ON CERTAIN ASPECTS OF OPTIMUM EXPLOITATION OF FISH POPULATIONS

by M. KRISHNAN KUTTY, *Biological Oceanography Division, National Institute of Oceanography, Cochin*-682018.

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Some aspects of the yield isopleth and eumetric fishing have been examined using the Beverton-Holt table of yield functions. The various yield curves have been described by approximating functions which permit an accurate estimation of the yield and the yield-maxima. Once the curves are fitted the yield calculations become unnecessary since they can be obtained accurately by using the approximate function with a minimum of additional computation. The use of an approximating function is a labour-saving method particularly when the derivative of the eumetric yield curve is required at different points, or when the eumetric yield and/or the optimum mesh size are needed at various fishing levels from the eumetric yield/fishing curves.

These methods can be applied even when the simple Beverton-Holt model is given in the general form to suit both isometric and allometric growth conditions or when the natural mortality varies with age. The methods described here also include those which are used for an accurate evaluation of the fishing level which permits eumetric fishing at a given mesh size or for finding the change in the fishing intensity needed to bring the yield and catch-per-unit effort to any desired level. The predictive value of the Beverton-Holt and Schaefer models in yield assessments and for maximising the profit are also discussed.

INTRODUCTION

An appropriate evaluation of a fishery requires an estimation of the changes in yield and catch-per-unit-effort at various levels of fishing and mesh selection. Hence, in this paper some of the properties of the various yield curves which describe such changes under different fishing conditions together with their significance in the optimum exploitation of a fishery, have been examined. Methods of numerical analysis, even though they are simple and accurate and commonly applied in other fields of studies, are seldom used in problems related to fisheries management. In this paper some of these methods have been used to describe the various yield curves so that accurate estimates of the yield and catch-per-unit effort can be made at various fishing levels or for finding the yield maxima with a minimum of additional computation. Since evaluation of fishing conditions by these methods is for maximising either the total return or the profit, suitability of the Beverton-Holt and Schaefer models in such studies, in view of the limitations inherent in the two models, has also been discussed.

BALANCED FISHING

The Beverton-Holt theory of balanced fihsing may be defined as the harvesting of a fish population for obtaining the maximum yield at any fishing intensity by selecting an appropriate mesh size. The selection of mesh size is based on the knowledge of growth and natural mortality. Earlier authors, especially Ricker (1945) and Allen (1953, 1954) have discussed the basic principles involved in obtaining the maximum yield within the practical limits of fishing intensity as against the potential yield. The potential yield can be obtained only by catching all the fish of a particular age (optimum age) in a population when growth and natural mortality just balance each other, resulting in maximum weight of the brood. But this is theoretically possible only at infinitely high fishing intensity. The eumetric fishing curve of Beverton and Holt enables one to read directly the appropriate size limit that will maximise the yield and the catch-per-unit-effort at a particular fishing intensity. This curve is asymptotic approaching the optimum age or size at infinite fishing intensity. The curve complementary to the eumetric fishing curve enables one to read the fishing level which will maximise the yield at any fixed mesh size.

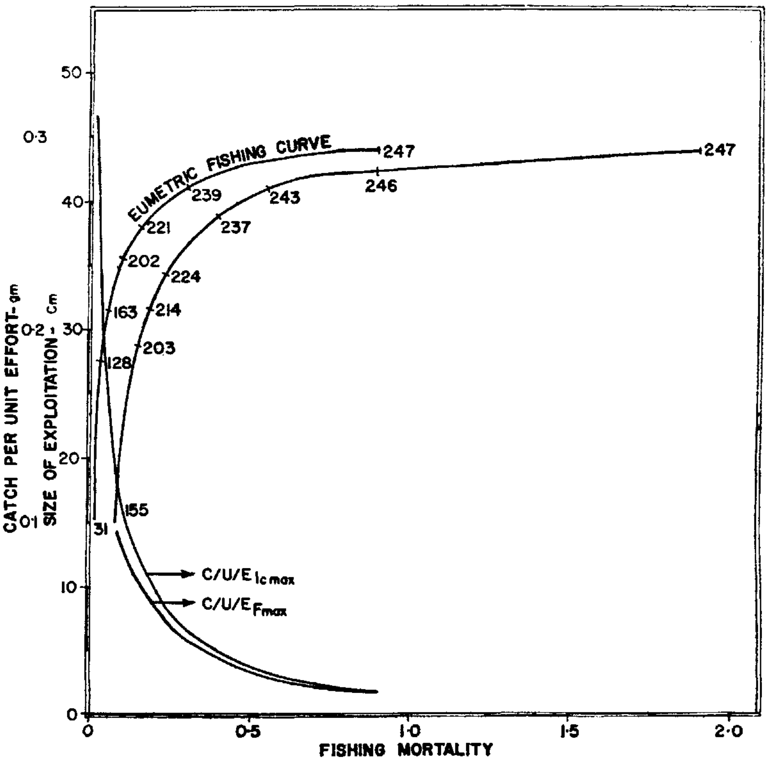

FIG. 1. Eumetric fishing curve, its complement and the corresponding catch-per-unit-effort for a set of population parameters. Conversion factor used to change Y' to $\frac{Y_w}{R}$ is 4354.99. Other parameters are L_∞=69 cm, M=0.1, K=0.066667, and catchability=0.0000526.

Similarly the eumertric yield curve and its complement would permit one to read the maximum yield at a fixed fishing level or mesh size.

Using the table of yield functions (Table I, $M/K=1.5$) prepared by Beverton and Holt (1966), the eumetric fishing curve, its complement and the relation between the catch-per-unit-effort (C/U/E) along these curves and the fishing mortality (F) have been drawn in Fig. 1, for a set of values of population parameters. A few estimates of the maximum yield per-recruit corresponding to a fixed fishing intensity or mesh

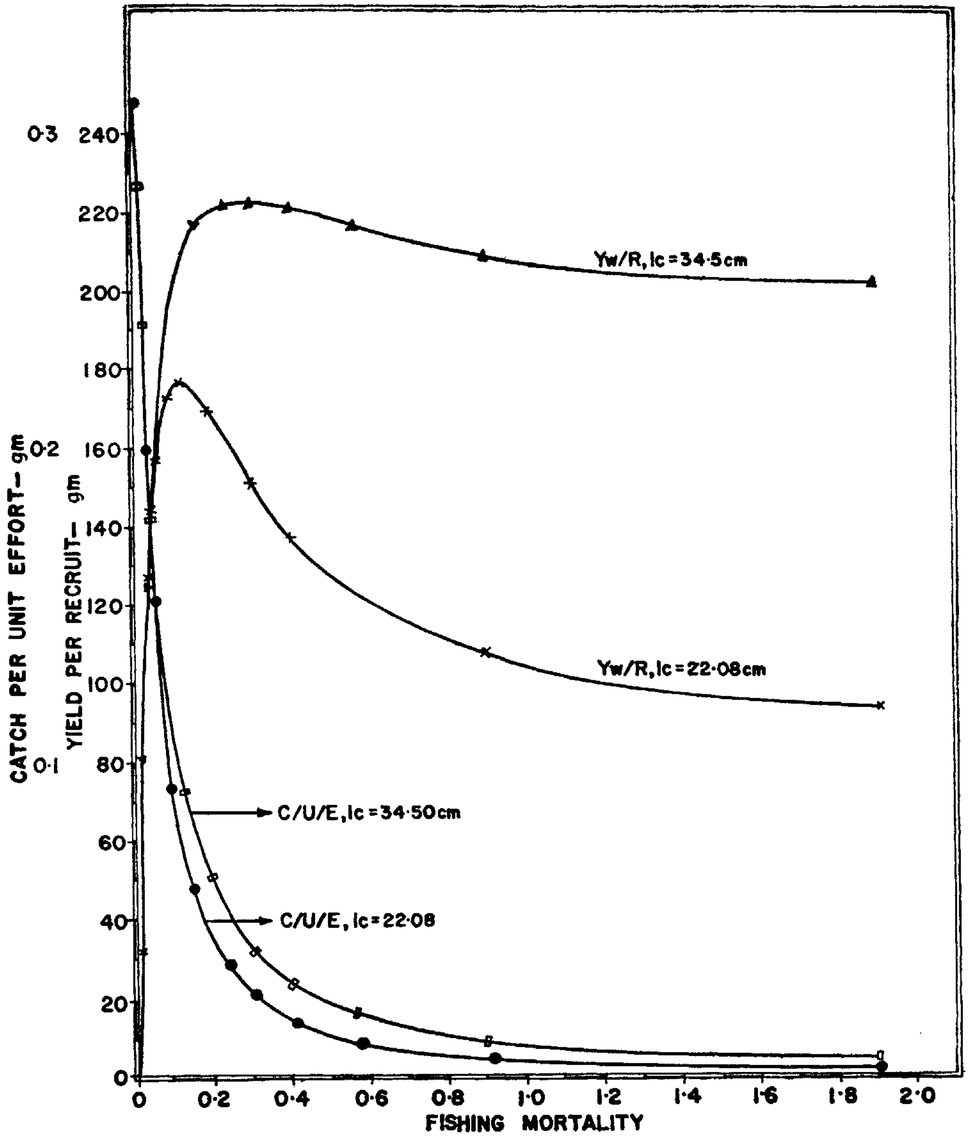

FIG. 2. Yield-intensity curves and the corresponding catch-per-unit-effort.

size are also given in the Figure. Since the eumetric fising/yield curves and their complements are all inversely exponential with an asymptote at infinitely high fishing mortality, the slope of these curves steadily decreases with increase in *F*.

In Figs. 2 and 3 the yield-intensity curves, the yield-mesh curves and their corresponding C/U/E have been drawn. Depending on the mesh size the yield-intensitv curves are either inversely exponential or they reach a maximum before declining to the asymptote, although this mode gradually disappears as the mesh

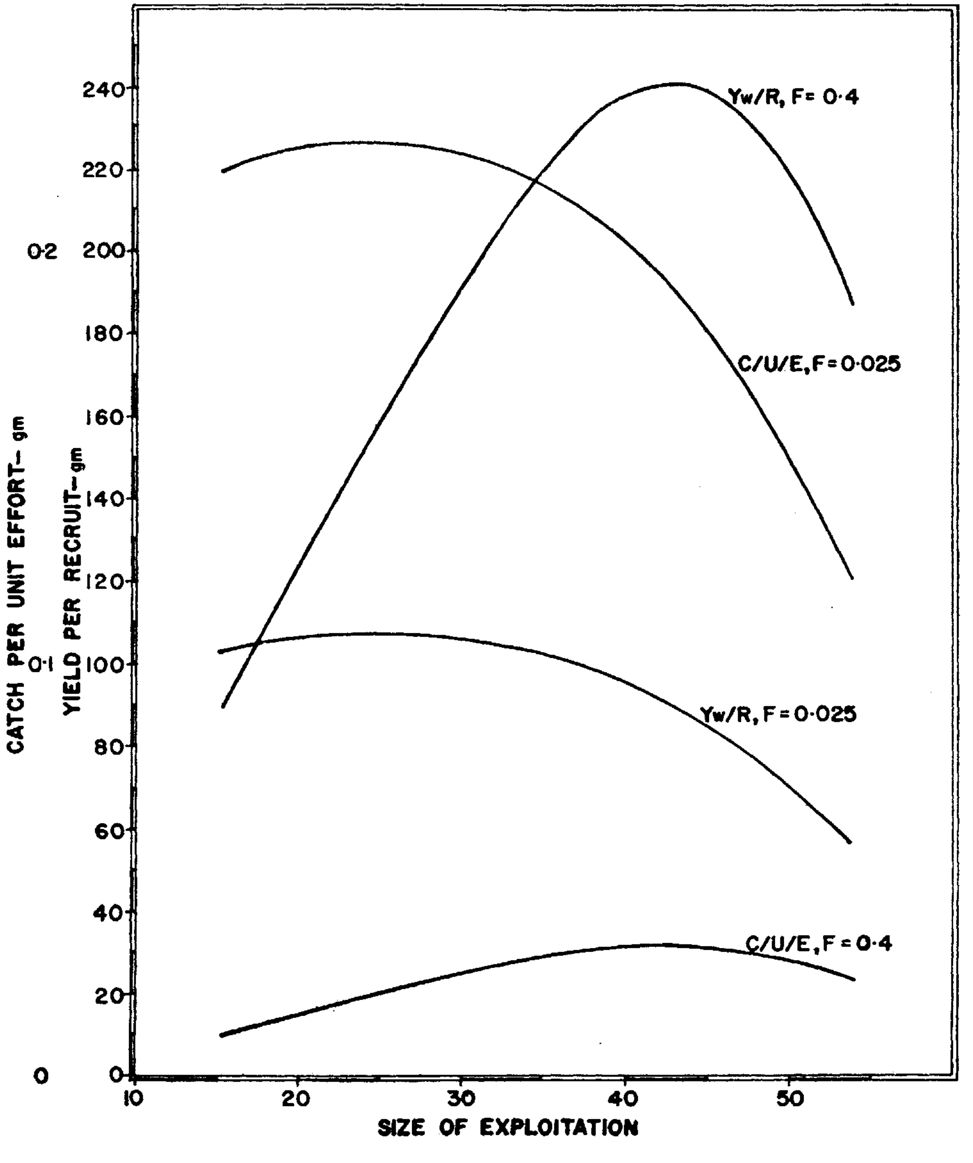

FIG. 3. Yield-mesh curves and the corresponding catch-per-unit-effort.

TABLE I

*Extra fishing intensity (units) required for one gramme increase in yield at different levels of the eumetric yield curve for a hypothetical population**

$\frac{F}{Z}$	F	Corresponding fishing intensity (Units)	Additional fishing intensity needed (Units)
0.15	0.01765	335.5	5.5
0.30	0.04286	814.8	11.1
0.45	0.08182	1555.5	26.2
0.60	0.150	2851.7	122.2
0.75	0.30	5703.4	403.4
0.90	0.90	17110.3	16325.6

*Values obtained partly by fitting a polynomial to the eumetric yield curve and applying Newton's method of solving equations and partly by the method of successive approximations based on Newton's advancing and receding formulae (*see* text for details).

size approaches the optimum size (Beverton 1954). The slope of the yieid-intensity curves, wheather it possesses a mode or not steadily decreases (*see* Beverto n and Holt 1957 and Gulland 1968 *a*). Hence the C/U/E corresponding to these curves also decreases with F and as pointed out by Gulland (1968 *b*) they declin e along a concave upward curve even when the yield curves take different shapes. Since the average biomass of the population in the exploited phase ($\overline{P'}_w$) along the eumetric yield curve and the yield-intensity curves decline exponentially and $Y_w = F\bar{P}'_w$, the corresponding C/U/E, i.e. $\frac{Y_w}{\tilde{f}}$ or $\frac{cY_w}{F}$ being proportional to $\bar{P}'_w$ also traces a concave upward curve, approaching zero as the fishing intensity is increased to infinity.

The catch-per-unit-effort corresponding to the yield-mesh curves (Fig. 3) shows the same trend as the yield values because the fishing mortality remains constant. In Fig. 4 the relative changes in the yield and C/U/E, at various gear selection, when the fishing level was altered so as to correspond to that indicated by the eumetric fishing curve, have been examined. This study is confined to the portion bounded by the eumetric yield curve and its complement. From Table I, the extra fishing effort required for a fixed increase in yield along the eumetric yield curve has been found to increase with the fishing pressure. The yield values given in Fig. 1 for this curve and its complement also showed that a significant reduction in the fishing intensity from a high level will have only a small effect on the yield. Hence at high fishing intensities, the catch being near the asymptote, the per cent decrease will be negligible. The corresponding C/U/E or $\frac{cY_w}{F}$ will therefore record a greater per cent increase, although the absolute increase may be small. At high fishing pressure the C/U/E being very low, this increase is particularly felt by the fishery. Thus in

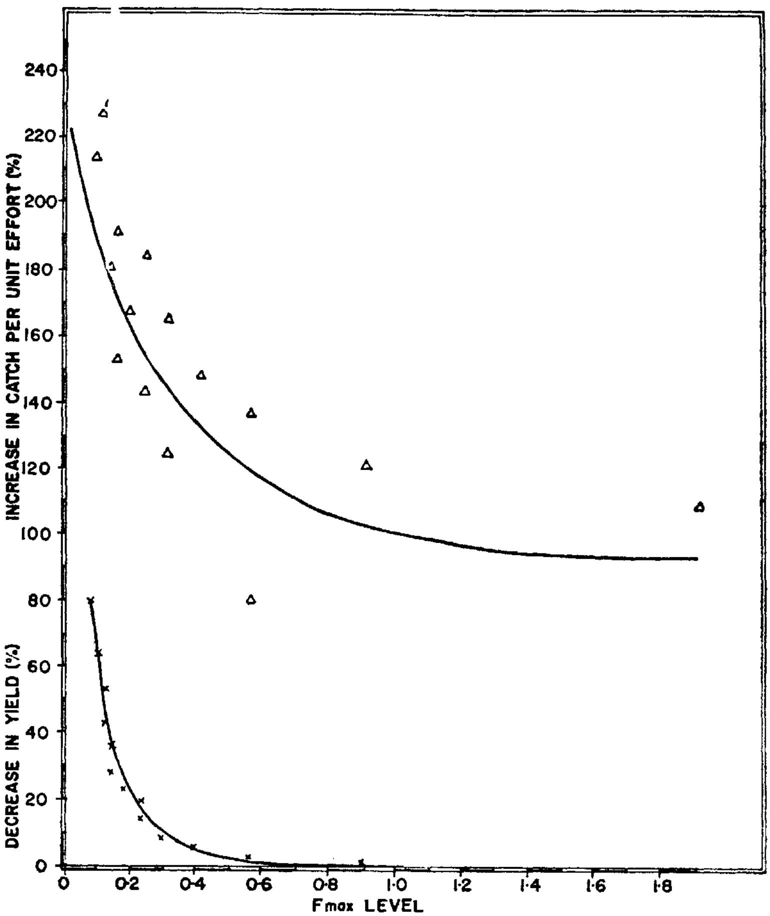

FIG. 4. Relative change in yield and catch-per-unit-effort at different mesh selection when the fishing level is reduced from F_{max} to that corresponding to the eumetric fishing curve for an imaginary population.

Fig. 4, the increase in C/U/E, especially at high mesh selection far outweighs the decrease in yield as the fishing mortality is reduced from the F_{max} level or the fishing level giving the maximum yield at a given mesh size to the $l_{c\ max}$ level, i.e. the level of fishing mortality, for which the above mesh size enables to obtain the maximum yield and catch-per-unit-effort.

Since the eumetric yield curve resembles an inverse exponential curve, with potential yield as the asymptote, the rate of increase in yield at any point may be approximately proportional to the difference between the yield at this point and the potential yield. The rate of change of yield along this curve is a function of both fishing intensity and gear selection. The relation of the latter with fishing intensity is described by the eumetric fishing curve. It is thus possible to relate the rate of change of yield and the fishing mortality by plotting the total derivative against the fishing mortality. The total derivative is obtained from the relation

$$\frac{dY_{w}l_{c\,max}}{dF'} = \frac{\partial Y_w}{\partial F'} + \frac{\partial Y_w}{\partial l_c} \times \frac{dl_c}{dF'} \qquad \ldots.(1)$$

where $\frac{\partial Y_w}{\partial F'}$ and $\frac{\partial Y_w}{\partial l_c}$ are the partial derivatives of the yield function with respect to F' and l_c ; and $\frac{dl_c}{dF'}$ is the derivative of the function relating l_c with F', i.e., the slope of the eumetric fishing curve. The latter function given by Krishnan Kutty (1970), the Holt's version of the Beverton-Holt yield equation, and the required derivatives and definition of the notations, have been given in Appendix I. Approximate derivatives obtained by fitting a polynominal to the eumetric yield curve were plotted in Fig. 5* to examine the relation between the rate of increase of yield along this curve and F. It registered a sharply declining exponential curve. In Fig. 5, the rate of decrease of the corresponding C/U/E also traced a concave upward curve, but the rate of approaching the abcissa is much lower and is due to the nature of the C/U/E and the yield curves and the relation between the two. Because, while the yield curve is very steep near the origin with potential yield as the asymptote, the C/U/E curve has zero for the asymptote. The derivative of the C/U/E curve is obtained by numerical differentiation using Newton's advancing difference formula (*see* p. 120). The rate of change in both cases is for unit F.

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*The rate of change of the yield-per-recruit, $\frac{d\frac{Y_w}{R}l_{c\,max}}{dF}$ plotted in Fig. 5 is of the eumetric yield curve having F for the abcissa. It is obtained from the relation

$$\frac{d=\frac{Y_w}{R}l_{c\,max}}{dF} = \frac{dY'_{max}}{dx} \times \frac{dx}{dE} \times \frac{dE}{dF} \times Q$$

where $\frac{dY'_{max}}{dx}$ is the slope of the fitted polynomial (*see* p. 117) with $x=\frac{E-E_0}{h}$ and hence $\frac{dx}{dE}=\frac{1}{h}$. And from the relation $E=\frac{F}{F+M}$, $\frac{dE}{dF}=\frac{M}{(F+M)^2}$. Since $Y'_{max}=f(x)$ and $\frac{Y_w}{R}l_{c\,max}$ $=Qf(x)$, where Q is the conversion factor, $\frac{d\frac{Y_w}{R}l_{c\,max}}{dx} = Q\frac{dY'_{max}}{dx}$. Hence $\frac{dY'_{max}}{dx}$ is also to be multiplied by Q to get the required derivative. Similarly the rate of decrease of C/U/E, $\frac{d(C/U/E)}{dF}=\frac{d(C/U/E)}{dx}\times\frac{dx}{dE}\times\frac{dE}{dF}$.

To obtain the slope of the yield curve having fishing intensity for the abcissa, use the relation

$$\frac{d\frac{Y_w}{R}l_{c\,max}}{d\tilde{f}} = \frac{d\frac{Y_w}{R}l_{c\,max}}{df} \times \frac{dF}{d\tilde{f}}$$

where $\frac{df}{d\tilde{f}}$ = catchability C, since $F = C\tilde{f}$

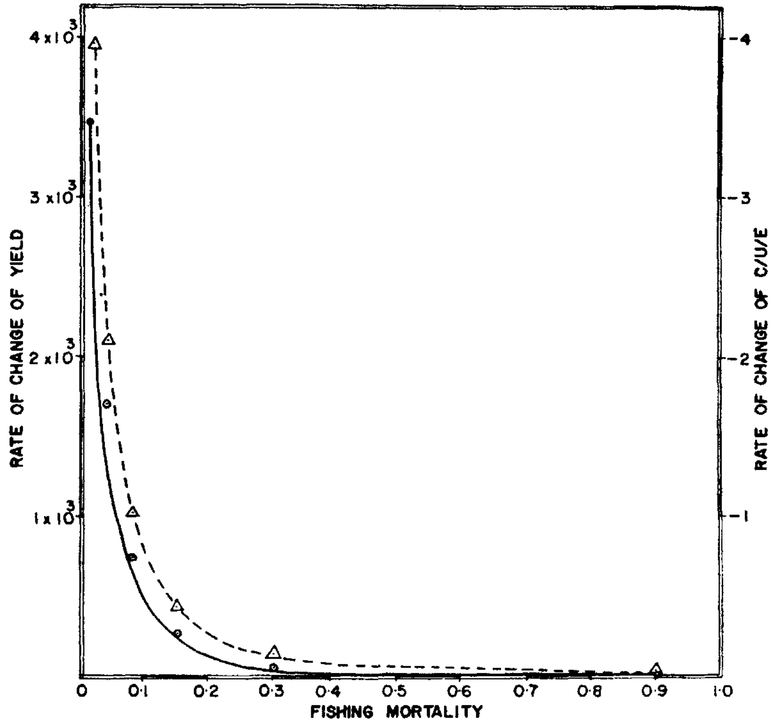

FIG. 5. Rate of change of yield and catch-per-unit-effort along the eumetric yield curve. —⊙—rate of increase of yield, —△—rate of decrease of C/U/E.

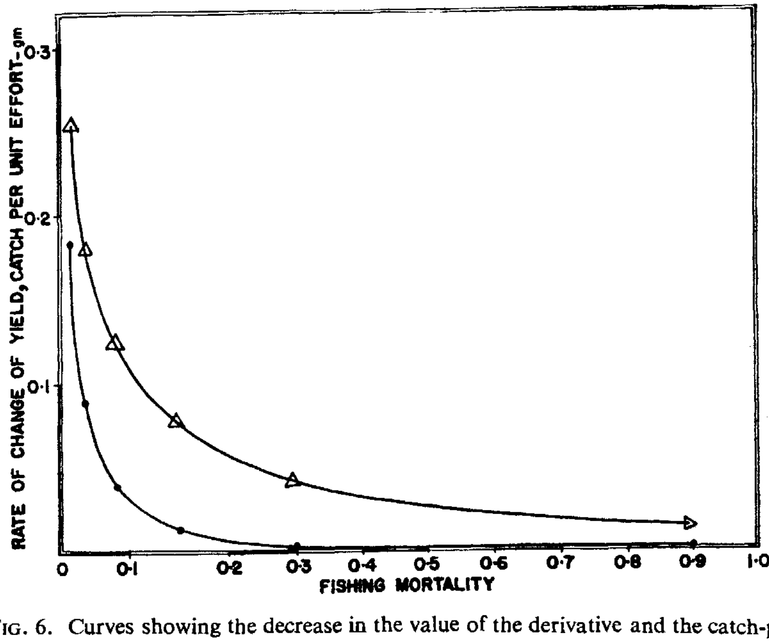

FIG. 6. Curves showing the decrease in the value of the derivative and the catch-per-unit-effort along the eumetric yield curve.—●—Marginal yield ; —△—catch-per-unit-effort.

The decrease in both the C/U/E and the slope of the eumetric yield curve, $\frac{d \frac{Y_w}{R} l_{c\,max}}{d\tilde{f}}$ with the increase in fishing mortality (Fig. 6) follows a concave upward curve with abcissa as the asymptote. But the C/U/E decreases more slowly than the slope of the eumetric yield curve. Gulland (1968 *a*) has recently made a more elegant study of this property together with its practical significance by examining the slope or the marginal yield along the yield intensity curves having fishing intensity for the abcissa. He points out that the marginal efficiency or the ratio between the marginal yield and the C/U/E decreases from one to zero as the fishing intensity is increased and acquires a negative value, if the fishing level becomes greater than that giving the maximum yield. This is true not only for the whole family of yield intensity curves but also with respect to the eumetric yield curve. In the latter case, the curve being inversely exponential, marginal efficiency does not acquire negative values. Instead, it approaches zero, as the fishing intensity reaches infinity. The marginal efficiency corresponding to any fishing level is highest along the eumetric yield curve. Nevertheless, since it is always less than one, marginal yield is also less than the C/U/E. Since the marginal efficiency tends to be very low at high fishing inten sities, significant increase in yield is not possible without a substantial increase in the fishing level and consequently decreases the already low C/U/E.

It should also be noted that while drawing the eumetric yield curve, Beverton and Holt (1957) assumed that any change in the mesh size does not by itself change the fishing mortality, although the latter is affected. The change in F due to any change in fishing power can readily be adjusted by reducing the fishing effort so that this aspect is omitted in a preliminary discussion of maximising the yield. It should, however, be considered for maximising the profit.

Estimation of the Conditions of Optimum Exploitation

The change in the mesh size or the fishing level is often not based on a scientific basis. Besides, because of free entry in the fishing industry, the overhead costs are treated as sunk costs so that the fishing level then tends to increase until the average cost equals the value of C/U/E (Gordon 1953; Gulland 1968 *b*). Hence, by controlling the fishing intensity and by suitable mesh regulation the profit can be maximised. The table of yield functions prepared by Beverton and Holt (1966) is very useful for determining the best combination of fishing pressure and the size of exploitation, without drawing an yield isopleth diagram or the eumetric fishing/yield curves. More precise methods of determining some of these parameters were described by Krishnan Kutty (1968 *a*, 1970*), and Krishnan Kutty and Qasim (1968). The determination of marginal efficiency (Gulland 1968 *a*) is also helpful in studying

*The equation given here for estimating $l_{c\,max}$ is independent of l_r and hence it can be used to find theoretical values of $l_{c\,max}$ which are actually less than l_r. Thus, for an F of 0.0001, $l_{c\,max}$ for plaice will be 3.16 cm and the corresponding $\frac{Y_w}{R} l_{c\,max}$ using Holt's version of the Beverton-Holt yield equation is 0.4106 g. The latter equation automatically adjusts for variations in the number of fish present at all sizes of exploitation, even for values which are less than l_r.

the possible changes in yield and catch-per-unit-effort following a small change in the fishing effort at any desired mesh size. Some simple and reliable methods of fitting the various related yield curves have been described below, using which the conditions of optimum exploitation can be examined for all levels of changes in the fishing effort and/or mesh size, even when the average growth pattern of the individual fish follows an allometric length-weight relation. More details on some of these methods are given by Freeman (1960), Hildebrand (1956) and Scarborough (1966).

Fitting of the eumetric fishing/yield curves and their complementaries. Fitting of the inverse exponential curve.

Although the exact algebraic expressions of the various yield curves are complex, the curves described by these equations are often simple. Hence, their fitting by simpler equations will reduce the labour involved in making the various estimations such as the optimum level of fishing or gear selection and the corresponding yield and the catch-per-unit-effort. The eumetric fishing/yield curves and their complementaries have the same general shape in being concave downward and approaching the asymptote at an infinite fishing intensity. The fitting of only the eumetric fishing and yield curves have been described below and the same can be applied to the other curves.

From the equation given by Krishnan Kutty (1970) for the eumetric fishing curve (*see* Appendix I), the optimum size of exploitation ($l_{c\ max}$) at any fishing mortality is independent of the size at recruitment and this curve, unlike the eumetric yield curve, does not pass through the origin. However, the equations

$$l_{c\ max} = l_{cy}\,(1 - e^{-qx}) \qquad \ldots(2)$$

and

$$l_{c\ max} = l_{cy}\,(1 - e^{-q\,(x-x_0)}) \qquad \ldots(3)$$

were fitted to the North Sea plaice data. In the above equation l_{cy} corresponds to the size of exploitation at which the potential yield is obtained x is the value of F in terms of the chosen unit of abcissa and x_0 is the abcissa at which $l_{c\ max}$ is zero and q is a constant.

Equation (2) is fitted by the method of Allen (1966) which gives the best least-squares estimates of the parameters. Nevertheless, the eumetric fishing curve could not be described adequately by this equation. The eumetric yield curve was also not satisfactorily described by the equivalent equation even though this curve passes through the origin.

The method adopted for fitting the equation (3) was the familiar Ford-Walford technique commonly employed for fitting the von Bertalanffy growth curve (*see* Beverton 1954). Using the North Sea plaice data, this equation was fitted to a set of $l_{c\ max}$ values corresponding to values of F, between 0.3 and 1.8, spaced at intervals of 0.3. The fitted equation describing the eumetric fishing curve is

$$l_{c\ max} = 49.13963\ \left(1 - e^{-0.634072(x+1.545069)}\right) \qquad \ldots(4)$$

where $x = \dfrac{F}{0.3}$ since the chosen unit of abcissa is 0.3. In equation (4) l_{cy} was obtained by plotting $l_{c\ max\ x+1}$ versus $l_{c\ max_x}$ and the constants q and x_0 by fitting

$$l_n\,(l_{cy} - l_{c\ max}) = (l_n\, l_{cy} + qx_0) - qx \qquad \ldots(5)$$

In equation (5) the unit of abcissa is 0.3. When the scale of abcissa is altered, the intercept remains constant but the slope changes. Thus when the unit of abcissa is changed to unity, so that F becomes the abscissa instead of $\frac{F}{0.3}$, the new slope q' in equation (5) is obtained from $0.634072 \times \frac{dx}{dF}$, where $\frac{dx}{dF} = \frac{1}{0.3}$, as $x = \frac{F}{0.3}$. The constant F_0 corresponding to x_0 is to be calculated from the new intercept, $(ln\, l_{cy} + q'F_0)$. Since the intercept remains unaltered by any change of scale of abcissa

$$(l_n\, l_{cy} + q'F_0) = (l_n\, l_{cy} + qx_0) \qquad \ldots.(6)$$

since $q' = \frac{q}{0.3}$, equation (6) will be satisfied only if $F_0 = x_0 \left(\frac{dF}{dx}\right.$ or the reciprocal of $\left.\frac{1}{0.3}\right)$. Thus by writing equation (4) in terms of F

$$l_{c\,max} = 49.13963 \left(1 - e^{-2.113573\,(F + 0.463521)}\right) \qquad \ldots.(7)$$

where F is the yearly instantaneous fishing mortality rate. Following the above procedure, the equation fitted to describe the eumetric yield curve is

$$\frac{Y_w}{R} l_{c\,max} = 437.6104 \left(1 - e^{-3.020353\,(F + 0.235861)}\right) \qquad \ldots.(8)$$

In equations (7) and (8) 49.13963 and 437.6104 are the estimates of l_{cy} and the potential yield-per-recruit, obtained in the course of fitting the respective equations and are less accurate than the values given by Krishnan Kutty (1970). Nevertheless, the best estimates of $l_{c\,max}$ and $\frac{Y_w}{R}\, l_{c\,max}$ for values of F, within the range used for fitting the respective curves, can be obtained only by substituting these estimates even if they are less accurate.

Tables II and III compare the theoretical values of $l_{c\,max}$ and $\frac{Y_w}{R} l_{c\,max}$ obtained by using equations (7) and (8) with the values used for fitting these curves. These values indicate that a reasonably good fit is obtained by these equations although extrapolation, particularly at low fishing intensity, has serious limitations. Allen's method* gave a better fit than equation (7), but involves slightly more computation.

Two other simple methods of fitting the eumetric fishing and the yield curves attempted below are : (*a*) by fitting a polynomial of n^{th} degree using a Table of matrices, and (*b*) by means of interpolation methods using finite differences.

Polynomial curve fitting—The method adopted is that described by Parker (1958) for equally spaced abcissas using inverse Vandermonde matrices. This method

*Dr. Allen Director, Pacific Biological Station, Nanaimo B.C., Canada, kindly fitted this equation while answering my queries on the fitting of equation (2) by other methods.

TABLE II

Comparison of $l_{c\ max}$ values obtained from different fitted equations with the original data

F	Original data* (cm)	Equation (7) (cm)	Equation (9) (cm)
0.1	**31.345**	34.2058	33.9720
0.3	40.0	39.3540	
0.5	**43.2264**	42.7274	43.2037
0.6	44.25	43.9490	
0.73	**45.2726**	45.1961	45.2721
0.9	46.25	46.3864	
1.2	47.455	47.6793	
1.5	48.24	48.3650	
1.65	**48.547**	48.5755	48.5212
1.8	48.8075	48.7287	

Data obtained by the method described by Krishnan Kutty (1970). Values in **bold figures** are not used for fitting the equations.

TABLE III

Comparison of $\frac{Y_w}{R} l_{c\ max}$ *values obtained from the different fitted equations with the original data.*

(Values in **bold figures** *are not used for fitting the equations)*

F	Original data (g)	Equation (8) (g)	Equation (10) (g)	Equations (11), (12), (14) respectively (g)
0.1	**223.6438**	278.9263	270.0094	
0.3	351.7430	350.8747		
0.5	**393.4884**	390.2036	392.8886	392.8886
0.6	404.5892	402.5609		
0.73	**414.42**	413.9427	414.6531	414.6532
0.9	422.7711	423.4471		
1.2	431.1211	431.8869		
1.5	435.5233	435.2976		
1.65	**436.9596**	436.1405	436.6287	436.6324
1.80	438.0738	436.6757		

is briefly described here and a part of the table of inverses given by Parker is reproduced in Appendix II for fitting third to sixth degree equations. Given $n+1$ equi-spaced arguments and the corresponding ordinates an equation of n^{th} degree can be fitted. For this, the abcissas are transformed so that they take successively the value 0, 1, 2, etc. The transformation is effected by subtracting the initial value of the abcissa, from each of the given abcissas, and dividing by the interval of tabulation, h. The successive columns of the appropriate table of inverses, i.e., the table containing $n+1$ rows and columns, are multiplied respectively by the ordinates y_0, y_1, y_2, etc., and summed by row. Each of these sums is then multiplied by $\frac{1}{n!}$.

The resulting values in the order are the required unknowns of the polynomial, the last value being the coefficient of x raised to n^{th} degree.

The fitted equations describing the eumetric fishing and yield curves of the North Sea plaice within the range $F=0.3$ and $F=1.8$ are

$$l_{c\ max} = 0.0075625\ x^5 - 0.120625\ x^4 + 0.7771875\ x^3 - 2.725625\ x^2 + 6.31150\ x + 40.0 \quad \ldots.(9)$$

and

$$\frac{Y_w}{R}\ l_{c\ max} = 0.12633333\ x^5 - 2.05286660\ x^4 + 13.297558\ x^3 - 44.749775\ x^2 + 86.22495\ x + 351.7430 \quad \ldots.(10)$$

where $x = \frac{F-0.3}{0.3}$, since the initial value of $F\,(F_0)$ is 0.3 and F is spaced at intervals of 0.3. Tables II and III give few estimates of $l_{c\ max}$ and $\frac{Y_w}{R}\ l_{c\ max}$ using equations (9) and (10). These estimates compare very well with the values in bold, and hence the method is accurate for any intermediate value of F lying between 0.3 and 1.8, but extrapolated values, especially at low F, were not reliable. When $l_{c\ max}$ values for a series of F are required, this method is better than the one described by Krishnan Kutty (1970). However, the estimates of $l_{c\ max}$ required for fitting equation (9) are to be obtained by the latter method and $\frac{Y_w}{R} l_{c\ max}$ values for fitting equation (10) were calculated by substituting the appropriate $l_{c\ max}$ values in Holt's version of the Beverton-Holt yield equation. The estimation of $l_{c\ max}$ using the Newton's method of solving equations is somewhat tedius when the growth is not isometric, but the computation required for obtaining $\frac{Y_w}{R} l_{c\ max}$ can be reduced by using either Jones's (1957) version of the yield equation (*see* also Wilimovsky and Wicklund 1963) or by approximate integration (*see* p. 135).

Polynomial curve fitting is also useful in calculations involving the derivatives of the yield function and for approximate solution of the equations. For instance, in applying the Newton's method of solving equations, given $y=f(x)$, computational labour can be reduced if the function is replaced by a polynomial. The approximating polynomial can be either in the form of a rational integral function described above or in the form of a difference equation. The construction and application of difference equations in the yield studies have been discussed below.

Estimation by the method of interpolation using finite differences

Krishnan Kutty (1968 *b*) has applied Newton's method of direct interpolation for equal intervals for estimating the yield-per-recruit when length-weight exponent is 3.5. Methods of direct and inverse interpolation are simple and accurate and are useful for estimating the various yield parameters required for determining the optimum fishing conditions. If tabulated values of a function for equally spaced values of abcissas are given, the value of the function corresponding to any intermediate value of x can be obtained to the desired accuracy, using any standard interpolation formula. The method of interpolation for unequally spaced abcissas is not considered here. According to Freeman, the principle involved in the interpolation method using finite differences is that the given set of data can be described by a polynomial with sufficient accuracy. When x is spaced equally and $y=f(x)$ is a polynomial, values of y will be so related that their successive differences decrease progressively to zero. Thus, if a third degree curve exactly fits the data, the third derivative and the third differences will be constant, and hence the fourth differences will be zero. With other functions, if they can be represented approximately by polynomials, as can often be done when $f(x)$ is continuous and single-valued, successive differences are taken until they are very nearly constant, and therefore, change only slowly. In such cases, five or six equi-spaced values of x, and the corresponding ordinates, are desirable so that the interpolation to fourth or fifth differences can be done. It is not a necessary condition that the maximum or minimum point, if present within the range of interpolation, should be given. Since interpolation is generally done with approximating functions, the accuracy of the interpolated value can be increased either by reducing the range of interpolation and the unit of abcissa or by retaining higher order differences for interpolation. Although direct interpolation based on a polynomial can be done even over non-monotonic range of the data, they are less suited for periodic functions except over restricted ranges.

For applying the interpolation formulae, a difference Table using the given values of the function, has to be prepared as shown in Table IV. It is convenient to change the scale of abcissa by taking the interval of tabulation, h as the unit of abcissa and changing the origin to the initial value of x, so that they now read 0, 1, 2,....... in the transformed scale. The successive abcissas may then be denoted in the order by $x_0, x_1, x_2, \ldots\ldots\ldots x_n$ and the corresponding ordinates by $y_0, y_1, y_2, \ldots\ldots y_n$. The difference table is then completed by taking the successive advancing y-differences

$$\triangle y_i = y_{i+1} - y_i,\ \triangle^2 y_i = \triangle y_{i+1} - \triangle y_i,\ \ldots\ \triangle^m y_i = \triangle^{m-1} y_{i+1} - \triangle^{m-1} y_i.$$

Using the above notation Newton's advancing (or forward) difference formula for interpolation is

$$y_k = y_0 + s_1 \triangle y_0 + s_2 \triangle^2 y_0 + s_3 \triangle^3 y_0 + \ldots \qquad \ldots(11)$$

TABLE IV

Difference table for estimating the maximum yield at $F=0.73$ for the North Sea plaice (The leading term and the successive leading differences are given in **bold figures**).

F	Transformed values of F	$\frac{Y_w}{R} l_{c\,max}$ (y_i)	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$	$\Delta^5 y_i$
0.3	$0(x_0)$	**351.7430**(y_0)					
			52.8462(Δy_0)				
0.6	$1(x_1)$	$404.5892(y_1)$		**—34.6643**$(\Delta^2 y_0)$			
0.73	$1.4333(x_k)$	?	$18.1819(\Delta y_1)$		**24.8324**$(\Delta^3 y_0)$		
0.9	$2(x_2)$	$422.7711(y_2)$		$—9.8319(\Delta^2 y_1)$		**—18.9483**$(\Delta^4 y_0)$	
			$8.3500(\Delta y_2)$		$5.8841(\Delta^3 y_1)$		**15.1603**$(\Delta^5 y_0)$
1.2	$3(x_3)$	$431.1211(y_3)$		$—3.9478(\Delta^2 y_2)$		$—3.7880(\Delta^4 y_1)$	
			$4.4022(\Delta y_3)$		$2.0961(\Delta^3 y_2)$		
1.5	$4(x_4)$	$435.5233(y_4)$		$—1.8517(\Delta^2 y_3)$			
			$2.5505(\Delta y_4)$				
1.8	$5(x_5)$	$438.0738(y_5)$					

where y_k is the value of the interpolant, y_0 is the value of $f(x)$ corresponding to the initial value of x or the leading term, $\triangle y_0$, $\triangle^2 y_0$, etc., are the successive leading differences and s_1, s_2, s_3, etc., are the binomial coefficients $\frac{x_k}{1!}$, $\frac{x_k(x_k-1)}{2!}$, $\frac{x_k(x_k-1)(x_k-2)}{3!}$, etc., where x_k is the argument of the interpolant in the transformed scale. If the maximum yield-per-recruit, $\frac{Y_w}{R}\ l_{c\ max}$ for $F = 0.73$ is to be interpolated using the data in Table IV, then $f(0.73)$ from equation (11) is

$$y_{1.4333} = 351.7430 + 1.4333 \times 52.8462 + \frac{1.4333(1.4333-1)}{2} \times -34.6643 +$$

$$\frac{1.4333(1.4333-1)(1.4333-2)}{3\times2} \times 24.8324 +$$

$$\frac{1.4333(1.4333-1)(1.4333-2)(1.4333-3)}{4\times3\times2} \times -18.9483 +$$

$$\frac{1.4333(1.4333-1)(1.4333-2)(1.4333-3)(1.4333-4)}{5\times4\times3\times2} \times 15.1603$$

$$= 414.6527\ g.$$

From Table III this is a close estimate of $\frac{Y_w}{R}\ l_{c\ max}$ for $F=0.73$.

The eumetric fishing curve over a certain range of F can also be similarly represented by a difference equation, if $l_{c\ max}$ values are available for equally spaced values of F.

In applying the interpolation formula, computation can sometimes be reduced, if y_i values are large by subtracting a constant and transforming them to smaller numbers. The coefficients of successive differences can also be obtained from a table of binomial coefficients.

Freeman has recommended the use of the central difference formula since the difference coefficients are smaller and these converge more rapidly so that generally with the same number of differences a more accurate y_k can be obtained. In applying the central difference formula, the difference coefficients on either side of the origin are required. The origin is thus shifted towards the centrally placed x within the range of the given abcissas with the interval of tabulation of x as the unit abcissa. The backward abcissas or the arguments whose values are less than that of the x corresponding to the origin are assigned progressively, according to the transformed scale, values of -1, -2, -3, etc. and the values 1, 2, 3, etc. for successively larger values of x or the forward abcissas. The successive tabular ordinates and abcissas may be denoted by the symbols given in Table V. Using this notation, the central difference formula known as the Gauss's forward formula is

$$y_k = y_0 + s_1\triangle y_0 + s_2\triangle^2 y_{-1} + (s+1)_3\triangle^3 y_{-1} + (s+1)_4\triangle^4 y_{-2} + \dots \qquad \dots(12)$$

and the Gauss's backward formula can be stated as

$$y_k = y_0 + s_1\triangle y_{-1} + (s+1)_2\triangle^2 y_{-1} + (s+1)_3\triangle^3 y_{-2} + (s+2)_4\triangle^4 y_{-2} + \dots \qquad \dots(13)$$

where y_0 is the ordinate at the origin, $\triangle y_0$, $\triangle y_{-1}$, $\triangle^2 y_{-1}$, etc., are the central differences, s_1, s_2, $(s+1)_2$, etc., are their coefficients and $s = x_k$. In both formulae the even differences like $\triangle^2 y_{-1}$, $\triangle^4 y_{-2}$, etc., fall on line with y_0 in a difference table (Table V) but the odd differences like $\triangle y_0$, $\triangle^3 y_{-1}$, etc. in the forward formula fall on line between y_0 and y_1 and in the backward formula they lie between y_0 and y_{-1}. If the argument of the interpolant lies between x_0 and x_1, the forward formula is preferred but if it lies between x_0 and x_{-1} then equation (13) is chosen for interpolation. The law of formation of these two equations as pointed out by Freeman, as we advance from each term to the next, leaving y_0, is that in the forward formula we *alternately* deduct unity from the subscript of y and add unity to the number whose factorial is the coefficient and in the backward formula we *alternately* add unity to the number whose factorial is the coefficient and deduct unity from the subscript of y.

Freeman has indicated that the central difference formula is advantageous only if (*a*) the argument of the interpolant lies near the central interval of the range of the given values of x, and (*b*) the neglected order of differences is nearly constant or changes only slowly. Hildebrand (1956) has pointed out that the difference path chosen for the interpolation should depart least for best results from an imaginary horizontal line through the argument of the interpolant, although for ordinary purposes the reduction in loss of accuracy afforded by the "preferred path" is of no great consequence. Hence, if best results are required it is preferable to choose abcissas on either side of the argument of the interpolant and perform the interpolation using the central difference formula.

Freeman has also suggested that it is not always necessary to interpolate with equation (11) which uses the leading term and the leading differences or with equations (12) and (13) which utilize the central differences. According to Sheppard's zig-zag rule, interpolation could be done starting with any given ordinate in Table IV and for choosing the successive difference coefficient from the difference Table, movements at each step could be made either upwards or downwards until the last single term is reached. The numerator of the coefficients of each of the difference terms, beginning from the second difference, is always obtained by subtracting successively from the argument of the interpolant, the abcissas involved in obtaining the preceding difference term and taking their product. For instance, referring to Table IV, if the interpolation is begun at $y_1 = 404.5892$ and -9.8319 and 5.8841 are the second and third differences lying in the difference path chosen for the interpolation, the coefficient of $\triangle^3 y_1$, in terms of the original abcissas, will be $\frac{(x-0.6)(x-0.9)(x-1.2)}{3 \times 2}$ because the preceding difference, $\triangle^2 y_1$ being -9.8319, is the difference of two $\triangle y_i$ values, which are in turn derived from the three consecutive ordinates corresponding to $x = 0.6$, 0.9 and 1.2. The coefficient of the first difference term is equal to the difference between the abcissa of the interpolant and the argument corresponding to the ordinate chosen for the interpolation, i.e. $x = 0.6$. If the transformed values of x are used, with the initial value of x taken as the origin

(Table IV), and if the interpolation is begun at $y_1=404.5892$, the coefficient of $\triangle^3 y_1$ will be $\frac{(x_k-1)(x_k-2)(x_k-3)}{3\times 2}$. If the origin is set at, say, $x=0.9$, with 0.3 as the unit abcissa and the interpolation is begun at $y_2=422.7711$ corresponding to $x=0.9$, the third order difference coefficient 5.8841 will be denoted by $\triangle^3 y_{-1}$ (see Table V) and its coefficient will be $\frac{(x_k-(-1))(x_k-0)(x_k-1)}{3\times 2}$.

When interpolation is to be done from the given tabular values alone, Hildebrand recommends the use of the difference equations formed by leading, central or receding differences if the intermediate value of x lies in the upper, middle or lower end of the range of interpolation, as can be judged from the difference table. This is because it is desirable to use a difference equation in which the successively introduced ordinates correspond to abcissas which are as near as possible to the argument of the interpolant. Newton's receding (or backward) difference formula makes use of the values in the end of the difference table (see Table IV) and in the notation given for writing the advancing difference formula, it can be written as

$$y_k=y_n+s_1\triangle y_{n-1}+(s+1)_2\triangle^2 y_{n-2}+(s+2)_3\triangle^3 y_{n-3}+\cdots\cdots \qquad \ldots(14)$$

where y_n is the last tabular ordinate, $\triangle y_{n-1}$, $\triangle^2 y_{n-2}$, etc...., are the lowest series of differences in the difference Table and s_1, $(s+1)_2$, etc., are the binomial coefficients of the difference terms where $s=(x_k-x_n)$.

Hildebrand (1956) and Scarborough (1966) discuss at some length the effect of round off and truncation errors on the interpolant. They have shown that if the values of the ordinates are subjected to round-off errors, they build up rapidly in successive differences and make higher differences fluctuate irregularly. If this is noticed, the interpolation should be stopped at the preceding order of differences. The effect of round-off errors can be regulated by rounding the initial data to the same decimal place (Scarborough 1966, p. 3) and by retaining one or more extra figures in the intermediate calculation. The truncation error is also generally reduced by retaining sufficient number of successive differences.

Horizontal and vertical sections of the yield isopleth

Fitting of the yield-intensity and yield-mesh curves may be useful for estimating the yield at a given mesh size for any F or the yield at a given F for any mesh size. This may be accomplished by the polynomial curve fitting, using the inverse matrices or by direct interpolation based on a suitable difference equation. The modal yield values with respect to these curves can be obtained by solving the respective derivatives of the yield equation as described by Krishnan Kutty (1970) and substituting the value of the root in the yield equation. But if the particular yield-mesh or intensity curve is already approximated by a polynomial of either form over a certain range containing the mode, the model yield value can be more easily obtained by finding the abcissa (see below) corresponding to the mode from the first derivative of the polynomial and substituting it in the approximating function. If a series of modal values are required it is advisable to approximate the eumetric yield curve or its complement as the case may be by a polynomial and obtain the maximum yield directly.

A polynomial is fitted to the y' values corresponding to $C=0.38$, 0.42, 0.46, 0.50 and 0.54 taken from the Beverton-Holt table of yield functions for $M/K=2.5$ and $E=0.7$. The mode is then calculated from this portion of the yield-mesh curve which, from the table, lies at $C=0.48$. The fitted polynomials using the inverse matrices and the central differences are

$$y' = 0{\cdot}00000125x^4-0.000016333x^3-0.00020425x^2+0.001247208x+0.019926 \quad \ldots(15)$$

and

$$y_k=0.00003(s+1)_4-0.000023(s+1)_3-0.000542s_2-0.000004s_1+0.021491 \quad \ldots(16)$$

where s_1, s_2, etc., are the binomial coefficients and $s=x_k$.

Equating the derivative of the equation (15) to zero and solving for x by Newton's method (*see p.* 128) using 2.625 as a first approximation of x, the abcissa which maximises y' is 2.49613. The corresponding y' from equation (15) is 0.021561. The maximum y' can be obtained similarly from the equation (16).

To find the abcissa corresponding to maximum y' the equation (16) can be expanded and written as

$$y' = 0.00003\,\frac{x^4-2x^3-x^2+2x}{24} - 0.000023\,\frac{x^3-x}{6} - 0.000542\,\frac{x^2-x}{2} - 0.000004x + 0.021491 \quad \ldots.(17)$$

Equating the derivative of the equation (17) to zero, i.e.,

$$0.00003\,\frac{4x^3-6x^2-2x+2}{24} - 0.000023\,\frac{3x^2-1}{6} - 0.000542\,\frac{2x-1}{2} - 0.000004=0 \quad \ldots.(18)$$

and solving the equation (18) for x by the Newton's method, the abcissa which maximises y' on second iteration is 0.4956. Substituting this value in the equation (16), the maximum y' is 0.021559. Both these estimates agree well with the tabled value.

Gulland (1961) and Allen (1967) have also described some simple methods for analysing the effect of mesh changes on the yield by keeping F as constant.

While handling the yield-intensity curves, it might be useful to be able to find the value of F at which a given l_c would become the $l_{c\ max}$. This would avoid the need for referring to the eumetric fishing curve although a few estimates of $l_{c\ max}$ for equally spaced values of x are required. It may also be necessary to estimate the change in the fishing intensity needed to alter the yield or the catch-per-unit-effort to a particular level either for a given mesh-size or along the eumetric yield curve. Since this is essentially a problem of solving for x corresponding to a particular value of the given function $f(x)$, some method, such as the inverse interpolation should be applied.

The term inverse interpolation is used for finding from a table of $y=f(x)$, the value of the argument corresponding to an intermediate value of $f(x)$. Even if the function $y=f(x)$ is known, the exact solution of x is difficult if $f(x)$ is higher than a quadratic. However, reliable estimates of x can easily be obtained by the numerical solution of equations. Very often when the function is not known, the numerical solution is done using an approximating function. Of the different methods, the successive approximation and newton's method of approximate solution of equations are discussed.

Successive approximation : Although this method is generally treated under inverse interpolation, it is essentially one of numerical solution of an equation by iteration. The function $y=f(x)$ is first equated to zero and then expressed in the form

$$x = F(x). \qquad \ldots.(19)$$

Progressively better estimates of x are then substituted in the right-hand side of equation (19) until the root is obtained to the desired accuracy. Taking the Gauss's forward difference equation, the first approximation of x, by neglecting terms containing second and higher order differences, will be

$$\hat{x}_1 = \frac{y_k - y_0}{\triangle y_0} \qquad \ldots.(20)$$

If the third and higher differences are neglected, the equation (12) becomes

$$y_k = y_0 + s_1 \triangle y_0 + s_2 \triangle^2 y_{-1} \qquad \ldots.(21)$$

so that the second approximation of x is

$$\hat{x}_2 = \frac{y_k - y_0}{\triangle y_0 + \frac{\hat{x}_1 - 1}{2} \triangle^2 y_{-1}} \qquad \ldots.(22)$$

By neglecting fourth and higher differences the third approximation of x is obtained from

$$\hat{x}_3 = \frac{y_k - y_0}{\triangle y_0 + \frac{\hat{x}_2 - 1}{2} \triangle^2 y_{-1} + \frac{(\hat{x}_2 + 1)\ (\hat{x}_2 - 1)}{6} \triangle^3 y_{-1}} \qquad \ldots.(23)$$

Higher approximations are obtained similarly.

Using the North Sea plaice data value of F is estimated by the above method for $l_{c\ max} = 45.28$ cm (Table V). Thus from the equation (20)

$$\hat{x}_1 = \frac{45.28 - 46.25}{1.205}$$
$$= -0{,}80498.$$

Substituting this value for $\hat{x}_1$ in the equation (22), $\hat{x}_2 = -\ 0.504557$. Continuing the third to fifth approximations, $\hat{x}_5 = -0.5613$. The unit of abcissa being 0.3 and the origin being at $F = 0.9$, on transforming back to the original scale

$$F = 0.9 - 0.5613 \times 0.3 = 0.7316$$

This is a good approximation since from Table II, when $F = 0.73$, $l_{c\ max} = 45.273$ cm.

Successive approximation based on Newton's advancing difference formula can be applied to find F at which the size at recruitment, l_r will be the optimum

TABLE V

Difference table for estimating the F by inverse interpolation for a given $l_{c\ max}$. (The differences used for successive approximation are given in **bold figures**)

F	Transformed F	$l_{c\ max}(y_i)$	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$	$\Delta^5 y_i$
0.3	$-2(x_{-2})$	$40.0(y_{-2})$					
			$4.25(\Delta y_{-2})$				
0.6	$-1(x_{-1})$	$44.25(y_{-1})$		$-2.25(\Delta^2 y_{-2})$			
	?	**45.28**	$2.00(\Delta y_{-1})$		$1.455(\Delta^3 y_{-2})$		
0.9	$0(x_0)$	**46.25**(y_0)		**−0.795**$(\Delta^2 y_{-1})$		**−1.08**$(\Delta^4 y_{-2})$	
			1.205(Δy_0)		**0.375**$(\Delta^3 y_{-1})$		**0.9075**$(\Delta^5 y_{-2})$
1.2	$1(x_1)$	$47.455(y_1)$		$-0.420(\Delta^2 y_0)$		$-0.1725(\Delta^4 y_{-1})$	
			$0.785(\Delta y_1)$		$0.2025(\Delta^3 y_0)$		
1.5	$2(x_2)$	$48.24(y_2)$		$-0.2175(\Delta^2 y_1)$			
			$0.5675(\Delta^2 y_2)$				
1.8	$3(x_3)$	$48.8075(y_3)$					

size of exploitation, $l_{c\ max}$. If extrapolation outside the range of given values of $l_{c\ max}$ is needed, it may generally be done for one unit of abcissa provided that the unit is small and the function runs smoothly near the ends of the range of the given values of x (Scarborough). The use of the equation (12) will also require a few theoretical estimates of $l_{c\ max}$ which are less than l_r.

Freeman has indicated that the accuracy in successive approximation is greater if the interval of tabulation is small and if the required root is close to the value of x corresponding to the ordinate at which interpolation begins. Also if $\triangle y_i$ has n significant Figures, $\hat{x}$ will also be accurate at best only to n significant digits although usually the last digit is doubtful.

Instead of using a difference equation as above, successive approximation can also be done using a polynomial (see Scarborough). Thus using the equation (9) for the above example

$$\hat{x}_n = \frac{45.28-40.0}{0.007563\hat{x}^4_{n-1}-0.120625\hat{x}^3_{n-1}+0.777188\hat{x}^2_{n-1}-2.725625\hat{x}_{n-1}+6.3115} \quad \ldots.(24)$$

From Table V when $l_{c\ max}$ is 45.28 cm F is approximately 0.75 so that in the transformed scale of the polynomial

$$\hat{x}_1 = \frac{0{\cdot}75-0.3}{0.3} = 1.5$$

Substituting 1.5 for $\hat{x}_{n-1}$ in the above equation

$$\hat{x}_2 = \frac{5.28}{3.603} = 1.4654$$

Continuing further approximations

$$\hat{x}_5 = 1.43979$$

and converting back to the original scale

$$F = 0.7319.$$

As in successive approximation using a difference equation, here also the terms containing the powers of x can be ignored initially, by successively introducing higher degree terms for iteration. But convergence will be somewhat slower because when *nth* degree term and above are omitted from a rational integral function the truncated function will be different from a polynomial of degree n-1 fitted to the same data. This is not the case when a difference equation is used for iteration.

Newton's method of approximate solution of equations—Krishnan Kutty (1970) has suggested this method to find the optimum size of exploitation, $l_{c\ max}$ from the equation describing the eumetric fishing curve. Given any value of the function $y=f(x)$, the corresponding abcissa is obtained by equating the function to zero and solving for x. For this, the Newton's method of solution by iteration uses the relation

$$\hat{x}_n = \hat{x}_{n-1} - \frac{f(\hat{x}_{n-1})}{f'(\hat{x}_{n-1})} \qquad \ldots (25)$$

where $\hat{x}_n$ and $\hat{x}_{n-1}$ are the *nth* and (*n*-1)*th* approximation of the root, $f(\hat{x}_{n-1})$ and $f'(\hat{x}_{n-1})$ are the values of the function and its derivative with respect to x obtained by substituting x_{n-1} for x. The first approximation is obtained graphically, and from the equation (25). the root is obtained to the desired accuracy by successive approximation. It may sometimes be necessary to replace the exact function by an approximate function if the former is unknown or for minimising computation. Thus if a Table of $y=f(x)$ is given for equally spaced values of x, the exact function can be replaced by a polynomial of either form discussed above. If $l_{c\ max}$ or the yield values are available over a range of F, the fishing mortality corresponding to any intermediate value of the ordinate can be solved easily by substituting the approximate function in the equation (25).

The calculations required for obtaining a value in Table I is given here in detail to illustrate the application of this method in yield studies. The fitted polynomial to y'_{max} values in Table IIb ($M/K=1.5$) of the Beverton-Holt Table of yield functions for the values of E between 0.15 and 0.90 at intervals of 0.15 is

$$y'_{max} = 0.000000892x^5 - 0.000016125x^4 + 0.000201958x^3 - 0.00254488x^2 + 0.01657715x + 0.019488 \qquad \ldots (26)$$

and the derivative

$$\frac{dy'_{max}}{dx} = 0.00000446x^4 - 0.0000645x^3 + 0.000605874x^2 - 0.00508976x + 0.01657715. \qquad \ldots (27)$$

Assuming that the conversion factor for changing y' to the corresponding yield-per-recruit is 4354.99, the $\frac{Y_w}{R}$, corresponding to y'_{max} of 0.033707 at E=0.3, is 146.793648 g. Since the additional effort needed to increase the $\frac{Y_w}{R}$ by one gram is to be determined, the corresponding y'_{max} is

$$y'_{max} = \frac{147.793648}{4354.99} = 0.03393662.$$

In order to find the corresponding abcissa from the equation (26), by using the Newton's method it is first equated to zero to get

$$0.000000892x^5 - 0.000016125x^4 + 0.000201958x^3 - 0.00254488x^2 + 0.01657715x - 0.01444862 = 0 \qquad \ldots (28)$$

Since the derivative of the equations (26) and (28) are the same and $\hat{x}_1=1$ (i.e., $E=0.3$, in the original scale), the values of equations (27) and (28) are 0.01203322 and -0.00022963. Substituting these values in equation (25)

$$\hat{x}_2 = 1.019083$$

Transforming back to the original scale,

$$E = 0.30286245.$$

Since the assumption that $M=0.1$ and catchability$=0.0000526$ are made for drawing up Table I and since $E = \frac{F}{F+M}$,

$$F = \frac{ME}{1-E} \qquad \ldots .(28)$$

and from this by substitution we get $F=0.043443$. The corresponding fishing effort is

$$\tilde{f} = \frac{0.043443}{0.0000526} = 825.9 \text{ units}$$

Limitations—When $f(x)$ is nonmonotonic over the x-interval, i.e., when it does not uniformly increase or decrease, so that its derivative vanishes at some point corresponding to a maximum or a minimum point in the curve, the successive approximation will not be accurate in this region of the curve (Hildebrand). Scarborough has pointed out that the Newton's method of solving the equations should not be applied if the curve is nearly horizontal in the neighbourhood of the root and the method fails in the region where $f'(x)$ is zero. Hilderbrand has also indicated that iteration by the Newton's method may not converge to the desired root if the curve representing the function has turning points or inflections in the interval between $x=\hat{x}_1$ and the true root or between $\hat{x}_1$ and $\hat{x}_2$. Hence, given the function $f(x)$ and the maximum value of the ordinate or the values near the mode, the corresponding abcissas cannot be obtained accurately by these two methods. Scarborough gives certain conditions of convergence to be satisfied for applying these two methods. Although these tests were found applicable to yield curves also, this difficulty can be avoided in yield studies by making the data monotonic by suitable transformations, as for instance, by choosing either the catch-per-unit-effort or $\frac{Y_w}{FR}$ values instead of yield-per-recruit; or as suggested by Scarborough the *regula falsi* method of iteration may be chosen.

The abcissa corresponding to a value near the maximum yield-per-recruit is obtained below by successive approximation using the tabular values given by Beverton and Holt (1957, p. 311) for the North Sea plaice. The transformation of the ordinates is effected by dividing the yield-per-recruit values by the corresponding F (Table VI). The successively approximated values of the abcissa corresponding to $\frac{Y_w}{FR}=1200$, based on the Gauss's backward formula with the origin at $F=0.3$ are $\hat{x}_1=-0.807$, $\hat{x}_2=-0.843$ (-0.85 is used as $\hat{x}_2$ for obtaining $\hat{x}_3$),

TABLE VI

Difference table for estimating the abcissa corresponding to an ordinate close to the maximum by successive approximation using the transformed yield-per-recruit values

Original data		Transformed data					
F	$\frac{Y_w}{R}$(g)	x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$
0.1	214.82	—2	2148.2000				
				—862.0500			
0.2	257.23	—1	1286.1500		415.6333		
				—446.4167		—218.3499	
0.3	251.92	0	839.7333		197.2834		121.6998
				—249.1333		—96.6501	
0.4	236.24	1	590.6000		100.6333		
				—148.5000			
0.5	221.05	2	442.1000				

$\hat{x}_3 = -0.8547$ (-0.8548 is used as $\hat{x}_3$ to get $\hat{x}_4$) and $\hat{x}_4 = -0.85$. On transforming $\hat{x}_4$ back to the original scale, $F=0.215$ and the yield-per-recruit is

$$\frac{Y_w}{R} = 1200 \times 0.215 = 258 \text{ g}.$$

Since a polynomial had already been fitted to the transformed data in Table VI $\frac{Y_w}{R}$ corresponding to $F=0.215$ is obtained for comparison from the polynomial instead of obtaining it by direct interpolation. The fitted polynomial is

$$\frac{Y_w}{FR} = 5.070825x^4 - 66.8166x^3 + 372.77066x^2 - 1173.0749x + 2148.20. \quad ..(30)$$

where $x=\frac{F-0.1}{0.1}$. From the equation (30), $\frac{Y_w}{FR}$, when $F=0.215$, is 1199.40227 and the corresponding yield is

$$\frac{Y_w}{R} = 1199.4 \times 0.215 = 257.87 \text{ g}.$$

The two estimates are in close agreement. The same result may also be obtained by the Newton's method of iteration.

In the above example even though -0.85, which happens to lie very close to the real root, was substituted for $\hat{x}_2$, $\hat{x}_3$ was found to be less accurate than -0.85 itself. This is because the successive iterates are generally closer approximations only when compared to the preceding estimate, but the substitution of a value close to the root will evidently reduce the error in subsequent estimates. For instace, if -0.843 was used for the third approximation, the third and fourth estimates of x would be -0.8569 and -0.8493 respectively, both being less accurate values. The third and fourth approximations seem to lie on either side of the root although $\hat{x}_4$ is closer to the root. Unlike the successive approximation, the Newton's method of iteration always gives a better estimate than the one which is actually substituted in the formula, provided that the method is applicable.

When the successive approximation was carried out on the original data (Table VI) using again the Gauss's backward formula with the origin at $F=0.3$, in order to find the abcissa which corresponds to the yield-per-recruit of 258 g, the successive iterates were found to diverge the values being $\hat{x}_1=-1.145$, $\hat{x}_2=-1.333867$ and $\hat{x}_3=4.78127$.

Freeman indicates that when direct interpolation to n th differences is accurate enough for y_k the inverse interpolation to n th differences may or may not be equally accurate. Thus a better interpolation of x from the table of $y=f(x)$ may sometimes be obtained by neglecting higher differences. Although such instances are not common he has briefly discussed how to determine whether or not higher differences should be ignored taking a third degree curve as an example. The same method may be applied when the exact form of the curve is not known. Thus, using the equation (12) when only the first three terms are used for estimating y_k, the error, by ignoring the third differences and above will be equal to the sum of the terms omitted, i.e., the error,

$$E_3 = \frac{x(x-1)(x+1)}{6} \triangle^3 y_{-1} + \frac{x(x-1)(x+1)(x-2)}{24} \triangle^4 y_{-2} + \epsilon \quad \ldots.(31)$$

where ϵ is the truncation error inherent in the equation (12). If the first four terms are taken, the error,

$$E_4 = \frac{x(x-1)(x+1)(x-2)}{24} \triangle^4 y_{-2} + \epsilon \quad \ldots.(32)$$

The term ϵ may be cancelled from (31) and (32) since it is common to both and giving new notations E'_3 and E'_4,

$$E'_3 = \frac{x(x-1)(x+1)}{6} \triangle^3 y_{-1} + E'_4 \quad \ldots.(33)$$

Rewriting the first term on the RHS of (33) in terms of E'_4,

$$E'_3 = \frac{x(x-1)(x+1)(x-2)}{24} \triangle^4 y_{-2} \left(\frac{4}{x-2} \cdot \frac{\triangle^3 y_{-1}}{\triangle^4 y_{-2}} \right) + E'_4 \quad \ldots.(34)$$

Hence

$$E'_3 = E'_4 \left(\frac{4}{x-2} \cdot \frac{\triangle^3 y_{-1}}{\triangle^4 y_{-2}} + 1 \right) \quad \ldots\ldots(35)$$

If the term $\left(\frac{4}{x-2} \cdot \frac{\triangle^3 y_{-1}}{\triangle^4 y_{-2}} \right)$ in (35) is negative and $| <2 |$, E'_3 will be less than E'_4. Hence the third and higher differences are to be ignored. Thus if we are interested to know whether n th differences and above should be neglected for a better interpolation of x, the truncated difference equation should have $n+2$ terms, the last term containing $(n+1)$th difference coefficient. And in

$$E'_n = E'_{4+1} \left\{ \frac{n+1}{\begin{array}{l}\text{[New factor added, when} \\ \text{compared to the } n\text{th term, to} \\ \text{to the numerator of the} \\ \text{coefficient of } (n+1)\text{th diffe-} \\ \text{rence term]}\end{array}} \times \frac{n \text{ th difference}}{(n+1)\text{th difference}} + 1 \right\} \ldots(36)$$

the first term in the bracket should be negative and $| <2 |$. When equation (11) or the advancing difference formula is used (36) becomes

$$E'_n = E'_{n+1} \left(\frac{n+1}{x-n} \cdot \frac{\triangle^n y_0}{\triangle^{n+1} y_0} + 1 \right) \quad \ldots\ldots(37)$$

where $\triangle^n y_0$ and $\triangle^{n+1} y_0$ are the nth and $(n+1)$th leading differences. The value of x to be substituted in equation (36) or (37) should be as close as possible to the required value of x.

The equation (36) is applied below to the data in Table V to verify whether the third difference interpolation is better than the interpolation to fourth differences. Since the value of x corresponding to $y_k = 45.28$ cm is expected to lie between 0 and

-1, $x=-0.5$ is substituted in the first term inside the square bracket in the equation (36) to get

$$\frac{5}{-0.5+2} \times \frac{-1.08}{0.9075} = -3.99$$

This quantity being numerically greater than 2, the interpolation is to be continued beyond third differences. This is as expected, judging from the example worked out under successive approximation.

Fitting of yield curves when natural mortality varies with age

The techniques described above for fitting the various yield curves can also be applied when growth is allometric and when natural mortality varies with age, but the computations of $\frac{Y_w}{R}$ and $l_{c\ max}$ are now more tedius. Moreover these are also subjected to slight errors in measurement.* Tabular values should be as accurate as possible if the above techniques were to yield reliable results. Hildebrand has pointed out that a polynomial approximation of a function obtained by fitting a set of tabular values exactly at all the points will have a curve that oscillates about the curve representing the true function if these values are not reliable. Besides, even when the deviation between the approximating function and the exact function is small throughout the interval, their slopes might still differ significantly. Any computation based on such an approximating function will not therefore be accurate, especially if the calculations involve the use of the derivatives of the function. Least-squares polynomial approximations are possible but these are cumbersome. However, in yield studies, it would generally be sufficient if the yield-per-recruit values for the purpose of fitting the yield curves are obtained by approximate integration instead of estimating them directly from the yield equation. If the assumption can be made that the growth in weight, like the mortality rates, is exponential within each time interval, then $\frac{Y_w}{R}$ can also be obtained with the help of Ricker's (1958*a*) yield equation which uses an exponential—average biomass, for computing the yield. The $l_{c\ max}$ values required for fitting the eumetric fishing curve can be obtained by fitting a polynomial to the yield-mesh curve, and equating its derivative to zero and solving for l_c. The $\frac{Y_w}{R}\ l_{c\ max}$ value for fitting the eumetric yield curve can be obtained directly from the polynomial fitted to the various yield-mesh curves by substituting the corresponding $l_{c\ max}$ values. If the polynomial is in the form of a

*If the growth in length follows the von Bertalanffy pattern, the general form of the simple Beverton-Holt yield equation given in Appendix I will be quite accurate for all values of the length-weight exponent although the model in this form becomes somewhat empirical. However, when the length-weight relation is anisometric and an allometric growth formula or a polynomial is to be fitted, the weight-length data should be extensive and should include all size groups, at least above the size at which recruitment occurs so as to avoid any serious errors in yield estimation (Beverton and Holt 1957, p. 280).

difference equation, the central difference formula may be preferred to obtain $\frac{Y_w}{R} l_{c\ max}$ by direct interpolation, using $l_{c\ max}$ in the transformed scale for x_k.

The advantages of approximate integration in the yield estimations are already pointed out by Krishnan Kutty (1968b). When the ordinate is a function of two variables, so long as one can be expressed as a function of the other, the values of definite integrals can be obtained by single integration. Hence, the ordinary methods of approximate integration can be applied, even when M varies with age, to obtain an accurate estimate of the yield provided that the integrand $\left(\frac{dy}{dx}\right)$ is known for as many equidistant abcissas as are necessary and it describes a smooth curve. If M is assumed to vary abruptly, as it is sometimes done, it is better to subdivide the interval of integration so that M is constant within each subdivision, and then perform the integration separately for each subinterval. Only two of the most common formulae are given below. By applying Simpson's rule of approximate integration, the total yield over the exploitable phase of the population is obtained from the relation

$$Y_w = \frac{h}{3}\left[y_0 + y_n + 2\left(y_2 + y_4 + y_6 + \ldots\ldots y_{n-2}\right) + 4\left(y_1 + y_3 + y_5 + \ldots\ldots y_{n-1}\right)\right] \quad \ldots\ldots(38)$$

where h is the unit by which the interval of integration is divided into n equal parts where n is an even number and hence has a width equal to $\frac{\lambda}{n}$ years where λ is the exploitable phase of the population and $y_0, y_1, y_2, \ldots\ldots y_n$ are the values of the integrand, FN_tW_t corresponding to $t_0 = t_p'$, $t_1 = t_p' + h$, $t_2 = t_p' + 2h, \ldots.$ $t_n = t_p' + nh$ or t_λ and N_t is the number of fish of age t, W_t is the weight of an individual fish at age t, t_p' is the age of first full exploitation and t_λ is the age at which the fish disappears from the fishery. Applying a more accurate formula known as Wedder's rule

$$Y_w = \frac{3h}{10}\Big[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + 2y_{12} + \ldots\ldots\ldots\ldots 2y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n\Big] \quad \ldots.(39)$$

The law of formation of this equation is that the successive coefficients of y_1, $y_2 \ldots. y_{n-1}$ are formed in order by the numbers in the set '5, 1, 6, 1, 5, 2', the set repeating itself until the coefficient of y_{n-1} is written out. The coefficients of y_0 and y_n are unity. One requirement in applying Wedder's rule is that the number of subdivisions n should be a multiple of six. Computation is made simpler by grouping all y_i which have the same coefficient.

Better methods than the above two have been described by Scarborough, but these are laborious. The equations (38) and (39) are quite accurate and the error in using them can be minimised by dividing the interval of integration into sufficient number of small units.

When natural mortality varies with age, and the age is also difficult to read, it may not be possible to apply Holt's version of the yield equation and one may have to use the original Beverton-Holt model. Likewise the approximate integration may only be possible if the integrand is of the form FN_tW_t, i.e., as a function in time t.

Discussion

The population models commonly employed in fisheries management are those of Beverton and Holt (1957) and Schaefer (1954). However, conservationists (mostly biologists) and the economists often disagree on the optimum level of fishing. The latter criticise the application of (a) Schaefer's model by pointing out that Schaefer's maximum yield can be obtained only at a fishing level in excess of the economically optimum level, and (b) the Beverton-Holt model by indicating that here the maximum yield is obtained only at a very high fishing intensity and that the economically optimum level of fishing cannot always be obtained from this model.

Some of the interesting properties and limitations of these two types of models have been discussed by Beverton and Holt (1956, 1957), Dickie (1962), Schaefer (1967) and Schaefer and Beverton (1963). Schaefer and Beverton (1963) are of opinion that the choice of the model often depends on the available information. The main difference between the two models is that the Schaefer's model is based on the assumption that the environment is limited and therefore the population growth is density-dependent. The assumption, on the other hand, in the simple Beverton-Holt model is that the population is largely density-independent with recruitment, growth and natural mortality remaining constant. The main limitation in Schaefer's model is that it ignores the effect of the different age composition on population growth and therefore on the yield, being concerned only with the total biomass present at any one time. The effect of mesh size on the yield cannot therefore be predicted by this model. Hence this model may be more suitable for short-lived populations where gear-selection is not an important consideration.

The assumption of Beverton and Holt that recruitment, growth and morta lity remain constant has been similarly questioned. Thus, Schaefer (1967) has indicated that the assumption of constant and density-independent growth and natural mortality rates are not realistic for many populations and that the recruitment often shows wide fluctuations. Although the growth of the fish is highly plastic (Beverton and Holt 1957), wide fluctuations are not of common occurrence in marine conditions (Beverton 1962). Much less is known regarding the variations in natural mortality although it is likely to be higher in the pelagic group such as the clupeoids. Fluctuations in recruitment are, to a large extent, due to the effect of environmental variations during the early life history stages (Gulland, 1962). Studies made by Beverton and Holt (1957), Ricker (1958*b*) and Larkin and Ricker (1964) indicate that when recruitment variations are density-independent, the best long-term average yield can still be obtained from the simple Beverton-Holt model. Beverton and Holt (1956) have also shown that over a large range of fishing mortality the population biomass changes only slightly so that over this range their model can still be applied to populations which are subjected to moderate density-dependent effects.

Recently Gulland (1968 *b*) has expressed the view that because of the assumptions involved in these two models, their predictive value with respect to most fisheries is not yet established except in rare instances like the blue whale stocks in the Antarctic which has a closely fitting yield parabola. The yellowfin tuna population also seems to be better represented by the logistic model (Schaefer 1967). Beverton's (1962) analysis of the North Sea plaice data similarly justifies the assumptions in the Beverton-Holt model at least with respect to this species. Extensive information on other populations is required to test these models. Nevertheless, Gulland's (1968 *a*, *b*) studies of (a) the relation between the catch-per-unit-effort and fishing intensity and, (b) the marginal efficiency, indicate that, for many marine fisheries, the predictions based on the Beverton-Holt model may be more realistic than those based on Schaefer's model.

If the main characteristics of the population are known and as long as the assumptions are reasonable, regulatory measures based on population models will at least be more reliable than when they are not, as the models make use of all the available informations related to the population. But Beverton and Holt (1956) and Dickie (1962) have suggested that the yield parabola, based on the logistic model, is only one in a whole family of curves resulting from the exploitation of the stock at different mesh sizes. It seems, therefore, reasonable to compare Schaefer's yield curve only with the yield-intensity curve of Beverton and Holt for the same mesh size. From the shapes of the two curves the respective yield maximum may be obtained only at different fishing levels. But in both cases, at the point where the yield is maximum, marginal yield is zero (Gulland 1968 *a*). Since maximum profit can be obtained only at the level of fishing intensity where marginal yield equals marginal cost (Gordon 1953), the argument put forward by the economists that the fishing intensity required to obtain Schaefer's maximum is not economically optimum, seems valid.

However, as suggested by Dickie (1962), the eumetric fishing curve of Beverton and Holt does not advocate fishery regulations for obtaining a unique maximum sustained yield, on the other hand it indicates the conditions of maximising the yield at each level of fishing. Thus by ignoring the uncertainties in biological conditions, the optimum fishing level normally lies along the eumetric fishing curve, because this curve gives the maximum catch-per-unit-effort at any fishing pressure. The next question is whether the eumetric fishing/yield curves can be used to obtain the economically optimum fishing conditions. Beverton and Holt (1957) have discussed the construction of the eumetric value curves based on the eumetric yield curve for arriving at the economically optimum fishing level. They conclude that generally the marginal cost of fishing and the price of the fish may be treated as constants over a large range of eumetric yield curve for a first approximation of the eumetric value-cost curves. The maximum profit level of fishing may thus be read directly from these curves or by noting the fishing intensity at which the marginal product curve cuts the marginal cost curve. If the marginal cost and price of the fish are constant the optimum fishing level can also be evaluated by successive approximation using the estimates of marginal product. The marginal product is obtained by multiplying the marginal yield by the price of the fish. The marginal yield-per-recruit may be estimated from the approximate derivatives such as the one given in Appendix I for plaice by converting it to $\dfrac{d\frac{Y_w}{R}}{d\tilde{f}}$. Finally, the marginal yield is obtained by multiplying $\dfrac{d\frac{Y_w}{R}}{d\tilde{f}}$ with $\bar{R}$ or the number of average recruits entering the fishery annually. It can be seen from Appendix I, that if the maximum profit level is estimated by equating the marginal product and marginal cost, it is much easier to use the approximate derivative than the total derivative of the eumetric yield curve. A method which is easier than equating the marginal product and marginal cost is to fit a suitable difference equation to the profit at different

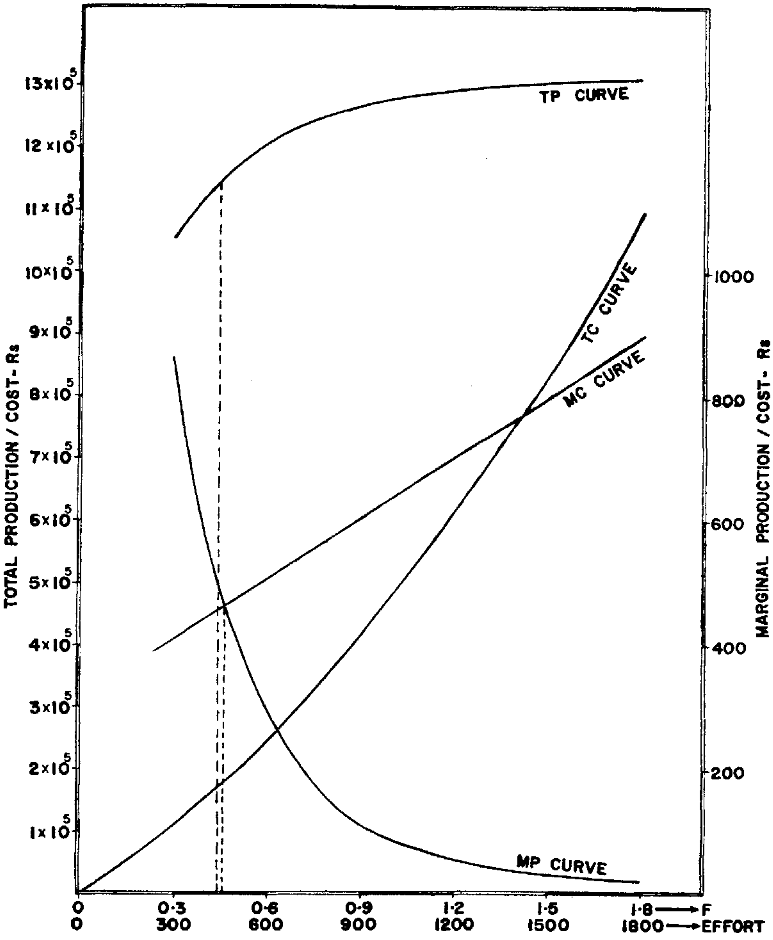

FIG. 7. Estimation of the fishing level of maximum profit for a hypothetical population. The $\frac{Yw}{R}$ given in Table III is used to obtain total production. The marginal product is evaluated from the approximate derivative of the eumetric yield curve given in Appendix I.

fishing mortalities and then equate its derivative to zero and solve for x by the Newton's method of iteration.

In Fig. 7 the total production along the eumetric yield curve and the total cost curve, with their marginal production and cost, are shown for hypothetical fishing and marketing conditions. The biological data of the North Sea plaice has been used for calculation and an annual recruitment of one million fish is assumed. The catchability is taken to be 0.001. The marginal cost is not assumed constant and is always higher than the average cost, since the total cost is estimated from the quadratic equation

$$\text{Total cost (Rs)} = 320\,\tilde{f} + 0.16\,\tilde{f}^2 \qquad \ldots.(40)$$

where f is the total number of fishing units. The price of fish is kept constant at Rs 3/- per kg. In Fig. 7 the maximum vertical distance measured between the total production (TP) and the total cost (TC) curves corresponding to the point where their tangents are parallel is at $F=0.45$. The marginal production (MP) and the marginal cost (MC) curves cut at $F=0.465$. The maximum profit level obtained by solving the derivative of the difference equation fitted to the profit data is at $F=0.48$. The estimates of maximum profit corresponding to these fishing levels are Rs. 979218, 979586 and 979488 respectively.

According to Beverton and Holt (1957) the fluctuations in the recruitment will be reflected more on the yield when the mesh size is high than when it is low. As long as changes in abundance are erratic, maximum long term average profit may still be obtained if an average estimate of the recruitment is used for calculating the total production. The analysis by Beverton (1962) and Gulland (1962) of the fluctuations in the recruitment give some support in using the best estimate of the average recruitment for making such evaluations of at least certain marine fisheries.

Scott (1962) has suggested that under certain conditions the eumertric yield curve may fail to give the conditions of optimum exploitation. According to him, the eumetric value-cost curve, based on the eumetric yield curve, does not consider all the economic conditions necessary for maximising the profit, since the production and cost curves are often functions of many variables. The production curve depends on the yield and price. Of these, yield is a function of growth, recruitment and natural and fishing mortalities. Likewise the price may also be dependent on a number of factors. The total cost may be a function of fishing effort, mesh size, etc. Maximising the profit, therefore, depends on both biological and economic conditions. The problem is thus reduced to finding the best combination of these variables which will maximise the difference between total production and total cost.

Scott also indicates that if the regulatory measures are introduced, there is often a time lag before the population settles to a new age distribution characteristic of the altered fishing intensity and/or mesh size. Hence, in a multiple age class fishery the C/U/E, which may now be less than that of the pre-regulatory level, improves only gradually to the expected level so that the cost of waiting should also be added to the total cost. Depending on the position of the pre-regulatory fishing level in the yield isopeth diagram, the C/U/E, on introducing regulation, may also increase gradually to the anticipated level without an initial decrease. Under conditions of *cacometric under-fishing* (*see* Beverton and Holt 1957) the immediate

TABLE VII

Per cent change in yield and catch-per-unit-effort (C/U/E) *for a hypothetical population using the Beverton-Holt table of yield functions. (Estimation of the immediate yield following regulation is given only for cases where regulation involves an increase in mesh size only.*

Data used : $L_\infty=69$ *cm,* $M=0.1$, $K=0.066667$, *catchability*$=0.0000526$.)

Mesh size alone is changed

	Mesh size is increased		Mesh size is decreased	
Base year	$F=0.3$	$l_c=22.08$ cm	$F=0.3$,	$l_c=55.2$ cm
	% change in yield and C/U/E			
Fully vulnerable size after regulation (l_c)	Immediate % decrease	Ultimate % increase	Fully vulnerable size after regulation (l_c)	Ultimate % increase
27.6	29.3	24.2	48.3	33.0
34.5	62.8	48.4	41.4($l_{c\ max}$)	43.2
41.4 ($l_{c\ max}$)	85.4	58.5		
Base year	$F=0.9$,	$l_c=22.08$ cm	$F=0.9$,	$l_c=55.2$ cm
27.6	73.6	43.3	48.3	27.3
34.5	97.0	92.9		
41.4	99.8	123.6	44.16 ($l_{c\ max}$)	31.2
44.16($l_{c\ max}$)	99.96	171.0		

*Mesh size is increased but F is reduced**

Base year	$F=0.3$,	$l_c=22.08$ cm		$F=0.9$,	$l_c=22.08$ cm
	When $l_c=27.6$ cm			When $l_c=27.6$ cm	
F	% change in yield	New C/U/E as a multiple of the base year	*F*	% change in yield	New C/U/E as a multiple of the base year
0.15	31.7	2.634	0.4	64.0	3.689
0.10	27.7	3.831	0.3	71.9	5.157
0.0667	16.2	5.226	0.15	82.4	10.094
0.0333	—14.8	7.678	0.0333	18.0	32.896
	when $l_c=34.5$ cm			when $l_c=34.5$ cm	
0.15	44.5	2.890	0.4	103.1	4,570
0.10	34.2	4.027	0.3	105.5	6.164
0.0818	26.8	4.651	0.15	100.1	12.004
			0.0818	75.6	19.323

**F* is reduced until the l_c becomes its $l_{c\ max}$

TABLE VII (*Contd.*)

	Both mesh size and F are reduced				
Base year	F=0.3, l_c=55.20 cm When l_c=48.3 cm			F=0.9, l_c=55.20 cm When l_c=48.3 cm	
F	% change in yield	New C/U/E as a multiple of the base year	F	% change in yield	New C/U/E as a multiple of the base year
0.15	16.1	2.322	0.3	17.8	3.535
0.10	1.7	3.050	0.15	2.8	6.169
0.0667	—15.0	3.823	0.10	—10.0	8.104
			0.0667	—24.7	10.158
	when l_c=41.4 cm			when l_c=41.4 cm	
0.15	31.0	2.619	0.30	26.08	3.804
0.10	17.7	3.531	0.15	16.0	6.959
0.0667	0.6	4.525	0.10	4.2	9.380
			0.0667	—10.9	12.023

increase in C/U/E can be even more than the ultimate increase. Such extra benefits may also be considered in maximising the profit.

In Table VII the per cent change in C/U/E after the introduction of regulation is given for a hypothetical example using the Beverton-Holt (1966) Table of yield functions (Table I, $M/K=1.5$). When mesh size is increased, there is an immediate decrease in yield and C/U/E, but the actual decrease will be less than what is shown in the Table, because following an increase in the mesh size fishing power also increases. An increase in fishing power will exert a beneficial influence on the catch-per-unit-effort, total cost and the marginal cost. If the effect of a larger mesh size is to reduce the construction and operational cost and if the price of the fish is constant, the economically optimum F and the mesh size and hence the eventual profit may also be expected to be slightly larger than that read from the eumetric fishing curve. When the minimum size of the fish is increased from 22.08 cm to 34.5 cm or more, and F is kept constant at 0.9, the immediate C/U/E almost drops down to zero, but the ultimate value is more than double of the original. A sharp increase in the C/U/E is noticed with increase in mesh size provided that the fishing level is reduced. When the mesh size is decreased, its immediate effect is to increase the C/U/E. From Fig. 4, even when the fishing intensity is high and a large meshed net is used a reduction in the fishing level sharply increases the catch-per-unit-effort. The comparison here is confined to the region bounded by the eumetric fishing curve and its complement. Since the fishing mortality tends to stabilise at a level, where the average cost=the value of the C/U/E, a reduction from this level to that of the eumetric fishing curve will result in a further increase in the catch-per-unit-effort. When both the mesh size and the fishing intensity are altered, many fold increase in C/U/E can be expected (Table VII). Hence, under certain fishing conditions, a significant increase in the C/U/E is possible from the regulations based on the eumetric yield curve.

To sum up, the eumetric value-cost curve of Beverton and Holt (1957) considers some of the important factors of production. It is also a valuable tool of prediction even when the population is moderately density-dependent. Further attempts to maximise the profit giving due consideration to all economic factors are only improvements in this basic approach. There is no approach which can ignore the biological characteristics of the population. It is often due to limitations in the data that one is forced to rely on the simplified eumetric value-cost curve.

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APPENDIX I

Although the following equations are derived from the simple Beverton-Holt model, these have been given a more general form so as to accommodate both isometric and allometric growth conditions. See also the foot note given under the section, "Fitting of yield curves when natural mortality varies with age".

(*a*) Holt's version of the Beverton-Holt yield equation. The yield per recruit,

$$\frac{Y_w}{R}=F'W_\infty\left(1-\frac{l_r}{L_\infty}\right)^{-M'}\sum_{n=0}^{d-1}(-1)^n r_n\frac{\left(1-\frac{l_c}{L_\infty}\right)^{M'+n}}{Z'+n}\left[1-\left(\frac{L_\infty-l_L}{L_\infty-l_c}\right)^{Z'+n}\right]$$

where r_n takes succesively the values of 1, $\frac{r}{1!}$, $\frac{r(r-1)}{2!}$, ...,

$\frac{r\,(r-1)\,(r-2)..............r-[(d-2)]}{(d-1)!}$, r=the exponent in the lentth-weight relation, d=number of terms obtained on expanding the Bentalanffy growth equation in weight, $F'=\frac{F}{K}$, $M'=\frac{M}{K}$, $Z'=\frac{Z}{K}$, F, M and Z are instantaneous fishing, natural and total mortality rates on yearly basis, K= rate of attaining the asymptote in Bertalanffy growth equation, W_∞= asymptotic weight, l_c= the size of first exploitation, l_L =the size at which the fish disappears from the fishery, L_∞= the asymptotic length and l_r=size at recruitment.

(*b*) The equation for the eumetric fishing curve given by Krishnan Kutty (1968, 1970) relating the size of exploitation and fishing mortality along the eumetric yield curve.

$$\left(\frac{L_\infty-l_c}{L_\infty-l_L}\right)^{z'}\sum_{n=0}^{d-1}(-1)^n r_n\frac{M'+n}{Z'+n}\left(1-\frac{l_c}{L_\infty}\right)^n=-F'\sum_{n=0}^{d-1}(-1)^n r_n\frac{\left[1-\frac{l_L}{L_\infty}\right]^n}{Z'+n}$$

(*c*) Slope of the yield intensity curve or the derivative of the yield function with respect to F'.

$$\frac{dY_w}{dF'}=W_\infty R\left[\frac{L_\infty-l_c}{L_\infty-l_x}\right]^{M'}\left(ln\left[\frac{L_\infty-l_L}{L_\infty-l_c}\right]\left\{\sum_{n=0}^{d-1}(-1)^{n+1}\right.\right.$$

$$\left.r_n\frac{F'}{Z'+n}\left[1-\frac{l_c}{L_\infty}\right]^n\left[\frac{L_\infty-l_L}{L_\infty-l_c}\right]^{z'+n}\right\}+$$

$$\left.\sum_{n=0}^{d1\,-}(-1)^n r_n\left[1-\frac{l_c}{L_\infty}\right]^n\frac{M'+n}{(Z'+n)^2}\left\{1-\left[\frac{L_\infty-l_L}{L_\infty-l_c}\right]^{z'+n}\right\}\right)$$

(*d*) Slope of the eumetric yield curve

$$\frac{dY_w\, l_{c\ max}}{dF'} = \frac{\partial Y_w}{\partial F'} + \frac{\partial Y_w}{\partial l_c} \times \frac{d\, l_c}{dF'}$$

Where $\frac{\partial Y_w}{\partial F'}$ is the slope of the yield intensity curve given above,

$$\frac{\partial Y_w}{\partial\, l_c} = \frac{W_\infty R}{L_\infty - l_c} \left[\frac{L_\infty - l_c}{L_\infty - l_r}\right]^{M'} \left[\left[\frac{L_\infty - l_L}{L_\infty - l_c}\right]^{z'} \sum_{n=0}^{d-1} (-1)^{n+1}\, r_n F' \left[1 - \frac{l_L}{L_\infty}\right]^n + \sum_{n=0}^{d-1} (-1)^{n+1}\, r_n\, \frac{F'(M'+n)}{Z'+n} \left(1 - \frac{l_c}{L_\infty}\right)^n \left\{ 1 - \left(\frac{L_\infty - l_L}{L_\infty - l_c}\right)^{z'+n} \right\} \right]$$

which is also the slope of the yield-mesh curve or the derivative of the yield equation with respect to l_c and

$$\frac{dl_c}{dF'} = \frac{\left\{ \sum_{n=0}^{d-1} (1-)^{n+1}\, r_n \frac{\left[1 - \frac{l_L}{L_\infty}\right]^n}{Z'+n} + \sum_{n=0}^{d-1} (-1)^n\, r_n \frac{F'\left[1 - \frac{l_L}{L_\infty}\right]^n}{(Z'+n)^2} \right\} - \left\{ \frac{\sum_{n=0}^{d-1} (-1)^{n+1}\, r_n \frac{(M'+n)\left[1 - \frac{l_c}{L_\infty}\right]^n}{(Z'+n)^2}}{\sum_{n=0}^{d-1} (-1)^n\, r_n \frac{(M'+n)\left[1 - \frac{l_c}{L_\infty}\right]^n}{Z'+n}} + ln\left[\frac{L_\infty - l_c}{L_\infty - l_L}\right] \right\}}{\left\{ \frac{\sum_{n=0}^{d-1} (-1)^{n+1}\, r_n \frac{n\,(M'+n)\left[1 - \frac{l_c}{L_\infty}\right]^{n-1}}{L_\infty (Z'+n)}}{\sum_{n=0}^{d-1} (-1)^n\, r_n \frac{(M'+n)\left[1 - \frac{l_c}{L_\infty}\right]^n}{Z'+n}} - \frac{F'+M'}{L_\infty - l_c} \right\}}$$

which is the slope of the eumetric fishing curve.

(*e*) Approximate derivative of the eumetric yield curve for North Sea plaice based on equation (10).

$$\frac{d\frac{Y_w}{R} l_{c\ max}}{dx} = 0.631665x^4 - 8.211468x^3 + 39.892674x^2 - 89.49955x + 86.22495$$

APPENDIX II

Table of inverse Vandermonde matrices, V_n^{-1} for fitting third to sixth degree equations. Reproduced from F. D. Parker's (1958) manuscript paper on curve fitting with the author's permission.

$$V_4^{-1}=\frac{1}{6}\begin{bmatrix} 6 & 0 & 0 & 0 \\ -11 & 18 & -9 & 2 \\ 6 & -15 & 12 & -3 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

$$V_5^{-1}=\frac{1}{24}\begin{bmatrix} 24 & 0 & 0 & 0 & 0 \\ -50 & 96 & -72 & 32 & -6 \\ 35 & -104 & 114 & -56 & 11 \\ -10 & 36 & -48 & 28 & -6 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}$$

$$V_6^{-1}=\frac{1}{120}\begin{bmatrix} 120 & 0 & 0 & 0 & 0 & 0 \\ -274 & 600 & -600 & 400 & -150 & 24 \\ 225 & -770 & 1070 & -780 & 305 & -50 \\ -85 & 355 & -590 & 490 & -205 & 35 \\ 15 & -70 & 130 & -120 & 55 & -10 \\ -1 & 5 & -10 & 10 & -5 & 1 \end{bmatrix}$$

$$V_7^{-1}=\frac{1}{720}\begin{bmatrix} 720 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1764 & 4320 & -5400 & 4800 & -2700 & 864 & -120 \\ 1624 & -6264 & 10530 & -10160 & 5940 & -1944 & 274 \\ -735 & 3480 & -6915 & 7440 & -4605 & 1560 & -225 \\ 175 & -930 & 2055 & -2420 & 1605 & -570 & 85 \\ -21 & 120 & -285 & 360 & -255 & 96 & -15 \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 \end{bmatrix}$$