COUETTE FLOW OF A DUSTY GAS BETWEEN TWO INFINITE COAXIAL CYLINDERS

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An exact solution of unsteady couette flow of an incompressible dusty gas in an annulus has been discussed. The gas is assumed to contain uniform distribution of dust particles. Initially the gas and dust particles are at rest and the flow is produced by the motion of the cylinders. The change in velocity profiles have been shown graphically when both the cylinders are moving with constant velocities.

Introduction

Saffman (1962) has formulated the basic equations for the flow of a dusty gas. Michael and Norey (1968) have discussed the flow of a dusty gas between rotating cylinders. Recently Singh (1973) has discussed the flow of a dusty gas contained in a coaxial cylinder in two cases when the cylinders were moving (i) in simple harmonic motion and (ii) with velocities which decrease exponentially, however, without using the initial condition.

In the present paper the flow of a dusty gas in the annular space between two concentric cylinders when both the cylinders are moving with arbitrary time-dependent velocities is discussed. It is assumed that initially the gas and dust particles are at rest and the flow is produced by the motion of cylinders. Using the technique of integral transforms, explicit formulae for the velocities of the gas and dust particles have been obtained. Two cases when cylinders are oscillating in their own plane and moving with constant velocities have been discussed as particular examples. The graphs have been drawn to represent change in velocity profiles of the gas and dust particles when cylinders are moving with constant velocities, which show that the profiles are pushed towards the boundary which moves at a faster rate.

Equation of Motion

The equations to represent the motion of a dusty viscous gas as given by Saffman (1962) are

$$\frac{\partial \vec{u}}{\partial T} + (\vec{u}.\nabla)\vec{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2 u + \frac{KN}{\rho}(\vec{v}-\vec{u}) \qquad ...(1)$$

$$m\left[\frac{\partial v}{\partial T} + (\vec{v}.\nabla)\vec{v}\right] = K(\vec{u}-\vec{v}) \qquad ...(2)$$

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$$\text{div}\ \vec{u} = 0 \qquad \ldots(3)$$

$$\frac{\partial N}{\partial T} + \text{div}\,(N\vec{v}) = 0, \qquad \ldots(4)$$

where $\vec{u}$ and $\vec{v}$ are velocities of fluid and dust particles respectively, p the fluid pressure, T the time, m the mass of a dust particle, N the number density of the dust particles, and K the Stokes' resistance coefficient which for spherical particles of radius r_1 is $6\pi\,\mu r_1$. μ, ν, ρ are viscosity, kinematic viscosity and density of the gas respectively.

FORMULATION AND METHOD OF SOLUTION

Let a and b be the radii of the inner and outer cylinders respectively. Taking centre of the cylinders as origin and z-axis along the axis of the cylinders, a frame of cylindrical polar system of coordinates is taken for reference (R, θ, z). Let u_R, u_θ, u_z and v_R, v_θ, v_z be the components of the velocities of the gas and dust particles in the direction of R, θ, and z respectively. For the present problem it is assumed that the pressure gradient is zero and flow is produced by the motion of the cylinders which are moving in z-direction. Since we have considered the motion of the cylinders in z-direction only, there is no displacement of the gas and particles in the direction of R and θ due to the motion of the cylinders. Thus the velocities of the gas and particles in the direction of R and θ are zero. Now if we take the number density of the dust particles to be constant, say N_o, equations (3) and (4) reduce to

$$\frac{\partial u_z}{\partial z} = 0$$

and

$$\frac{\partial v_z}{\partial z} = 0$$

which implies that u_z and v_z are independent of z. They are now functions of R and the time T only. Thus Eqs. (1) and (2) reduce to

$$\frac{\partial u_z}{\partial T} = \nu\left(\frac{\partial^2 u_z}{\partial R^2} + \frac{1}{R}\frac{\partial u_z}{\partial R}\right) + \frac{K\,N_o}{\rho}\left(v_z - u_z\right) \qquad \ldots(5)$$

and

$$\frac{\partial v_z}{\partial T} = \frac{K}{m}\left(u_z - v_z\right). \qquad \ldots(6)$$

Introducing the following non-dimensional quantities

$r = R/a,\ \bar{z} = z/a,\ t = T\nu/a^2,\ u = u_z\,a/\nu$

and

$v = v_z\,a/\nu$

Eqs. (5) and (6) reduce to

$$\frac{\partial u}{\partial t} = \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + \beta\,(v - u) \qquad \ldots(7)$$

$$\frac{\partial v}{\partial t} = \gamma'(u-v), \qquad \text{...(8)}$$

$$\text{where } \beta = \frac{f}{\gamma} = \frac{N_o\, Ka^2}{\rho\nu}, f = \frac{N_o m}{\rho}, \gamma = \frac{m\nu}{Ka^2} \text{ and } \gamma' = \frac{1}{\gamma} \qquad \text{...(9)}$$

Since the flow is produced by the motion of cylinders, let us assume that inner and outer cylinders start to move from rest with velocities $\phi(t)$ and $\psi(t)$, where $\phi(t)$ and $\psi(t)$ are functions of time.

We have to solve Eqs. (7) and (8) under the following initial and boundary conditions :

Initial condition :

$$\left.\begin{aligned} u(r,\, t) &= 0 \quad \text{for } t \leqslant 0 \\ v(r,\, t) &= 0 \quad \text{for } t \leqslant 0 \end{aligned}\right\} \qquad \text{...(10)}$$

Boundary conditions :

$$\left.\begin{aligned} u(r,\, t) &= \phi\,(t) \text{ at } r = 1,\ t>0 \\ u(r,\, t) &= \psi\,(t) \text{ at } r = \sigma,\ t>0 \end{aligned}\right\} \qquad \text{...(11)}$$

where $\sigma = b/a > 1$.

Introducing the finite Hankel transform defined by

$$u_H(t, p_i) = \int_1^\sigma u.r\ B_o\,(p_i\ r)\ dr \qquad \text{...(12)}$$

$$v_H(t, p_i) = \int_1^\sigma v.r\ B_o\ (p_i\ r)\ dr \qquad \text{...(13)}$$

where

$$B_o(p_i\ r) = J_o(p_i r)\ Y_o(p_i) - Y_o(p_i\ r)\ J_o(p_i).$$

$J_o\ (p_i\ r)$, $Y_o\ (p_i\ r)$ are Bessel functions of first and second kind of order zero and argument $p_i\ r$ and p_i is a positive root of equation.

$$J_o(p_i\ \sigma)\ Y_o(p_i) - Y_o(p_i\ \sigma)\ J_o(p_i) = 0. \qquad \text{...(14)}$$

From Eqs. (7), (8) and using (11) (Tranter 1962), we get

$$\frac{\partial u_H}{\partial t} = \frac{2}{\pi}\left[\frac{J_o\,(p_i)}{J_o\,(p_i\sigma)}\,\psi(t)-\phi(t)\right]- p_i^2\, u_H+\beta\,(v_H-u_H), \qquad \text{...(15)}$$

$$\frac{\partial v_H}{\partial t} = \gamma'\,(u_H-v_H) \qquad \text{...(16)}$$

and the condition (10) becomes

$$u_H = 0 \quad \text{at } t = 0, \quad v_H = 0 \quad \text{at } t = 0. \qquad \ldots(17)$$

We apply Laplace transform defined by

$$\bar{u}_H = \int_0^\infty u_H e^{-st}\, dt, \quad \bar{v}_H = \int_0^\alpha v_H e^{-st}\, dt,$$

under the transformed condition (17). The Eqs. (15) and (16) to

$$s\,\bar{u}_H = \frac{2}{\pi}\left[\frac{J_o(p_i)}{J_o(p_i\sigma)}\bar{\psi}(s) - \bar{\phi}(s)\right] - p_i^2\,\bar{u}_H\,\beta(\bar{v}_H - \bar{u}_H), \qquad \ldots(18)$$

and $s\,\bar{v}_H = \gamma'\,(\bar{u}_H - \bar{v}_H)$, ...(19)

where $\bar{\phi}(s)$ and $\bar{\psi}(s)$ are the Laplace transforms of $\phi(t)$ and $\psi(t)$.

Eliminating $\bar{u}_H$ and $\bar{v}_H$ between Eqs. (18) and (19), we get

$$\bar{u}_H = \frac{2}{\pi}\left[\frac{J_o(p_i)}{J_o(p_i\sigma)}\bar{\psi}(s) - \bar{\phi}(s)\right] \times \frac{s+\gamma'}{s^2 + (\gamma'+\beta+p_i^2)\,s + \gamma' p_i^2} \qquad \ldots(20)$$

$$\bar{v}_H = \frac{2}{\pi}\left[\frac{J_o(p_i)}{J_o(p_i\sigma)}\bar{\psi}(s) - \bar{\phi}(s)\right] \times \frac{\gamma'}{s^2 + (\gamma'+\beta+p_i^2)\,s + \gamma' p_i^2}, \qquad \ldots(21)$$

Inverting the Laplace transform in Eqs. (20) and (21) by well-known convolution theorem, we get

$$u_H = \frac{2}{\pi(\alpha_1 - \alpha_2)}\left[\frac{J_o(p_i)}{J_o(p_i\sigma)} F_2(p_i, t) - F_1(p_i, t)\right] \qquad \ldots(22)$$

and

$$v_H = \frac{2\gamma'}{\pi(\alpha_1 - \alpha_2)}\left[\frac{J_o(p_i)}{J_o(p_i\sigma)} G_2(p_i, t) - G_1(p_i, t)\right] \qquad \ldots(23)$$

where

$$\begin{matrix}\alpha_1 \\ \alpha_2\end{matrix} = -\frac{1}{2}\left|(\gamma'+\beta+p_i^2) \mp \sqrt{\{(\gamma'+\beta+p_i^2)^2 - 4\gamma' p_i^2\}}\right., \qquad \ldots(24)$$

$$\left.\begin{aligned} F_1(p_i, t) &= \int_o^t \phi(t-x)\left\{(\alpha_1+\gamma')\, e^{\alpha_1 x} - (\alpha_2+\gamma')\, e^{\alpha_2 x}\right\} dx \\ F_2(p_i, t) &= \int_o^t \psi(t-x)\left\{(\alpha_1+\gamma')\, e^{\alpha_1 x} - (\alpha_2+\gamma')\, e^{\alpha_2 x}\right\} dx \end{aligned}\right] \qquad \ldots(25)$$

$$\left.\begin{aligned} G_1(p_i, t) &= \int_0^t \phi(t-x)\,(e^{\alpha_1 x} - e^{\alpha_2 x})\,dx \\ G_2(p_i\, t) &= \int_0^t \psi(t-x)\,(e^{\alpha_1 x} - e^{\alpha_2 x})\,dx \end{aligned}\right] . \quad \ldots(26)$$

Finally, the inversion formula is applied for Hankel transform to Eqs. (22) and (23) (Tranter 1962) to obtain

$$u = \pi \sum_i \frac{p_i^2 J_o^2(p_i\sigma)\, B_0(p_i r)}{\left\{ J_o^2(p_i) - J_o^2(p_i\sigma) \right\}} \left[\frac{J_o(p_i)}{J_o(p_i\sigma)} F_2(p_i, t) - F_1(p_i, t) \right] \frac{1}{\alpha_1 - \alpha_2} \quad \ldots(27)$$

and

$$v = \pi \sum_i \frac{p_i^2 J_o^2(p_i\sigma)\, B_o(p_i r)}{\left\{ J_o^2(p_i) - J_o^2(p_i\sigma) \right\}} \left[\frac{J_o(p_i)}{J_o(p_i)\sigma} G_2(p_i, t) - G_1(p_i, t) \right] \frac{\gamma'}{\alpha_1 - \alpha_2} \quad \ldots(28)$$

where the summation is taken over the positive roots of Eq. (14).

Special Cases

Case (i)–Cylinders oscillating in their own planes

In this case let

$\phi(t) = \sin \delta_1 t$ and $\psi(t) = \sin \delta_2 t$

From Eqs. (25) and (26) we get

$$F_j(t, p_i) = \frac{\alpha_1 + \gamma'}{\alpha_1^2 + \delta_j^2} \left(\delta_j e^{\alpha_1 t} - (\alpha_1 \sin \delta_j t + \delta_j \cos \delta_j t) \right)$$

$$- \frac{\alpha_2 + \gamma'}{\alpha_2^2 + \delta_j^2} \left(\delta_j e^{\alpha_2 t} - (\alpha_2 \sin \delta_j t + \delta_j \cos \delta_j t) \right)$$

$$= D_j \text{ (say)}, \quad j = 1, 2 \quad \ldots(29)$$

and

$$G_j(t, p_i) = \frac{1}{\alpha_1^2 + \delta_j^2} \left(\delta_j e^{\alpha_1 t} - (\alpha_1 \sin \delta_j t + \delta_j \cos \delta_j t) \right)$$

$$- \frac{1}{\alpha_2^2 + \delta_j^2} \left(\delta_j \, e^{\alpha_2 t} - (\alpha_2 \sin \delta_j t + \delta_j \cos \delta_j t) \right)$$

$$= E_j \text{ (say)}, \quad j = 1,2. \qquad \ldots(30)$$

Substituting Eqs. (29) and (30) in Eqs. (27) and (28), we get the velocities of the gas and dust particles in this particular case which are given by

$$u = \pi \sum_i \frac{p_i^2 \; J_0^2 (p_i \sigma) \; B_0(p_i r)}{\left\{ J_0^2 (p_i) - J_0^2 (p_i \sigma) \right\}} \left[\frac{(J_0 p_i)}{J_0(p_i \sigma)} D_2 - D_1 \right], \qquad \ldots(31)$$

$$v = \pi \sum_i \frac{p_i^2 \; J_0^2 (p_i \sigma) \, B_0 (p_i r)}{\left\{ J_0^2 (p_i) - J_0^2 (p_i \sigma) \right\}} \left[\frac{J_0(p_i)}{J_0(p_i \sigma)} E_2 - E_1 \right]. \qquad \ldots(32)$$

Case (ii)–Cylinders moving with constant velocities

In this case let

$$\phi(t) = u_1 \text{ and } \psi(t) = u_2 \qquad \ldots(33)$$

where u_1 and u_2 are constants.

Substituting Eq. (33) in Eqs. (25), (26) and on simplifying we get

$$u = \pi \sum_i \frac{p_i^2 \; J_0^2 \; (p_i \sigma)}{\left[J_0^2 (p_i) - J_0^2 \; (p_i \sigma) \right]} \left[\frac{J_0 \, (p_i)}{J_0 \, (p_i \sigma)} u_2 - u_1 \right] \frac{1}{p_i^2} \Big[(\alpha_1 - \alpha_2)$$

$$+ (p_i^2 + \alpha_2) \, e^{\alpha_1 t} - (p_i^2 + \alpha_1) \, e^{\alpha_2 t} \Big] \frac{B_0 \, (p_i r)}{(\alpha_1 - \alpha_2)} \qquad \ldots(34)$$

and

$$v = \pi \sum_i \frac{p_i^2 \; J_0^2 (p_i \sigma)}{\left[J_0^2 \, (p_i) - J_0^2 (p_i \sigma) \right]} \left[\frac{J_0 \, (p_i)}{J_0 \, (p_i \sigma)} u_2 - u_1 \right] \frac{1}{p_i^2} \Big[(\alpha_1 - \alpha_2)$$

$$+ \alpha_2 e^{\alpha_1 t} - \alpha_1 \, e^{\alpha_2 t} \Big] \frac{B_0 \, (p_i r)}{(\alpha_1 - \alpha_2)} . \qquad \ldots(35)$$

It can easily be shown that

$$\pi \sum_i \frac{J_0^2(p_i \sigma)\, B_0(p_i r)}{\left[J_0^2(p_i) - J_0^2(p_i \sigma) \right]} \left[\frac{J_0(p_i)}{J_0(p_i \sigma)}\, u_2 - u_1 \right] = u_1 + (u_2 - u_1) \frac{\log r}{\log \sigma} \quad \ldots(36)$$

Substituting from Eq. (36) in Eqs. (34) and (35), we get

$$u = u_1 + (u_2 - u_1) \frac{\log r}{\log \sigma} - \pi \sum_i \frac{J_0^2(p_i\sigma) B_0(p_i r)}{\left\{ J_0^2(p_i) - J_0^2(p_i\sigma) \right\}} \left[\frac{J(p_i)}{J_0(p_i\sigma)}\, u_2 - u_1 \right] \times \left\{ \frac{\left(p_i^2 + \alpha_1\right) e^{\alpha_2 t} - \left(p_i^2 + \alpha_2\right) e^{\alpha_1 t}}{\alpha_1 - \alpha_2} \right\}, \quad \ldots(37)$$

$$v = u_1 + (u_2 - u_1) \frac{\log r}{\log \sigma} - \pi \sum_i \frac{J_0^2(p_i\sigma)\, B_o(p r_i)}{\left\{ J_0^2(p_i) - J_0^2(p_i\sigma) \right\}} \left[\frac{J_0(p_i \sigma)}{J_0(p_i\sigma)}\, u_2 - u_1 \right] \times \left\{ \frac{\alpha_1 e^{\alpha_2 t} - \alpha_2 e^{\alpha_1 t}}{\alpha_1 - \alpha_2} \right\}. \quad \ldots(38)$$

Equations (36) and (38) determine the velocities of the gas and dust particles when both cylinders are moving with constant velocities.

Discussion

Equations (27) and (28) represent the velocities of the gas and dust particles for the general case. From these expressions various particular cases can easily be obtained for several values of $\phi(t)$ and $\psi(t)$. Singh (1973) has given expressions for two particular cases. For the case when both the cylinders are moving with constant velocities the steady flow cannot be derived from his results.

The velocity profiles for the gas and dust particles are plotted in Figs. 1—4 when cylinders are moving with constant velocities. From these graphs it is observed that the profiles are pushed towards the boundary which moves at a faster rate. As the time increases since the start of the motion the velocities approach their steady state. It is evident from the figures that the gas moves faster than the dust particles. Thus in this case behaviour of the velocities is similar to the case when cylinders execute simple harmonic motion, which has been discussed by Devi Singh.

It is found that if the masses of dust particles are small their influence on gas is reduced and in the limit as $m \to 0$ the gas becomes ordinarily viscous and we get the flow of a viscous gas through the annulus.

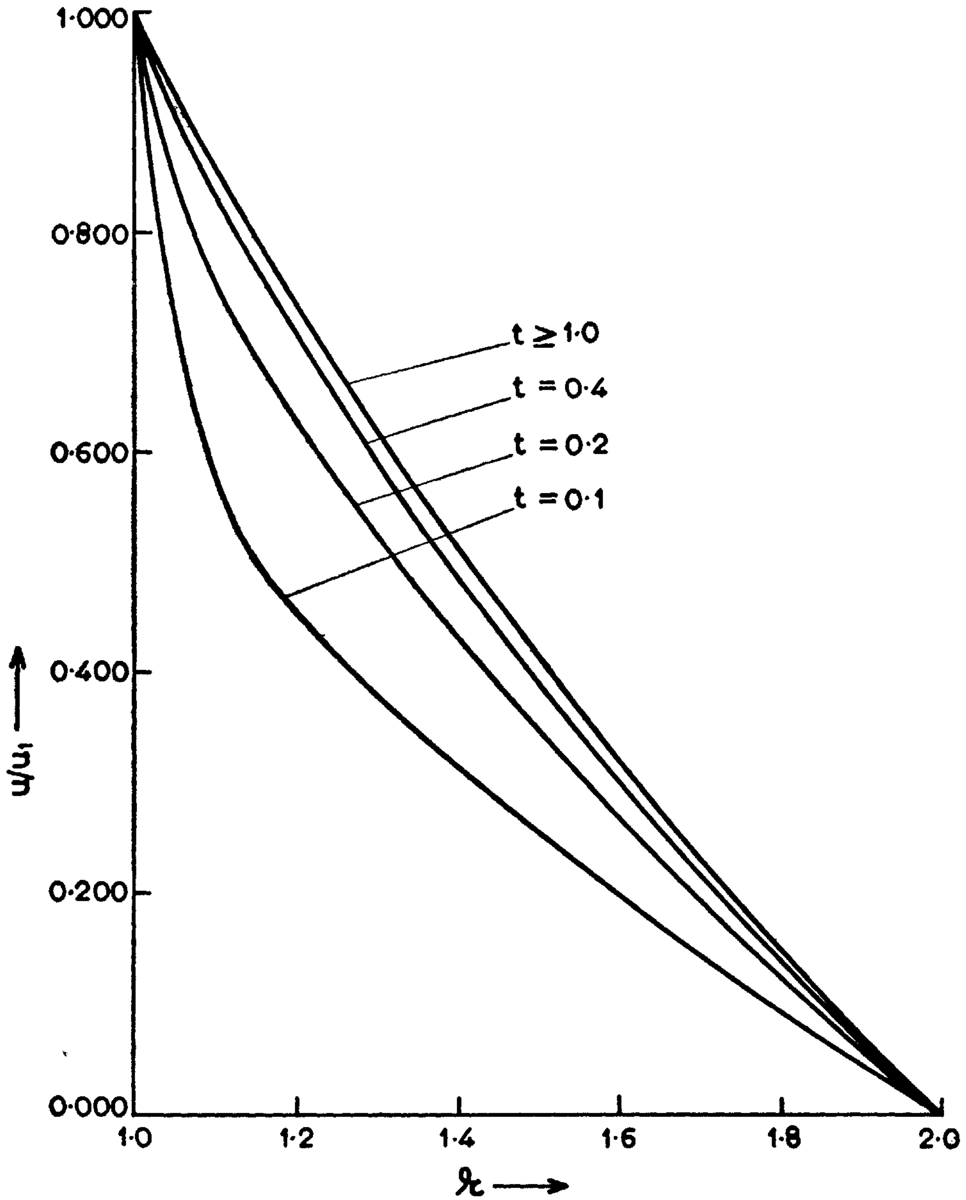

FIG. 1. Velocity profiles of gas at different times for $u_2 = 0$ ($\gamma = 0.8$, $f = 0.2$).

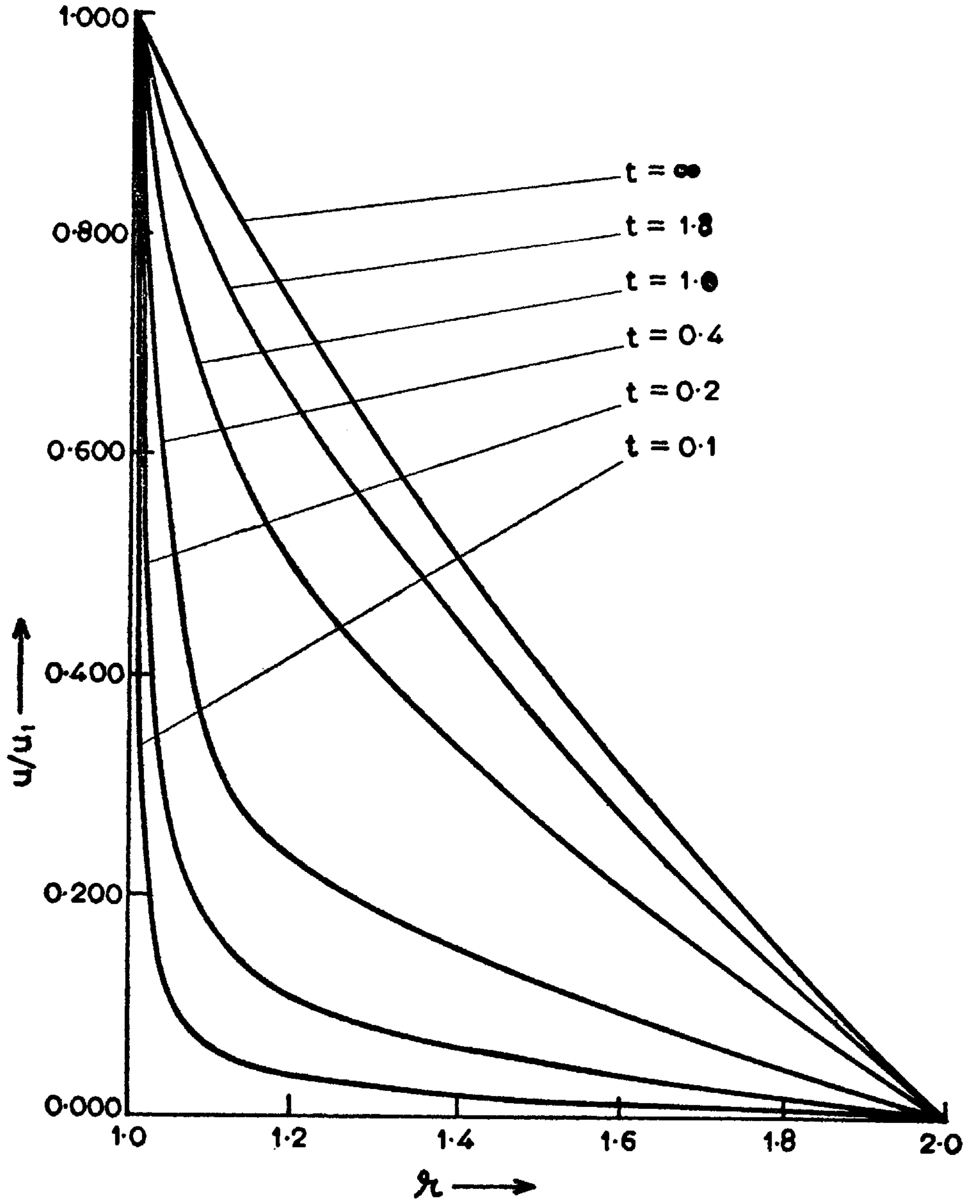

FIG. 2. Velocity profiles of dust particles at different times for $u_2 = 0$ ($\gamma = 0.8, f = 0.2$).

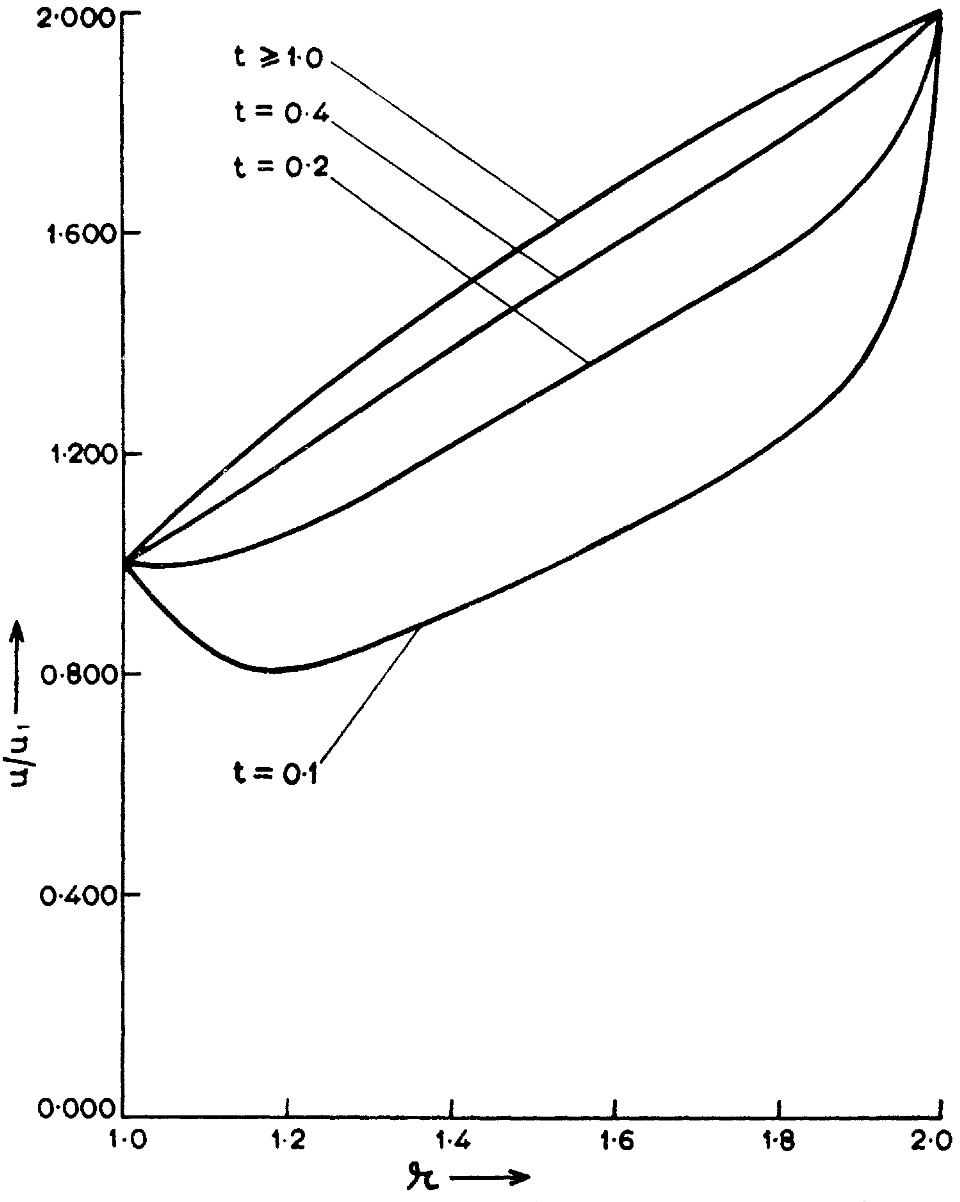

FIG. 3. Velocity profiles of gas at different times for $u_2/u_1 = 2$ ($\gamma = 0.8, f = 0.2$).

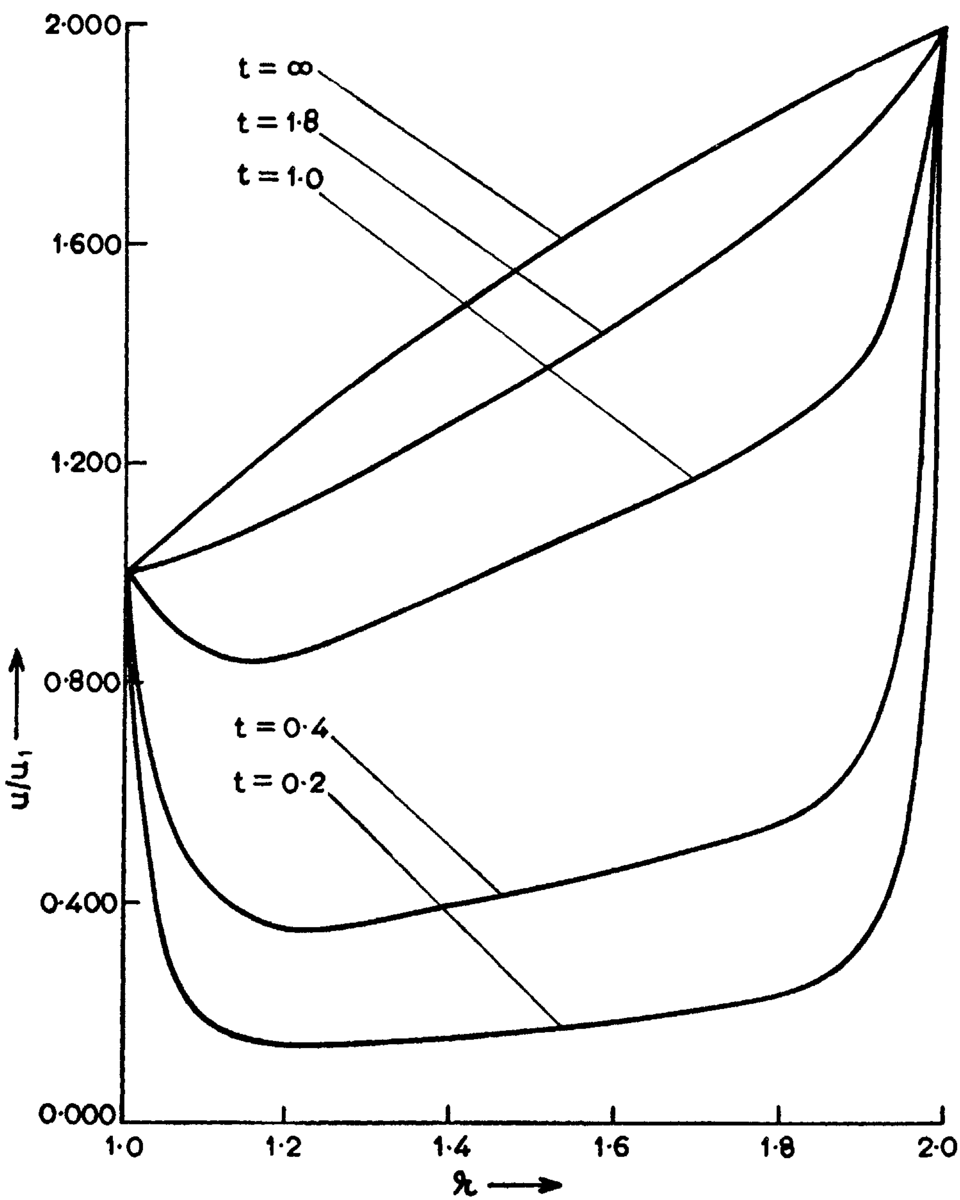

FIG. 4. Velocity profiles of dust particles at different times for $u_2/u_1 = 2$ ($\gamma = 0.8$, $f = 0.2$).

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