SLOWING-DOWN DENSITY OF NEUTRONS IN A CYLINDRICAL SHELL OF FINITE HEIGHT WITH EXACT BOUNDARY CONDITIONS IN THE AGE THEORY

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In this study, the problem of slowing-down density of neutrons for a cylindrical shell of finite height with exact boundary conditions on all the surfaces in the age theory has been solved by operational method.

INTRODUCTION

Sneddon (1951) has solved a number of problems of slowing-down density of neutrons with approximate boundary conditions (i.e., the slowing-down density is zero at the boundaries). He has also pointed out the corrections, making use of the correct boundary conditions instead of an approximate one for a semi-infinite medium bounded by vacuum.

Marshak (1947) has estimated the distance of extrapolated end-point (at which slowing-down density is zero) from the boundary as

$$\tfrac{2}{3}\, L(u)\, [1 - \langle \cos \phi \rangle]^{-1},$$

where $L(u)$ is the total mean free path, $u = \log E_0/E$ (called lethargy), E_0 is the initial energy of the neutrons, E is the energy at any successive time, $\langle \cos \phi \rangle$ is the average of the cosines of the angle of deflection produced by one collision (measured in the laboratory system).

Marchi and Zgrablich (1966) have discussed the slowing-down density of neutrons in a cylindrical shell of infinite height and having sources of neutrons in it. They have considered exact boundary conditions on the cylindrical surfaces and the age theory without capture and time variation.

In this paper the slowing-down density of neutrons for a cylindrical shell of finite height and having sources of neutrons in it under the exact boundary conditions on all the four surfaces has been solved and considered the age theory with capture and time variation.

FORMULATION OF THE PROBLEMS AND ITS SOLUTION

In this section we consider the age theory with weak capture and time variation. Considering the problem in which slowing-down medium is a cylindrical shell of finite height whose axis is coincident with z-axis defined in cylindrical polar coordinates

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by $0 \leqslant z \leqslant q$ and $a \leqslant r \leqslant b$, where a and b are the internal and external radii respectively, with symmetry with respect to z-axis, and having sources of neutrons in it. The slowing -down density $X(r, z, \theta, t)$ of neutrons at any point of the cylinder is the solution of the fundamental age differential equation of the form

$$\frac{\partial}{\partial \theta} X(r, z, \theta, t) + \frac{3}{v\, L_t(u)} \frac{\partial X}{\partial t} + \frac{3\,[1 - h(u)]X}{L(u)\, L_t(u)}$$

$$= \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial X}{\partial r}\right) + \frac{\partial^2 X}{\partial z^2} + T(r, z)\, \delta(\theta)\, \delta(t), \quad \ldots (1)$$

where $X(r, z, \theta, t)$ represents the number of neutrons per unit volume per unit time which reaches the age θ, the "symbolic age" θ is defined by

$$\theta = \tfrac{1}{3} \int_o^u \frac{L^2(u')\, du'}{[1 - <\cos \phi>]},$$

the function $T(r, z)\, \delta(\theta)\, \delta(t)$ is related to the source function $S(r, z, u)\, \delta(t)$ by the equation $T(r, z)\, \delta(\theta)\, \delta(t) = 4\pi\, S(r, z)\, \delta(u)\, \delta(t)\, \partial u/\partial\theta$, where $\delta(\theta)$, $\delta(t)$ denotes the Dirac-delta function of age θ, and time t respectively, and

$$L_t(u) = \frac{L_s(u)}{[1 - <\cos \phi>]},$$

where $L_s(u)$ is the mean free path for scattering.

Also $h(u) = \dfrac{L(u)}{L_s(v)}$ and $v = |V|$,

where V is the velocity of neutrons when its energy is E.

Making the substitution (Sneddon 1951, p. 238)

$$X(r, z, \theta, t) = X(r, z, \theta)\; F_1(h)\; F_2(t), \quad \ldots (2)$$

where $F_1(h)$ and $F_2(t)$ are chosen to be of the form

$$F_1(h) = \exp\left[-\int_o^u \left[1 - h(u')\right] \frac{du'}{\xi}\right] \quad \ldots (3)$$

$$F_2(t) = \delta\left[t - \int_o^u \frac{L(u')\, du'}{v\xi}\right], \quad \ldots (4)$$

the Eq. (1) reduces to,

$$\frac{\partial}{\partial \theta} X(r, z, \theta) = \frac{\partial^2 X}{\partial r^2} + \frac{1}{r} \frac{\partial X}{\partial r} + \frac{\partial^2 X}{\partial z^2} + T(r, z)\, \delta(\theta) \quad \ldots (5)$$

The exact boundary conditions are

$$\left[\frac{\partial}{\partial r} X(r, z, \theta) - k_1 X(r, z, \theta)\right]_{r=a} = 0, \text{ for all } z \text{ and all } \theta, \quad \ldots \quad (6)$$

$$\left[\frac{\partial}{\partial r} X(r, z, \theta) + k_1 X(r, z, \theta)\right]_{r=b} = 0, \text{ for all } z \text{ and all } \theta, \quad \ldots \quad (7)$$

$$\left[\frac{\partial}{\partial r} X(r, z, \theta) - k_2 X(r, z, \theta)\right]_{z=0} = 0, \text{ for all } r \text{ and all } \theta, \quad \ldots \quad (8)$$

$$\left[\frac{\partial}{\partial r} X(r, z, \theta) + k_2 X(r, z, \theta)\right]_{z=q} = 0, \text{ for all } r \text{ and all } \theta, \quad \ldots \quad (9)$$

where

$$\frac{1}{k_1} = \frac{2}{3} \frac{L(u_1)}{[1 - <\cos\phi>]}, \quad \frac{1}{k_2} = \frac{2}{3} \frac{L(u_2)}{[1 - <\cos\phi>]} \quad \ldots \quad (10)$$

are the distances between the extrapolated end points and the boundaries. $L(u_1)$ and $L(u_2)$ are the total mean free paths, $u_1 = \log E_0 / E_1$ and $u_2 = \log E_0/E_2$ (called lethargies), E_0 is the initial energy of the neutrons, E_1 and E_2 are the energies at any two successive time.

Cinelli (1965) has defined finite Hankel transform as

$$\bar{f}(\lambda_i) = \int_a^b r f(r)\, C_m(r, \lambda_i)\, dr, \; a < r < b \quad \ldots \quad (11)$$

where

$$C_m(r, \lambda_i) = J_m(\lambda_i r)\,[\lambda_i Y'_m(\lambda_i a) + h_1 Y_m(\lambda_i a)]$$
$$= Y_m(\lambda_i r)\,[\lambda_i J'. ((\lambda_i a) + h_1 J_m(\lambda_i a)], \quad \ldots \quad (12)$$

$J_m(\lambda_i r)$ and $Y_m(\lambda_i r)$ are Bessel functions of the first kind and second kind respectively and of order m, and λ_i is a root of the equation

$$[\lambda_i Y'_m(\lambda_i a) + h_1 Y_m(\lambda_i a)]\,[\lambda_i J'_m(\lambda_i b) + h_2 J_m(\lambda_i b)]$$
$$= [\lambda_i Y'_m(\lambda_i b) + h_2 Y_m(\lambda_i b)]\,[\lambda_i J'_m(\lambda_i a) + h_1 J_m(\lambda_i a)] \quad \ldots \quad (13)$$

Inversion theorem of (11) is

$$f(r) = \frac{\pi^2}{2} \Sigma_{\lambda_i} \lambda_i^2 \left[\lambda_i J'_m\left(\lambda_i b\right) + h_2 J_m\left(\lambda_i b\right)\right]^2 \bar{f}(\lambda_i) \frac{C_m(r, \lambda_i)}{F_m(\lambda_i)} \quad \ldots \quad (14)$$

where

$$F_m(\lambda_i) = \left[h_2^2 + \lambda_i^2 \left\{1 - \left(\frac{m}{\lambda_i b}\right)^2\right\}\right] \left[\lambda_i J'^m\left(\lambda_k a\right) + h_1 J_m\left(\lambda_i a\right)\right]^2$$

$$-\left[h_1^2 + \lambda_i^2 \left\{1 - \left(\frac{m}{\lambda_i a}\right)^2\right\}\right]\left[\lambda_i J'_m\left(\lambda_i b\right) + h_2 J_m\left(\lambda_i b\right)\right]^2 \quad \ldots \quad (15)$$

and the summation is taken over the positive roots of the equation (13).

The operational property of (11) is

$$\int_a^b r\left[\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{m^2}{r^2} f\right] C_m(r, \lambda_i)\, dr$$

$$= \frac{2}{\pi}\left[\alpha\, [f'(b) + h_2 f(b)] - [f'(a) + h_1 f(a)]\right] - \lambda_i^2 \bar{f}(\lambda_i) \quad \ldots \quad (16)$$

where

$$\alpha = \frac{[\lambda_i J'_m(\lambda_i a) + h_1 J_m(\lambda_i a)]}{[\lambda_i J'_m(\lambda_i b) + h_2 J_m(\lambda_i b)]}\,. \quad \ldots \quad (17)$$

Substituting

$$h_1 = -k_1,\ h_2 = k_2 \text{ and } m = 0, \quad \ldots \quad (18)$$

and applying Eq. (11) to the Eq. (5) for the variable r and using (6), (7) and (16) we obtain

$$\frac{\partial}{\partial\theta}\bar{X}(\lambda_i, z, \theta) + \lambda_i^2\ \bar{X}(\lambda_i, z, \theta) = \frac{\partial^2}{\partial z^2} X(\lambda_i, z, \theta) + \bar{T}(\lambda_i\, z,)\, \delta(\theta), \quad \ldots \quad (19)$$

where

$$\bar{T}(\lambda_i, z) = \int_a^b r\, T(r, z) C_0(r, \lambda_i)\, dr. \quad \ldots \quad (20)$$

Kaplan and Sonnemann (1959) have defined the finite transform as

$$\bar{B} \equiv \bar{B}(r, \theta, \alpha_n) = \int_0^q B(r,\theta,z)\left[\sin \alpha_n z - \frac{\alpha_n}{h_3} \cos \alpha_n z\right] dz. \quad \ldots \quad (21)$$

Inversion theorem of (21) is

$$B(r, \theta, z) = 2\Sigma_{\alpha_n} \frac{\bar{B}(r,\theta,\alpha_n)}{N_n}\left[\sin \alpha_n z - \frac{\alpha_n}{h_3} \cos \alpha_n z\right] \quad \ldots \quad (22)$$

where

$$N_n = q\left[1 + \frac{\alpha_n}{h_3}\right] - \frac{1}{\alpha_n} \sin \alpha_n q\left[\frac{\alpha_n}{h_3} + 1\right] \times$$

$$\left[\cos \alpha_n q + \frac{\alpha_n}{h_3} \sin \alpha_n q \right] \quad \dots \quad (23)$$

and the summation is taken over the positive roots of the equation

$$\frac{\alpha_n}{h_3} \left[\alpha_n \tan \alpha_n q - h_4 \right] + \left[\alpha_n + h_4 \tan \alpha_n q \right] = 0 \quad \dots \quad (24)$$

Operational property of (21) is

$$\bar{B} \left[\frac{\partial^2 B}{\partial z^2} \right] = \frac{\alpha_n}{h_3} \left[\frac{\partial B}{\partial z} + h_3 B \right]_{z=0} + \left[\sin \alpha_n q - \frac{\alpha_n}{h_3} \cos \alpha_n q \right] \times$$

$$\left[\frac{\partial B}{\partial z} + h_4 B \right]_{z=q} - \alpha_n^2 \bar{B}(r,\theta,\alpha_n) \quad \dots \quad (25)$$

Substituting

$$\frac{1}{h_3} = - k_2 . h_4 = k_2, \quad \dots \quad (26)$$

and applying (21) to the equation (19) for the variable z and using (8), (9) and (25) we obtain

$$\frac{\partial}{\partial \theta} \bar{X} (\lambda_i,\alpha_n,\theta) + p \, \bar{\bar{X}} (\lambda_i,\alpha_n,\theta) \, \bar{\bar{T}} (\lambda_i,\alpha_n) \; \delta(\theta), \quad \dots \quad (27)$$

where

$$p = \lambda_i^2 + \alpha_n^2 . \quad \dots \quad (28)$$

and

$$\bar{\bar{T}} (\lambda_i, \alpha_n) = \int_0^q \bar{T} (\lambda_i, z) \left[\sin \alpha_n z + k_2 \alpha_n \cos \alpha_n z \right] dz$$

$$= \int_a^b \int_0^q r \, T (r, z) \, C_o (r, \lambda_i) \times$$

$$\left[\sin \alpha_n z + k_2 \alpha_n \cos \alpha_n z \right] dr \, dz. \quad \dots \quad (29)$$

Equation (27) is an ordinary differential equation, and its solution is

$$\bar{\bar{X}} (\lambda_i, \alpha_n, \theta) = \bar{\bar{T}} (\lambda_i, \alpha_n) \, e^{-p\theta}. \quad \dots \quad (30)$$

Using inversion theorem (14) for $m = 0$, and (22) the slowing down density of neutrons is obtained as

$$X(r, z, \theta) = \pi^2 \Sigma_{\lambda_i} \Sigma_{\alpha_n} \lambda_i^2 \Big[\lambda_i J_0' (\lambda_i b) + k_1 J_o (\lambda_i b) \Big]^2 \times$$

$$\frac{C_0(r, \lambda_i)}{F_0 (\lambda_i) N_n} \Big[\sin \alpha_n z + k_2 \alpha_n \cos \alpha_n z \Big] e^{-\nu\theta},$$

$$\int_a^b \int_0^q r\, T(r, z)\, C_0 (r, \lambda_i) [\sin \alpha_n z + k_2 \alpha_n \cos \alpha_n z]\, dr\, dz \qquad \ldots (31)$$

Using (31) and (2), we obtain the slowing-down density of neutrons $X(r, z, \theta, t)$ as

$$X(r,z,\theta,t) = \pi^2 F_1(h) F_2(t)\, \Sigma_{\lambda_i} \Sigma_{\alpha_n} \lambda_i^2 \Big[\lambda_i J_0' \Big(\lambda_i b \Big) + k_1 J_0 \Big(\lambda_i b \Big) \Big]^2 \times$$

$$\frac{C_o (r, \lambda_i)}{F_o(\lambda_i) N_n} \Big[\sin \alpha_n z + k_2 \alpha_n \cos \alpha_n z \Big] e^{-\nu\theta} \int_a^b \int_0^q r\, T(r, z)\, C_o (r, \lambda_i) \times$$

$$\Big[\sin \alpha_n z + k_2 \alpha_n \cos \alpha_n z \Big]\, dr\, dz, \qquad \ldots (32)$$

where the summation over the subscripts λ_i and α_n are taken over all the positive roots of the Eq. (13) with $m = 0$, $h_1 = - k_1$, $h_2 = k_1$ and (24) with $1/h_3 = - k_2$, $h_4 = k_2$ respectively, and N_n is the value of (23) with $1/h_3 = - k_2$.

Special Cases

(*i*) If

$$k_2 = 0,$$
$$q \to \infty,$$

we get the boundary value problem of a type similar to that considered recently by Marchi and Zgrablich (1966) for the exact boundary conditions on cylindrical surfaces of an infinitely long cylindrical shell using a new finite Hankel transform defined by them (Marchi & Zgrablich 1964).

(*ii*) If

$$a = 0,$$

we get the boundary value problem for a solid cylinder using finite Hankel transform defined by Sneddon (1951, p. 89).

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