THEORETICAL CONSIDERATION OF SLOWING-DOWN DENSITY OF NEUTRONS IN A SPHERICAL SYMMETRY

by B. S. MEHTA, *Department of Mathematics, Government College, Shahpura, (Rajasthan)*

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In this paper the problem of slowing-down density of neutrons in a spherical symmetry with different sources of neutrons in it with exact boundary conditions in the age theory has been solved. Solutions for the approximate boundary conditions have been derived as special cases.

INTRODUCTION

Slowing-down density of neutrons with approximate boundary conditions (i.e., the slowing-down density is zero at the boundaries) has been discussed by Sneddon (1951) for different symmetries and sources. Marshak (1947) has estimated the distance of extrapolated end-point from the boundary. The slowing-down density of neutrons in an infinite cylindrical shell with exact boundary conditions on the cylindrical surfaces has been discussed by Marchi and Zgrablich (1966).

In Part A of this paper we present some results of slowing-down density in a spherical shell ($a<r<b$) with sources of neutrons in it with the exact boundary conditions in the age theory . In Part B, results for the solid sphere have been investigated under different boundary conditions. In both the parts, results for approximate boundary conditions are deduced by changing the parameters.

PART 'A'

FORMULATION OF THE PROBLEM

The fundamental age of differential equation of slowing-down density of neutrons for the present problem is

$$\frac{\partial}{\partial \theta} X(r, \theta) = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial X}{\partial r}\right) + S(r)\, \delta(\theta), \qquad \ldots(1)$$

where $X(r, \theta)$ represents the number of neutrons per unit volume per unit time which reaches the age θ, $S(r)\,\delta(\theta)$ is prescribed by the known sources of neutrons in the material, $\delta(\theta)$ is the Dirac-delta function of θ, and $S(r)$ is the function of r only.

Boundary conditions are

$$\left[\frac{\partial X}{\partial r}(r, \theta) - kX(r, \theta)\right]_{r=a} = 0, \text{ for all } \theta \qquad \ldots(2)$$

$$\left[\frac{\partial X}{\partial r}(r, \theta) + kX(r, \theta)\right]_{r=b} = 0, \text{ for all } \theta \qquad \ldots(3)$$

and

$$\frac{1}{k} = \frac{\frac{2}{3} L(u)}{[1 - < \cos \phi >]} \qquad \ldots(4)$$

is the distance between extrapolated end-point and the boundary, $L(u)$ is the total mean free path, $u = \log \frac{E_0}{E}$ (called lethargy), E_0 is the initial energy of the neutrons, E is the energy at any successive time, and $< \cos \phi >$ is the average of the cosines of the deflection produced by one collision (measured in the laboratory system).

Making the substitution

$$X(r, \theta) = r^{-1/2} Z(r, \theta), \qquad \ldots(5)$$

Eqs. (1) to (3) take the form

$$\frac{\partial}{\partial \theta} Z(r, \theta) = \frac{\partial^2 Z}{\partial r^2} + \frac{1}{r}\frac{\partial Z}{\partial r} - \frac{1}{4r^2} Z(r, \theta) + S_0(r)\,\delta(\theta), \qquad \ldots(6)$$

$$\left[\frac{\partial Z}{\partial r}(r, \theta) - \alpha_1 Z(r, \theta)\right]_{r=a} = 0, \qquad \ldots(7)$$

$$\left[\frac{\partial Z}{\partial r}(r, \theta) + \alpha_2 Z(r, \theta)\right]_{r=b} = 0, \qquad \ldots(8)$$

where

$$\alpha_1 = k + \frac{1}{2a}, \; \alpha_2 = k - \frac{1}{2b}, \text{ and } S_0(r) = r^{1/2} S(r) \text{ respectively.}$$

SOLUTION

Cinelli (1965) has defined finite Hankel transform as

$$\bar{f}(\lambda_i) = \int_a^b r\, f(r)\, C_m(r, \lambda_i)\, dr, \; a \leqslant r \leqslant b \qquad \ldots(9)$$

where

$$C_m(r, \lambda_i) = J_m(\lambda_i r)\,[\lambda_i Y'_m(\lambda_i a) + h_1 Y_m(Y_i a)] - Y_m(\lambda_i r)\,[\lambda_i J'_m(\lambda_i a) + h_1 J_m(\lambda_i a)]. \qquad \ldots(10)$$

$J_m(\lambda_i r)$ and $Y_m(\lambda_i r)$ are Bessel functions of the first and second kind respectively and of order m and λ_i is a positive root of the equation

$$[\lambda_i Y'_m(\lambda_i a) + h_1 Y_m(\lambda_i a)]\,[\lambda_i J'_m(\lambda_i b) + h_2 J_m(\lambda_i b)] = [\lambda_i Y'_m(\lambda_i b) + h_2 Y_m(\lambda_i b)]\,[\lambda_i J'_m(\lambda_i a) + h_1 J_m(\lambda_i a)] \qquad \ldots(11)$$

Inversion series of Eq. (9) is

$$f(r) = \frac{\pi^2}{2} \sum_{\lambda_1} \lambda_i^2 \, [\lambda_i J_m'(\lambda_i b) + h_2 J_m(\lambda_i b)]^2 \, \bar{f}(\lambda_i) \, \frac{C_m(r, \lambda_i)}{F_m(\lambda_i)}, \qquad \ldots(12)$$

where

$$F_m(\lambda_i) = \left[h_2^2 + \lambda_i^2 \left\{ 1 - \left(\frac{m}{\lambda_i b} \right)^2 \right\} \right] [\lambda_i J_m'(\lambda_i a) + h_1 J_m(\lambda_i a)]^2$$

$$- \left[h_1^2 + \lambda_i^2 \left\{ 1 - \left(\frac{m}{\lambda_i a} \right)^2 \right\} \right] [\lambda_i J_m'(\lambda_i b) + h_2 J_m(\lambda_i b)]^2 \qquad \ldots(13)$$

and the summation is taken over the positive roots of the Eq. (11).

The operational property of Eq. (9) is

$$\int_a^b r \left[\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} - \frac{m^2}{r^2} f \right] C_m(r, \lambda_i) \, dr$$

$$= \frac{2}{\pi} \left[\alpha \, [f'(b) + h_2 f(b)] - [f'(a) + h_1 f(a)] \right] - \lambda_i^2 \bar{f}(\lambda_i), \qquad \ldots(14)$$

where

$$\alpha = \frac{[\lambda_i J'_m(\lambda_i a) + h_1 J_m(\lambda_i a)]}{[\lambda_i J_m'(\lambda_i b) + h_2 J_m(\lambda_i b)]}. \qquad \ldots(15)$$

Substituting

$$h_1 = -\alpha_1, \; h_2 = \alpha_2 \text{ and } m = 1/2 \qquad \ldots(16)$$

and using Eqs. (9), (14), (7) and (8), Eq. (6) takes the form

$$\frac{\partial}{\partial \theta} \bar{Z}(\lambda_i, \theta) + \lambda_i^2 \bar{Z}(\lambda_i, \theta) = S_0(\lambda_i) \, \delta(\theta), \qquad \ldots(17)$$

where

$$\bar{S}_0(\lambda_i) = \int_a^b r^{3/2} S(r) \, C_{1/2}(r, \lambda_i) \, dr. \qquad \ldots(18)$$

Eq. (17) is an ordinary differential equation, its solution is

$$\bar{Z}(\lambda_i, \theta) = \bar{S}_0(\lambda_i) \, e^{-\lambda_i^2 \theta}. \qquad (19)\ldots$$

Inverting Eq. (19) by Eq. (12) we obtain,

$$X(r, \theta) = \frac{\pi^2}{2r^{1/2}} \sum_{\lambda_1} \lambda_i^2 \, [\lambda_i J'_{1/2}(\lambda_i b) + \alpha_2 J_{1/2}(\lambda_i b)]^2$$

$$\times \frac{C_{1/2}(r, \lambda_i)}{F_{1/2}(\lambda_i)} \, e^{-\lambda_i^2 \theta} \int_a^b r^{3/2} S(r) \, C_{1/2}(r, \lambda_i) \, dr \qquad \ldots(20)$$

and the summation is taken over the position roots of the Eq. (11) with $h_1 = -\alpha_1$, $h_2 = \alpha_2$, and $m = 1/2$.

Special Cases

(*i*) *Point Source at the Centre of the Shell*— Since the actual source is not in the medium, we determine the effective sources, which are given by

$$\frac{S_0\,\delta\,(r-a)}{4\,\pi\,a^2},$$

if $S_0\,\delta\,(r)$ is the source at the centre of the shell The problem in this case reduces to solve the age differential equation

$$\frac{\partial}{\partial\theta}\,Z(r,\theta) = \frac{\partial^2 Z}{\partial r^2} + \frac{1}{r}\frac{\partial Z}{\partial r} - \frac{1}{4r^2}Z(r,\theta) + \frac{r^{1/2}\,S_0\,\delta\,(r-a)}{4\pi\,a^2}\,\delta(\theta) \qquad ...(21)$$

with the boundary conditions

$$\frac{\partial Z}{\partial r}(r,\theta)\;\Big|_{r=a} = 0 \qquad ...(22)$$

$$\left[\frac{\partial Z}{\partial r}(r,\theta) + \alpha_2\,Z(r,\theta)\right]_{r=b} = 0. \qquad ...(23)$$

If

$$h_1 = -\alpha_1 = 0,\ h_2 = \alpha_2 \text{ and } m = 1/2$$

the solution for this case becomes

$$X(r,\theta) = \frac{\pi\,S_0}{8(ra)^{1/2}}\Sigma_{\lambda_i}\lambda_i^3\Big[\lambda_i\,J'_{1/2}(\lambda_i\,b) + \alpha_2\,J_{1/2}(\lambda_i\,b)\Big]^2\frac{C_{1/2}(r,\lambda_i)}{F_{1/2}(\lambda_i)}$$

$$\times\Big[J_{1/2}(\lambda_i\,a)\,Y'_{1/2}(\lambda_i\,a) - Y_{1/2}(\lambda_i\,a)\,J'_{1/2}(\lambda_i\,a)\Big]e^{-\lambda_i^2\,\theta}, \qquad ...(24)$$

where the values of Eqs. (10), (12), (13) and (14) are to be taken for $h_1 = -\alpha_1 = 0$, $h_2 = \alpha_2$ and $m = 1/2$ and the summation is taken over the positive roots of the equation

$$Y'_{1/2}(\lambda_i\,a)\,[\lambda_i\,J'_{1/2}(\lambda_i\,b) + \alpha_2\,J_{1/2}(\lambda_i\,b)] = J'_{1/2}(\lambda_i\,a)\,[\lambda_i\,Y'_{1/2}(\lambda_i\,b) + \alpha_2\,Y_{1/2}(\lambda_i\,b)] \qquad ...(25)$$

(*ii*) *Cavity in an Infinite Medium : Point Source at the Centre of the Cavity* — The case of the cavity in an infinite medium is obtained by letting the outer radius go to infinity. For this case we obtain the effective sources over the surface of the cavity as in case (*i*). The boundary conditions are that $Z(r,\theta)$ tends to zero as $r \to \infty$ and that $\frac{\partial Z}{\partial r}(r,\theta) = 0$, when $r = a$, then we get the boundary value problem considered by Sneddon (1951, p. 229).

(*iii*) If

$h_1 = -\alpha_1 \to \infty$, $h_2 = \alpha_2 \to \infty$ and $m = 1/2$,

we obtain the boundary value problem under the boundary conditions, that the density of neutrons tends to zero as r tends to a and r tends to b, which can be solved by using finite Hankel transform considered by Sneddon (1951, p. 85).

(*iv*) If

$h_1 = -\alpha_1 = 0$, $h_2 = \alpha_2 = 0$ and $m = 1/2$,

we obtain the result under the boundary conditions, that the flux of the neutrons density tends to zero as r tends to a and r tends to b.

PART 'B'

Slowing-Down Density of Neutrons in a Sphere of Radius b

The solid sphere is obtained by letting the inner radius of the shell considered in Part A approach zero. The slowing-down desnity of neutrons $X(r, \theta)$, satisfies the Eq. (1) and the physical condition in this case is given by Eq. (3).

In this case we use the transform given by Sneddon (1951, p. 83) as

$$\bar{f}\ (\lambda_i) = \int_0^b rf\,(r)\, J_m\,(r\,\lambda_i)\; dr,\ 0 \leq r \leq b,\ m > -1/2, \qquad \ldots(26)$$

where $f(r)$ is continuous function and satisfies Dirichlet's conditions in $0 \leq r \leq b$ and λ_i is a root of the equation

$$J_m\,(\lambda_i\, b) = -\frac{\lambda_i\, J'_m\,(\lambda_i\, b)}{h_2}. \qquad \ldots(27)$$

Inversion series of Eq. (26) is

$$f\,(r) = \frac{2}{b^2} \sum_{\lambda_i} \frac{\bar{f}(\lambda_i)}{\left[1 + \frac{1}{h^2_2}\left(\lambda^2_i - \frac{m^2}{b^2}\right)\right]} \frac{J_m\,(r\,\lambda_i)}{[J'_m\,(b\,\lambda_i)]^2} \qquad \ldots(28)$$

and the sum is taken over the positive roots of Eq.. (27).

The operational property of Eq. (26) is

$$\int_0^b r\left[\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{m^2}{r^2} f\right] J_m\,(r\,\lambda_i)\; dr$$

$$= bJ_m\,(\lambda_i\, b)\,[f'\,(b) + h_2 f\,(b)] - \lambda_i^2\,\bar{f}\,(\lambda_i) \qquad \ldots(29)$$

Solutions

If

$$h_2 = \alpha_2 \text{ and } m = 1/2 \qquad \ldots(30)$$

and use is made of Eqs. (26), (29) and (8), Eq. (6) takes the form

$$\frac{\partial}{\partial\theta}\,\bar{Z}\,(\lambda_i, \theta) + \lambda_i^2\,\bar{Z}(\lambda_i, \theta) = \bar{S}_0\,(\lambda_i)\;\delta\;(\theta), \qquad \ldots(31)$$

where

$$\bar{S}_0(\lambda_i) = \int_0^b r^{3/2} S(r) J_{1/2}(r\lambda_i)\, dr \qquad \ldots(32)$$

Solution of the ordinary differential equation (31) is

$$\bar{\bar{Z}}(\lambda_i, \theta) = \bar{\bar{S}}(\lambda_i)\, e^{-\lambda_i^2 \theta}. \qquad \ldots(33)$$

Inverting Eq. (33) by Eq. (28) we obtain

$$\chi(r, \theta) = \frac{2}{b^2 r^{1/2}} \sum_{\lambda_i} \frac{J_{1/2}(r\lambda_i)}{\left[1 + \frac{1}{\alpha_2^2}\left(\lambda^2{}_i - \frac{1}{4b^2}\right)\right]} \frac{1}{\left[J'_{1/2}(b\lambda_i)\right]^2}$$

$$\times e^{-\lambda^2{}_i \theta} \int_0^b r^{3/2} S(r) J_{1/2}(r\lambda_i)\, dr, \qquad \ldots(34)$$

where the sum is taken over the positive roots of Eq. (27) with $h_2 = \alpha_2$.

Special Cases

(*i*) If

$h_2 = \alpha_2 \to \infty$ and $m = 1/2$,

the solution for the case, that the slowing-down density of neutrons tends to zero as r tends to b is

$$X(r, \theta) = \frac{2}{b^2 r^{1/2}} \sum_{\lambda_i} \frac{J_{1/2}(r\lambda_i)}{[J'_{1/2}(b\lambda_i)]^2} e^{-\lambda^2{}_i \theta}$$

$$\times \int_0^b r^{3/2} S(r) J_{1/2}(r\lambda_i)\, dr, \qquad \ldots(35)$$

where the sum is taken over the positive roots of the equation

$$J_{1/2}(\lambda_i b) = 0 \qquad \ldots(36)$$

(*ii*) If

$h_2 = \alpha_2 = 0$, and $m = 1/2$,

the solution for the case that the flux of neutrons density tends to zero as r tends to be is

$$X(r, \theta) = \frac{2}{b^2 r^{1/2}} \sum_{\lambda_i} \frac{1}{\left[1 - \frac{1}{4b^2\lambda_i^2}\right]} \frac{J_{1/2}(r\lambda_i)}{[J_{1/2}(b\lambda_i)]^2}$$

$$\times e^{-\lambda^2{}_i \theta} \int_0^b r^{3/2} S(r) J_{1/2}(r\lambda_i)\, dr \qquad \ldots(37)$$

where the sum is taken over the positive roots of the equation

$$J'_{1/2}(\lambda_i b) = 0 \qquad \ldots(38)$$

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