STABILITY OF CERTAIN TYPES OF FORCE-FREE PERIODIC FIELDS

by M. R. Raghavachar, *Department of Applied Mathematics, Indian Institute of Science, Bangalore 12*

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The stability of a certain type of periodic force-free fields having neutral lines, investigated by Schatzman on the basis of energy principle, is discussed. The normal mode technique is employed restricting to neighbourhood of neutral lines. The neutral lines contribute to instability in agreement with Schatzman. Repetition of stability analysis employing energy principle revealed that every cell appears to be stable contrary to Schatzman's conclusion that instabilities can be present under certain conditions. Moreover, the present study provides the growth rate of the instabilities.

1. Introduction

It was pointed out by Lüst and Schlüter (1954) that cosmic magnetic fields occurring in regions of low density should be force-free. As the ionized matter is highly conducting large currents can flow giving rise to large Lorentz force predominating over other forces such as arising from pressure gradient and gravitation or inertia. Consequently the Lorentz force should vanish to maintain steady states over considerably long time scales. Such types of fields have been characterized by Chandrasekhar (1956), Bhatnagar (1957) and Menzel (1964). In particular, Menzel has given a class of periodic force-free fields which could support quiescent prominences. Stability of the force-free fields has been investigated by Trehan (1957), Woltjer (1958) and Chakraborty and Bhatnagar (1960). Schatzman (1960) has discussed the stability of the periodic force-free fields of the type

$$\vec{H} = H_0\left[-\frac{l}{\alpha}\cos \kappa x \sin ly,\ \frac{\kappa}{\alpha}\sin \kappa x \cos ly,\ \cos \kappa x \cos ly\right] \quad .. \quad (1.1)$$

suggested by Menzel, where $\kappa^2+l^2 = \alpha^2$, α being the constant of proportionality of current to magnetic field. We note in particular that these fields have neutral lines. He has employed the energy principle given by Bernstein *et al.* (1958) and obtained certain necessary conditions for instability.

In the present study we discuss the stability of an ideally conducting homogeneous medium in presence of fields of the type (1.1) by normal mode technique, restricting ourselves to neighbourhood of neutral lines, as the general discussion is mathematically very complicated. For the above fields the neutral lines are given by

$$x = \frac{\pi}{2\kappa}(2m+1), \quad y = \frac{\pi}{2l}(2n+1); \quad m; n = 0, \pm 1, \pm 2, \ldots \qquad (1.2)$$

We find that neutral lines contribute to instability which agrees with the result obtained by Schatzman. But the conclusion drawn by him that there can be instabilities present along z-axis if $r^2 < \frac{\alpha^2 B^2}{32\pi} \frac{\sin^2 2\omega}{\gamma p_0}$ does not seem to be correct as can be verified by repeating the Schatzman stability analysis on the basis of energy principle. Contrary to Schatzman's conclusion we find that in this case there is always stability.

2. Linearized Hydromagnetic Equations

We shall work through the physical quantities rendered dimensionless according to the following scheme:

$$(x', y', z') = \alpha(x, y, z), \quad t' = (\alpha V_A)t$$

$$\delta\rho' = \frac{\delta\rho}{\rho_0}, \quad \delta p' = \frac{\delta p}{\gamma p_0}, \quad \vec{v}' = \frac{\vec{v}}{V_A}, \quad \vec{h}' = \frac{\vec{h}}{H_0}$$

where $V_A = \sqrt{\frac{H_0^2}{4\pi\rho_0}}$ is the Alfvén speed, p_0, ρ_0 being the pressure and density of the medium in equilibrium.

The linearized hydromagnetic equations in dimensionless form are:

$$\frac{\partial}{\partial t}(\delta\rho) + \operatorname{div} \vec{v} = 0 \qquad (2.1)$$

$$\delta p = \delta\rho \qquad (2.2)$$

$$\frac{\partial \vec{v}}{\partial t} = -S^2 \nabla(\delta p) + \vec{H}_0 \times \vec{h} + (\operatorname{curl} \vec{h}) \times \vec{H}_0 \qquad (2.3)$$

$$\frac{\partial \vec{h}}{\partial t} = \operatorname{curl}(\vec{v} \times \vec{H}_0) \qquad (2.4)$$

where $S^2 = (c_s^2/V_A^2)$, $\vec{H}_0$ is dimensionless steady state magnetic field, and $c_s = (\gamma p_0/\rho_0)^{\frac{1}{2}}$ is the sound velocity in the undisturbed medium.

3. Dispersion Relation

Taking perturbed quantities proportional to

$$\exp(i\omega t + ikz)$$

ω and k being dimensionless frequency and wave number respectively eqns. (2.1)–(2.4) reduce to after eliminating $\vec{h}$, δp and $\delta\rho$:

$$(S^2+H_y^2+H_z^2)u_{xx}+H_y^2u_{yy}+2\left(H_z\frac{\partial H_z}{\partial x}+H_y\frac{\partial H_y}{\partial x}\right)u_x+\left[2ikH_yH_z-H_y\frac{\partial H_x}{\partial x}\right.$$

$$\left.+H_y\frac{\partial H_y}{\partial y}\right]u_y+\left[\omega^2-H_y\frac{\partial H_z}{\partial x}+H_z\frac{\partial H_y}{\partial x}+ikH_z\left(ikH_z-\frac{\partial H_x}{\partial x}\right)+H_z\frac{\partial^2 H_z}{\partial x^2}\right.$$

$$\left.+H_y\frac{\partial^2 H_y}{\partial x^2}+H_y\left(ik\frac{\partial H_z}{\partial y}-\frac{\partial^2 H_x}{\partial x\partial y}\right)\right]u-H_xH_yv_{xx}+(S^2+H_z^2)v_{xy}-H_xH_yv_{yy}$$

$$-\left[ikH_yH_z-H_y\frac{\partial H_y}{\partial y}+H_y\frac{\partial H_x}{\partial x}\right]v_x+\left[-H_yH_z-ikH_zH_x+H_z\frac{\partial H_z}{\partial x}-2H_y\frac{\partial H_x}{\partial y}\right]v_y$$

$$+\left[-H_y\frac{\partial H_z}{\partial y}-H_z\left(ikH_z-\frac{\partial H_y}{\partial y}\right)-ikH_z\frac{\partial H_x}{\partial y}+H_z\frac{\partial^2 H_z}{\partial x\partial y}-H_y\left(ik\frac{\partial H_z}{\partial x}-\frac{\partial^2 H_y}{\partial x\partial y}\right)\right.$$

$$\left.-H_y\frac{\partial^2 H_x}{\partial y^2}\right]v-H_zH_xw_{xx}-H_yH_zw_{xy}+\left[H_xH_y-H_z\frac{\partial H_x}{\partial x}+ikH_y^2+ikS^2\right]w_x$$

$$+\left[H_y^2-H_z\frac{\partial H_y}{\partial x}-ikH_yH_x\right]w_y+\left[ikH_yH_z+k^2H_xH_z+ikH_y\frac{\partial H_y}{\partial x}\right.$$

$$\left.-ikH_y\frac{\partial H_x}{\partial y}\right]w=0 \quad \ldots \qquad (3.1)$$

$$-H_xH_yu_{xx}+\left(S^2+H_z^2\right)u_{xy}-H_xH_yu_{yy}+\left[H_xH_z-2H_x\frac{\partial H_y}{\partial x}+H_z\frac{\partial H_z}{\partial y}-ikH_yH_z\right]u_x$$

$$-\left[H_x\left(ikH_z-\frac{\partial H_x}{\partial x}\right)+H_x\frac{\partial H_y}{\partial y}\right]u_y+\left[ikH_z^2+H_x\frac{\partial H_z}{\partial x}-H_x\frac{\partial^2 H_y}{\partial x^2}\right.$$

$$\left.-H_x\left(ik\frac{\partial H_z}{\partial y}-\frac{\partial^2 H_x}{\partial x\partial y}\right)-ikH_z\frac{\partial H_y}{\partial x}\right]u+H_x^2v_{xx}+(S^2+H_x^2+H_z^2)v_{yy}$$

$$+\left[H_x\left(ikH_z-\frac{\partial H_y}{\partial y}\right)+H_x\frac{\partial H_x}{\partial x}+ikH_xH_z\right]v_x+2\left(H_x\frac{\partial H_x}{\partial y}+H_z\frac{\partial H_z}{\partial y}\right)v_y$$

$$+\left[\omega^2-H_z\frac{\partial H_x}{\partial y}+H_x\frac{\partial H_z}{\partial y}+H_x\left(ik\frac{\partial H_z}{\partial x}-\frac{\partial^2 H_y}{\partial x\partial y}\right)+H_x\frac{\partial^2 H_x}{\partial y^2}+H_z\frac{\partial^2 H_z}{\partial y^2}\right.$$

$$\left.+ikH_z\left(ikH_z-\frac{\partial H_y}{\partial y}\right)\right]v-H_xH_zw_{xy}-H_zH_yw_{yy}-\left[H_x^2+ikH_xH_y+H_z\frac{\partial H_x}{\partial y}\right]w_x$$

$$+\left[ikS^2-H_xH_y+ikH_x^2-H_z\frac{\partial H_y}{\partial y}\right]w_y+\left[-ikH_xH_z-ikH_x\frac{\partial H_y}{\partial x}+ikH_x\frac{\partial H_x}{\partial y}\right.$$

$$\left.+k^2H_yH_z\right]w=0 \quad \ldots \qquad (3.2)$$

$$-H_xH_zu_{xx}-H_yH_zu_{xy}+\left[ikS^2-H_xH_y-H_y\frac{\partial H_z}{\partial y}+ikH_y^2-2H_x\frac{\partial H_z}{\partial x}\right]u_x$$

$$-ikH_xH_yu_y+\left[-H_x\frac{\partial H_y}{\partial x}-H_y\left(ikH_z-\frac{\partial H_x}{\partial x}\right)-H_y\frac{\partial^2 H_z}{\partial x\partial y}+ikH_y\frac{\partial H_y}{\partial x}\right.$$

$$\left.-ikH_x\left(ikH_z-\frac{\partial H_x}{\partial x}\right)-H_x\frac{\partial^2 H_z}{\partial x^2}\right]u-H_yH_zv_{yy}-H_xH_zv_{xy}+\left[H_x^2-ikH_xH_y\right.$$

$$\left.-H_x\frac{\partial H_z}{\partial y}\right]v_x+\left[ikS^2+H_xH_y-2H_y\frac{\partial H_z}{\partial y}+ikH_x^2-H_x\frac{\partial H_z}{\partial x}\right]v_y+\left[H_x\left(ikH_z-\frac{\partial H_y}{\partial y}\right)\right.$$

$$\left.+H_y\frac{\partial H_x}{\partial y}-H_y\frac{\partial^2 H_z}{\partial y^2}-ikH_y\left(ikH_z-\frac{\partial H_y}{\partial y}\right)+ikH_x\frac{\partial H_x}{\partial y}-H_x\frac{\partial^2 H_z}{\partial x\partial y}\right]v+H_x^2w_{xx}$$

$$+2H_xH_yw_{xy}+H_y^2w_{yy}+\left[H_y\frac{\partial H_x}{\partial y}+H_x\frac{\partial H_x}{\partial x}\right]w_x+\left[H_y\frac{\partial H_y}{\partial y}+H_x\frac{\partial H_y}{\partial x}\right]w_y$$

$$+\left[\omega^2-k^2S^2-k^2H_y^2-k^2H_x^2\right]w=0 \qquad \ldots \qquad (3.3)$$

where H_x, H_y, H_z are given by

$$H_x=-b\cos ax\sin by,\quad H_y=a\sin ax\cos by,\quad H_z=\cos ax\cos by \qquad (3.4)$$

with $a=\kappa/\alpha$, $b=l/\alpha$ and $a^2+b^2=1$.

As we shall limit ourselves to the neighbourhood of neutral lines, we put

$$x=\frac{\pi}{2a}+\xi,\quad y=\frac{\pi}{2b}+\eta \qquad \ldots \qquad (3.5)$$

where ξ and η are small. We then have

$$\left.\begin{aligned} H_x&=ab\xi-\frac{ab^3}{2}\xi\eta^2-\frac{a^3b}{6}\xi^3+\ldots\\ H_y&=-ab\eta+\frac{a^3b}{2}\xi^2\eta+\frac{ab^3}{6}\eta^3+\ldots\\ H_z&=ab\xi\eta-\frac{a^3b}{6}\xi^3\eta-\frac{ab^3}{6}\xi\eta^3+\ldots\end{aligned}\right\} \qquad \ldots \qquad (3.6)$$

Substituting for H_x, H_y and H_z in (3.1), (3.2), (3.3) and retaining up to quadratic terms in ξ and η we get

$$\begin{aligned}&(C_s^2+\eta^2)u_{\xi\xi}+\eta^2u_{\eta\eta}+2\eta u_\eta+(W^2-2ik\xi\eta)u+\xi\eta v_{\xi\xi}\\ &\quad+C_s^2v_{\xi\eta}+\xi\eta v_{\eta\eta}+2\eta v_\xi+ik\eta^2v+(ik\eta^2-2\xi\eta+ikC_s^2)w_\xi\\ &\quad+(\eta^2+ik\xi\eta)w_\eta=0 \qquad \ldots \qquad (3.7)\end{aligned}$$

$$\begin{aligned}&\xi\eta u_{\xi\xi}+C_s^2u_{\xi\eta}+\xi\eta u_{\eta\eta}+2\xi u_\eta-ik\xi^2u\\ &\quad+\xi^2v_{\xi\xi}+(C_s^2+\xi^2)v_{\eta\eta}+2\xi v_\xi+(W^2+2ik\xi\eta)v\\ &\quad+(ik\xi\eta-\xi^2)w_\xi+(ikC_s^2+2\xi\eta+ik\xi^2)w_\eta=0 \qquad \ldots \qquad (3.8)\end{aligned}$$

$$ik(C_s^2+\eta^2)u_\xi+ik\xi\eta u_\eta+ik\xi u+ik\xi\eta v_\xi$$
$$+(C_s^2+\xi^2)ikv_\eta+ik\eta v+\xi^2 w_{\xi\xi}-2\xi\eta w_{\xi\eta}$$
$$+\eta^2 w_{\eta\eta}+\xi w_\xi+\eta w_\eta+(W^2-k^2C_s^2-k^2\xi^2-k^2\eta^2)w=0 \quad (3.9)$$

where

$$C_s^2=\frac{S^2}{a^2b^2},\quad W^2=\frac{\omega^2}{a^2b^2}$$

Consistent with our approximation we assume

$$\left.\begin{aligned} u &= a_0+a_1\xi+a_2\eta+a_3\xi^2+a_4\xi\eta+a_5\eta^2+\ldots \\ v &= b_0+b_1\xi+b_2\eta+b_3\xi^2+b_4\xi\eta+b_5\eta^2+\ldots \\ w &= c_0+c_1\xi+c_2\eta+c_3\xi^2+c_4\xi\eta+c_5\eta^2+\ldots \end{aligned}\right\} \quad (3.10)$$

Substituting the above expressions for u, v and w in eqns. (3.7)–(3.9) and equating to zero the coefficients of $\xi^r\eta^s$, $0\leqslant r+s\leqslant 2$, we obtain the following linear homogeneous equations determining a's, b's and c's:

$$2C_s^2a_3+W^2a_0+C_s^2b_4+ikC_s^2c_1=0 \quad (3.11)$$

$$W^2a_1+2ikC_s^2c_3=0 \quad (3.12)$$

$$(W^2+2)a_2+2b_1+ikC_s^2c_4=0 \quad (3.13)$$

$$W^2a_3=0 \quad (3.14)$$

$$(W^2+2)a_4-2ika_0+6b_3+2b_5-2c_1+ikc_2=0 \quad (3.15)$$

$$2a_3+(W^2+6)a_5+2b_4+ikb_0+ikc_1+c_2=0 \quad (3.16)$$

$$C_s^2a_4+2C_s^2b_5+W^2b_0+ikC_s^2c_2=0 \quad (3.17)$$

$$2a_2+(W^2+2)b_1+ikC_s^2c_4=0 \quad (3.18)$$

$$W^2b_2+2ikC_s^2c_5=0 \quad (3.19)$$

$$2a_4-ika_0+(W^2+6)b_3+2b_5-c_1+ikc_2=0 \quad (3.20)$$

$$2a_3+6a_5+(W^2+2)b_4+2ikb_0+ikc_1+2c_2=0 \quad (3.21)$$

$$W^2b_5=0 \quad (3.22)$$

$$ikC_s^2a_1+ikC_s^2b_2+(W^2-k^2C_s^2)c_0=0 \quad (3.23)$$

$$2ikC_s^2a_3+ika_0+ikC_s^2b_4+(W^2-k^2C_s^2+1)c_1=0 \quad (3.24)$$

$$ikC_s^2a_4+2ikC_s^2b_5+ikb_0+(W^2-k^2C_s^2+1)c_2=0 \quad (3.25)$$

$$ika_1+ikb_2+(W^2-k^2C_s^2+4)c_3-k^2c_0=0 \quad (3.26)$$

$$2ika_2+2ikb_1+(W^2-k^2C_s^2)c_4=0 \quad (3.27)$$

$$ika_1+ikb_2+(W^2-k^2C_s^2+4)c_5-k^2c_0=0 \quad (3.28)$$

Equations (3.14) and (3.22) give $a_3=0$ and $b_5=0$. We are left with 16 unknowns and 16 equations. These form three mutually exclusive sets, say,

A, B and C. The set A consisting of the three equations (3.13), (3.18), (3.27) in the unknowns a_2, b_1 and c_4; the set B consisting of the five equations (3.12), (3.19), (3.23), (3.26), (3.28) in the unknowns a_1, b_2, c_0, c_3 and c_5, and the set C consisting the eight equations (3.11), (3.15), (3.16), (3.17), (3.20), (3.21), (3.24), (3.25) in the unknowns a_0, a_4, a_5, b_0, b_3, b_4, c_1 and c_2.

Each of the three sets leads to dispersion relation when we apply the condition for the non-trivial solution of the equations.

The set A gives the dispersion relation

$$W^2 = k^2C_s^2-4$$

or in physical parameters

$$\frac{\omega^2}{k^2} = c_s^2 - \frac{4\kappa^2 l^2}{\alpha^2 k^2} V_A^2 \quad \ldots \quad (3.29)$$

This relation represents the sonic modes modified by the presence of inhomogeneous magnetic field. These modes are stable or unstable according as

$$k \gtrless \frac{2\kappa l}{\alpha} \frac{V_A}{c_s}.$$

The set B breaks up into the following two relations:

$$\text{(i)} \quad W^2 - k^2C_s^2 + 4 = 0 \quad \ldots \quad (3.30)$$

$$\text{(ii)} \quad W^4 + 2(2-k^2C_s^2)W^2 + k^4C_s^4 = 0 \quad \ldots \quad (3.31)$$

The first relation is the same as obtained in set A, while the second gives on solving

$$\omega^2 = \left(k^2c_s^2 - \frac{2\kappa^2 l^2}{\alpha^2} V_A^2\right) \pm \frac{2\kappa l}{\alpha} V_A \sqrt{\frac{\kappa^2 l^2}{\alpha^2} V_A^2 - k^2 c_s^2} \quad \ldots \quad (3.32)$$

Now, if $k^2 < \frac{\kappa^2 l^2}{\alpha^2} \frac{V_A^2}{c_s^2}$, the two values of ω^2 are real and negative, giving rise to growing and decaying modes while if $k^2 > \frac{\kappa^2 l^2}{\alpha^2} \frac{V_A^2}{c_s^2}$, the values of ω^2 are complex conjugate, again leading to instability.

Finally, the set C gives the dispersion relation which breaks up into the following two relations:

$$(W^2+7)[W^4+(1-k^2C_s^2)W^2+k^2C_s^2] + W^2(W^2+1) \pm 4kC_s^2(W^2+3) = 0 \quad (3.33)$$

We can easily show that each of these factors, cubic on W^2 possesses at least one negative root < -3, thus giving growing and decaying modes.

From the above discussion, we conclude that there is always instability in the neighbourhood of neutral lines. This result is in agreement with the conclusion drawn by Schatzman that neutral lines contribute to instability. However, by employing normal mode technique, we are able to obtain magnitudes of growth rates of instability present around neutral lines.

4. Study of Stability through Energy Method

In this section we again investigate the stability of the system (1.1) on the basis of energy principle given by Bernstein *et al.* (1958) and also employed by Schatzman.

The energy principle states that if

$$\delta W = -\tfrac{1}{2}\int \vec{\xi}\,.\,\vec{F}\left(\vec{\xi}\right) d\tau \qquad \ldots \quad \ldots \quad \ldots \quad (4.1)$$

where $\vec{\xi}$ denotes an arbitrary displacement from the equilibrium,

$$\vec{F}\left(\vec{\xi}\right) = \gamma p_0 \text{ grad}\left(\text{div}\,\vec{\xi}\right) + \left(\text{curl}\,\vec{B}_0\right)\times\vec{Q} - \vec{B}_0\times\text{curl}\,\vec{Q},$$

$$\vec{Q} = \text{curl}\left(\vec{\xi}\times\vec{B}_0\right)$$

and the integration is performed over the smallest cell bounded by the neutral lines, we have stability or instability according as $\delta W >$ or < 0.

Following Schatzman, we assume

$$\vec{\xi} = \sum_m \sum_n (a_{mn}, b_{mn}, c_{mn}) \exp i(m\kappa x + nly + rz) \qquad \ldots \quad \ldots \quad (4.2)$$

The integral (4.1) reduces to

$$-\frac{K}{2}\sum_m \sum_n \left[\vec{\xi}_{mn}\,.\,\vec{F}\left(\vec{\xi}^*_{mn}\right) + \vec{\xi}^*_{mn}\,.\,\vec{F}\left(\vec{\xi}_{mn}\right)\right] \qquad \ldots \quad \ldots \quad (4.3)$$

where K denotes a positive constant.

If $\vec{R} = \vec{\xi}\times\vec{B}_0$, the contribution to (δW) from magnetic part is

$$\tfrac{1}{2}\left\{\vec{R}^*\,.\left[\alpha\text{ curl }\vec{R} - \text{curl curl }\vec{R}\right] + \vec{R}\,.\left[\alpha\text{ curl }\vec{R}^* - \text{curl curl }\vec{R}^*\right]\right\} \quad \ldots \quad (4.4)$$

Taking $\qquad \vec{R}_{mn} = (A_{mn}, B_{mn}, C_{mn}) \exp i(m\kappa x + nly + rz) \qquad \ldots \quad (4.5)$

the m, nth component of (4.4) reduces to, after dropping subscripts from A, B, C, A^*, B^* and C^*,

$$\begin{aligned} &mn\kappa l(AB^* + A^*B) + nlr(BC^* + B^*C) + m\kappa r(AC^* + A^*C) \\ &-(l^2n^2 + r^2)AA^* - (m^2\kappa^2 + r^2)BB^* - (m^2\kappa^2 + n^2l^2)CC^* \\ &+\alpha i[r(AB^* - A^*B) + m\kappa(BC^* - B^*C) + nl(CA^* - C^*A)] \quad \ldots \quad (4.6)\dagger \end{aligned}$$

We can express the components of $\vec{R}$ in terms of the components of $\vec{\xi}$ to obtain A, B, C as follows:

$$\begin{aligned} A_{mn} = \frac{H_0}{4}\Big[&(b_{m-1,\,n-1} + b_{m-1,\,n+1} + b_{m+1,\,n-1} + b_{m+1,\,n+1}) \\ &-\frac{\kappa}{\alpha i}(c_{m-1,\,n-1} + c_{m-1,\,n+1} - c_{m+1,\,n-1} - c_{m+1,\,n+1})\Big] \quad \ldots \quad (4.7) \end{aligned}$$

† This expression differs from that of Schatzman. Perhaps some error has crept in his work due to some slip.

$$B_{mn} = -\frac{H_0}{4}\left[\frac{l}{\alpha i}(c_{m-1,\,n-1}-c_{m-1,\,n+1}+c_{m+1,\,n-1}-c_{m+1,\,n+1})\right.$$

$$\left.+(a_{m-1,\,n-1}+a_{m-1,\,n+1}+a_{m+1,\,n-1}+a_{m+1,\,n+1})\right] \quad .. \quad (4.8)$$

$$C_{mn} = \frac{H_0}{4\alpha i}[\kappa(a_{m-1,\,n-1}+a_{m-1,\,n+1}-a_{m+1,\,n-1}-a_{m+1,\,n+1})$$

$$+l(b_{m-1,\,n-1}-b_{m-1,\,n+1}+b_{m+1,\,n-1}-b_{m+1,\,n+1})] \quad .. \quad (4.9)$$

Substituting for A, B, ... in terms of a, b, ... and collecting the terms with index mn, we finally obtain the following expression for (δW):

$$\frac{4}{KH_0^2}(\delta W)_{mn} = \frac{C_s^2}{V_A^2}|m\kappa a+nlb+rc|^2+\frac{l^2}{\alpha^2}(r^2+n^2\kappa^2)|a|^2$$

$$+\frac{\kappa^2}{\alpha^2}(r^2+m^2l^2)|b|^2+\left[r^2+(m^2+n^2)\frac{\kappa^2l^2}{\alpha^2}\right]|c|^2$$

$$+|m\kappa a+nlb|^2+\frac{\kappa^2}{\alpha^2}|rc+ma|^2+\frac{l^2}{\alpha^2}|rc+nb|^2 \quad .. \quad (4.10)$$

where C_s, V_A are the sound speed and Alfvén speed respectively. As the expression (4.10) shows (δW) is positive definite and, therefore, the system as a whole is stable. This is in contradiction to the result obtained by Schatzman where he has shown that instabilities are present in z-direction if

$$r^2 < \frac{\alpha^2 B_0^2}{32\pi}\,\frac{\sin^2 2\omega}{\gamma p_0}, \quad \omega = \tan^{-1}\frac{l}{\kappa}.$$

5. Conclusion

By the energy principle which in a way provides a gross treatment, every cell appears to be stable but the contribution to δW by the neighbourhood of a neutral line is negative as proved by Schatzman or by the present investigation in § 3. This explains the motivation of the present investigation. Moreover, the present study provides the growth rate of the instabilities.

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