)+\frac{Wt^2}{2} \quad (23)$$

S_1 is $2\pi i$ times the sum of the residues of the function

$$\frac{-Me^{pt}[K_1(\alpha pr_2)I_1(\alpha pr_1)-I_1(\alpha pr_2)K_1(\alpha pr_1)]}{hp^3\left[I_1(\alpha pr_1)K_1(\alpha pr_2)-I_1(\alpha pr_2)K_1(\alpha pr_1)+K_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}\,dr-I_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}\,dr\right]}$$

lying to the left of the line $R(p) = \gamma$. The poles of this are at $p = 0$ and at the zeros of

$$f(p) = \left[I_1(\alpha pr_1)K_1(\alpha pr_2)-I_1(\alpha pr_2)K_1(\alpha pr_1)+K_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}\,dr-I_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}\,dr\right].$$

These zeros are at $p = i\alpha_n/\alpha$, as in Carslaw and Jaeger (1959), where $\pm\alpha_n$ $(n = 1, 2, 3, \ldots)$ are the roots of

$$\left[J_1(\alpha_n r_1)\{iy_1(\alpha_n r_2)-J_1(\alpha_n r_2)\}-J_1\ \alpha_n r_2\{iy_1(\alpha_n r_1)-J_1(\alpha_n r_1)\}+\{iy_1(\alpha_n r_2)-J_1(\alpha_n r_2)\}\int_{r_1}^{r_2}\frac{J_1(\alpha_n r)}{r}\,dr-J_1(\alpha_n r_2)\int_{r_1}^{r_2}\frac{\{iy_1(\alpha_n r_1)-J_1(\alpha_n r_1)\}}{r}\,dr\right] = 0$$

where J_1 and Y_1 are Bessel functions of first order. The residue at $p = 0$ is zero. The residue at $p = i\alpha_n/\alpha$ is

$$\left[p\frac{d}{dp}\left\{hI_1(\alpha pr_1)K_1(\alpha pr_2)-hI_1(\alpha pr_2)K_1(\alpha pr_1)+hK_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}\,dr-hI_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}\,dr\right\}\right]_{p=id_n/\alpha} = hi\frac{\pi}{2}(A_1+iB_1)$$

where

$$A_1 = \left[\{J_1(\alpha_n r_2)-\alpha_n r_2 J_0(\alpha_n r_1)\}\int_{r_1}^{r_2}\frac{J_1(\alpha_n r)}{r}\,dr+\{\alpha_n r_2 J_0(\alpha_n r_2)-J_1(\alpha_n r_2)\}\int_{r_1}^{r_2}\frac{y_1(\alpha_n r)}{r}\,dr\right]$$

$$B_1 = \Big[\alpha_n r_1\{J_0(\alpha_n r_1)Y_1(\alpha_n r_2)-J_1(\alpha_n r_2)Y_0(\alpha_n r_1)\}-\alpha_n r_2\{J_0(\alpha_n r_2)Y_1(\alpha_n r_1)-J_1(\alpha_n r_1)Y_0(\alpha_n r_2)\}+3\{J_1(\alpha_n r_1)Y_1(\alpha_n r_2)-J_1(\alpha_n r_2)Y_1(\alpha_n r_1)\}$$
$$+\{\alpha_n r_2 Y_0(\alpha_n r_2)-Y_1(\alpha_n r_2)\}\int_{r_1}^{r_2}\frac{J_1(\alpha_n r)}{r}\,dr+\{\alpha_n r_2 J_0(\alpha_n r_2)-J_1(\alpha_n r_2)\}\int_{r_1}^{r_2}\frac{J_1(\alpha_n r)}{r}\,dr\Big].$$

Similar results for other two integrals S_2 and S_3 can be obtained. Hence from eqn. (23)

$$(u)_{r=r_1} = \text{Real part of}\left\{\sum_{\eta=1}^{\infty}\frac{-Mie^{\frac{id_n t}{\alpha}}(A_1-iB_1)[J_1(\alpha_n r_1)Y_1(\alpha_n r_2)-J_1(\alpha_n r_2)Y_1(\alpha_n r_1)]}{h(A_1^2+B_1^2)}\right\}$$
$$+\text{Real part of}\left\{\sum_{n=1}^{\infty}\frac{(V_0+N)ie^{\frac{i\alpha_n t}{\alpha}}(A_1-iB_1)[J_1(\alpha_n r_1)Y_1(\alpha_n r_2)-J_1(\alpha_n r_2)Y_1(\alpha_n r_1)]}{h(A_1^2+B_1^2)}\right\}$$

$$+ \text{Real part of} \left\{ \sum_{n=1}^{\infty} \frac{Le^{\frac{i\alpha_n t}{\alpha}}(A_1 - iB_1)\left[J_1(\alpha_n r_1)\int_{r_1}^{r_2} \frac{\{-J_1(\alpha_n r) + iY_1(\alpha_n r)\}}{r}dr + \{-J_1(\alpha_n r_1) + iY_1(\alpha_n r_1)\}\int_{r_1}^{r_2}\frac{J_1(\alpha_n r)}{r}dr\right]}{(A_1^2 + B_1^2)} \right\} + \frac{Wt^2}{2}. \qquad (24)$$

Similarly in this manner, the radial pressure or circumferential tension can be evaluated. But these evaluations as well as the previous one involve the evaluation of the definite integrals

$$\int_{r_1}^{r_2} \frac{J_1(\alpha_n r)}{r}dr \quad \text{and} \quad \int_{r_1}^{r_2} \frac{Y_1(\alpha_n r)}{r}dr.$$

By using the series expansions for $J_1(\alpha_n r)$ and $Y_1(\alpha_n r)$ (*see* Erdeyli 1953), these integrals can be evaluated.

Approximate Evaluation of the Displacement of the Inner Boundary $r = r_1$ for Large Values of Time

$$(\bar{u})_{r=r_1} = \frac{-M[K_1(\alpha pr_2)I_1(\alpha pr_1) - I_1(\alpha pr_2)K_1(\alpha pr_1)]}{hp^3\left[I_1(\alpha pr_1)K_1(\alpha pr_2) - I_1(\alpha pr_2)K_1(\alpha pr_1) + K_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}dr - I_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}dr\right]}$$

$$+ \frac{(V_0+N)[K_1(\alpha pr_2)I_1(\alpha pr_1) - I_1(\alpha pr_2)K_1(\alpha pr_1)]}{hp\left[I_1(\alpha pr_1)K_1(\alpha pr_2) - I_1(\alpha pr_2)K_1(\alpha pr_1) + K_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}dr - I_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}dr\right]}$$

$$+ \frac{L\left[I_1(\alpha pr_1)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}dr - K_1(\alpha pr_1)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}dr\right]}{p^3\left[I_1(\alpha pr_1)K_1(\alpha pr_2) - I_1(\alpha pr_2)K_1(\alpha pr_1) + K_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}dr - I_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}dr\right]} + \frac{W}{p^3}. \quad \cdot\cdot \quad \cdot\cdot \quad (25)$$

Its evaluation for large values of time will be possible if we expand it in powers of p [*see* Carslaw and Jaeger (1959)]. p is small for large values of time. Therefore, in the expansion of the relevant quantities in powers of p, we neglect squares and higher powers of p.

The expansions for $I_1(\alpha pr)$ and $K_1(\alpha pr)$ are given by Carslaw and Jaeger (1959).

$$I_1(\alpha pr) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}\alpha pr)^{2m+1}}{m!\ \Gamma(m+2)}$$

$$K_1(\alpha pr) = \{\log(\tfrac{1}{2}\alpha pr)+\gamma\}I_1(\alpha pr)-\tfrac{1}{2}\sum_{m=1}^{\infty} \frac{(\frac{1}{2}\alpha pr)^{2m+1}}{m!\ (m+1)!}\left[\sum_{\frac{1}{\alpha}=1}^{m+1}\alpha+\sum_{\frac{1}{\alpha}=1}^{m}\alpha\right].$$

Restricting to the kind of approximation referred to above, we can write

$$I_1(\alpha pr) \sim \tfrac{1}{2}(\alpha pr)$$

$$K_1(\alpha pr) \sim a_1(\alpha pr)+\frac{b_1}{(\alpha pr)}$$

where a_1 and b_1 are numerical quantities.
Thus

$$\int_{r_1}^{r_2} \frac{I_1(\alpha pr)}{r}\,dr \sim \tfrac{1}{2}\alpha p(r_2-r_1)$$

$$\int_{r_1}^{r_2} \frac{K_1(\alpha pr)}{r}\,dr \sim a_1\alpha p(r_2-r_1)+\frac{b_1}{\alpha p}\left(\frac{1}{r_1}-\frac{1}{r_2}\right).$$

So

$$I_1(\alpha pr_2)\int_{r_1}^{r_2} \frac{K_1(\alpha pr)}{r}\,dr \sim \tfrac{1}{2}r_2 b_1\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

$$K_1(\alpha pr_2)\int_{r_1}^{r_2} \frac{I_1(\alpha pr)}{r}\,dr \sim \tfrac{1}{2}\frac{b_1}{r_2}(r_2-r_1)$$

$$K_1(\alpha pr_2)I_1(\alpha pr_1)-I_1(\alpha pr_2)K_1(\alpha pr_1) \sim \tfrac{1}{2}b_1\left(\frac{r_1}{r_2}-\frac{r_2}{r_1}\right)$$

$$I_1(\alpha pr_1)\int_{r_1}^{r_2} \frac{K_1(\alpha pr)}{r}\,dr \sim \tfrac{1}{2}a_1\alpha^2p^2r_1r_2-\tfrac{1}{2}a_1(\alpha pr_1)^2+\frac{b_1}{2}-\frac{b_1r_1}{2r_2}$$

$$K_1(\alpha pr_1)\int_{r_1}^{r_2} \frac{I_1(\alpha pr)}{r}\,dr \sim \tfrac{1}{2}a_1p^2\alpha^2r_1r_2-\tfrac{1}{2}a_1(p\alpha r_1)^2+\frac{b_1r_2}{2r_1}-\frac{b_1}{2}.$$

Hence from eqn. (25) it follows that

$$(\bar{u})_{r=r_1} \sim \frac{M}{hp^3}\cdot\frac{\left(\dfrac{r_1}{r_2}-\dfrac{r_2}{r_1}\right)}{\left\{\left(\dfrac{r_1}{r_2}-\dfrac{r_2}{r_1}\right)+\dfrac{r_2-r_1}{r_2}-r_2\left(\dfrac{1}{r_1}-\dfrac{1}{r_2}\right)\right\}}$$

$$+\frac{(V_0+N)}{hp}\cdot\frac{\left(\frac{r_1}{r_2}-\frac{r_2}{r_1}\right)}{\left\{\left(\frac{r_1}{r_2}-\frac{r_2}{r_1}\right)+\frac{r_2-r_1}{r_2}-r_2\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right\}}$$

$$+\frac{L}{p^3}\cdot\frac{\left[2-\left(\frac{r_2}{r_1}-\frac{r_1}{r_2}\right)\right]}{2\left(1-\frac{r_2}{r_1}\right)}+\frac{W}{p^3}$$

or

$$(\bar{u})_{r=r_1}\sim\frac{M}{2hp^3}\left(1+\frac{r_1}{r_2}\right)+\frac{(V_0+N)}{2hp}\left(1+\frac{r_1}{r_2}\right)+\frac{L(2r_1r_2-r_2^2-r_1^2)}{2r_2(r_1-r_2)p^3}+\frac{W}{p^3}.$$

Taking the inverse transform, the mechanical displacement of the inner boundary of the disk $r=r_1$, for large values of time, is given by

$$(u)_{r=r_1}\sim\frac{Mt^2}{4h}\left(1+\frac{r_1}{r_2}\right)+\frac{L(2r_1r_2-r_2^2-r_1^2)t^2}{4r_2(r_1-r_2)}+\frac{Wt^2}{2}$$
$$+\left[\frac{(V_0+N)}{2h}\left(1+\frac{r_1}{r_2}\right)\right]H(t).$$

Thus to conclude we say that the mechanical response owing to a step voltage input under thermal conditions is partly step in character, for large values of time. The similar result can also be had from (24). In the absence of temperature the result agrees with some known results.

Acknowledgement

I am grateful to Dr. R. R. Giri of the Department of Mathematics, Jadavpur University, Calcutta, for his kind help and active guidance.

References

Cady, W. G. (1946). Piezoelectricity. McGraw-Hill Book Co. Inc., New York.

Carslaw, H. S., and Jaeger, J. C. (1959). Conduction of Heat in Solids, 2nd ed., Clarendon Press, Oxford.

Erdeyli, A. (1953). Higher Transcendental Functions, 2. McGraw-Hill Book Co. Inc., New York.

Giri, R. R. (1966). Note on the study of responses in a piezoelectric plate transducer when the input is partly periodic and partly periodic transient. *Revue roum. Sci. Techn.*, **11**, 253.

Mason, W. P. (1950). Piezoelectric Crystals and their Applications to Ultrasonics. D. Van Nostrand Co. Inc., New York.

Redwood, M. (1961*a*). Transient performance of a piezoelectric transducer. *J. acoust. Soc. Am.*, **33**, 527.

——— (1961*b*). Piezoelectric generation of an electrical impulse. *J. acoust. Soc. Am.*, **33**, 1386.

Sinha, D. K. (1962). Note on responses in a piezoelectric plate transducer with a periodic step input. *Indian J. theor. Phys.*, **10**, 21.

——— (1965). A note on mechanical responses in a piezoelectric transducer owing to an impulsive voltage input. *Proc. natn. Inst. Sci. India*, **31**, 395.