COLLINEATION AND MOTION IN RELATIVISTIC THERMODYNAMICS

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In this paper it has been shown that Ricci collineation along flow vector implies motion in the space-time of relativistic thermodynamics. The theory of relativistic thermodynamics used here is Cattneo's theory.

INTRODUCTION

Let g_{ab} denote the metric tensor in the 4-dimensional space-time V_4 of relativistic thermodynamics with the signature $(- - - +)$. We will represent partial derivative by a comma and covariant derivative by a semicolon. Roman indices $a, b..$ will take values 1 to 4. Since metric tensor is covariant constant,

$$g_{ab;c} = 0.$$

Following Cattaneo's theory the stress energy tensor for the relativistic thermodynamical fluid is given by

$$T_{ab} = (\rho + p)\, u_a u_b - p g_{ab} - Q_a u_b - Q_b u_a, \qquad ..(1)$$

where ρ is the total energy density of the fluid, p is the pressure, and Q_a are the components of the heat flux vector and U^a is the time like flow vector such that

$$Q_a\, u^a = 0 \qquad ..(2a)$$

$$u_a\, u^a = 1 \qquad ..(2b)$$

$$Q_a\, Q^a = -Q^2 \qquad ..(2c)$$

Since for the thermodynamical fluid stress tensor is given by Eq. (1), Ricci tensor is obtained from Eq. (1) in the form

$$R_{ab} = \psi\,(\rho + p)\, u_a u_b - \frac{\psi}{2}\,(\rho - p)\, g_{ab} - Q_a u_b - Q_b u_a$$

$$R_{ab} = A u_a u_a - B g_{ab} - \chi Q_a u_b - \chi Q_b u_a \qquad ..(3)$$

$$A = \psi(\rho + p) \qquad ..(4)$$

$$B = \frac{\psi}{2}(\rho - p). \qquad ..(5)$$

Equations (4) and (5) yield

$$A - B = \frac{\psi}{2}\,(\rho + 3p). \qquad ..(6)$$

In consequence of Eqs. (2) and (3) we get

$$R_{ab}\, u^a = (A - B)\, u_b - Q_b$$
$$R_{ab}\, u^b u^b = (A - B)$$

$$R_{ab}\, Q^a = -BQ_b + Q^2 u_b$$
$$R_{ab} Q^a Q^b = BQ^2.$$

A vector Q^a is said to be harmonic (Nordvedt & Pagel 1962) if

$$Q_{a;b} - Q_{ab} = 0 \qquad ..(7)$$

$$Q^{\;a}_{;a} = 0$$

Ricci Collineation

Katzine *et al.* (1969) defined Ricci collineation as the point transformation

$$X^i \rightarrow X^i + \xi^i \delta t,$$

which keeps the form of the Ricci tensor invariant along the field vector ξ^i, that is for Ricci collineation

$\pounds_{\xi}\, R_{a}, = 0$ when $\pounds$ denotes the Lie derivative.

When a space-time admits groups of motions it is obvious that it admits the corresponding Ricci collineation

i.e., $\frac{\pounds}{k}\, g_{ab} = 0 \Rightarrow \frac{\pounds}{k}\, R_{ab} = 0.$

However, even if the converse is not true, in general. We have shown that in this case converse is also true under certain assumptions.

Theorem 1 — *In relativistic thermodynamical non-Zeldovich fluid if* Q^a *is harmonic and* $p_{;k}\, u^k = 0$, then

$\frac{\pounds}{u}\, R_{ab} = 0$ *implies that the world lines of material particles are geodesic.*

Proof : $$\frac{\pounds}{u}\, R_{ab} = R_{ab;k}\, u^k + R_{kb}\, u^k_{\;:a} + R_{ak}\, u^k_{\;;b} \qquad ..(8)$$

With the help of Eq. (3) and the assumption that the heat flux vector is harmonic i.e.,

$$Q_{a;b} - Q_{b;a} = 0$$

Eg. (8) reduces to

$$\frac{\pounds}{u}\, R_{ab} = A_{;k}\, u_a u_b u^k - B_{jk}\, g_{ab} u^k + \dot{u}_a\, (Au_b = \chi Q_b)$$
$$+ \dot{u}_b\, (Au_a - \xi\chi Q_a) - B\,(u_{a;b} + u_{b;a}) \qquad ..(9)$$

where $\quad \dot{u}_a \underset{==}{\text{def}}\, u_{a;k} u^k.$

$$u^a u^b \;\frac{\pounds}{u}\, R_{ab} = (A - B)_{;k} u^k$$

$$\underset{u}{\pounds} R_{ab} = 0 \Rightarrow u^a u^b \underset{u}{\pounds} R_{ab} = 0.$$

Hence in consequence of Eq. (6)

$$(A - B)_{;k} u^k = 0 \Rightarrow (\rho + 3p)_{;k} u^k = 0.$$

The hypothesis $p_{;k} u^k = 0$.

$$\rho_{;k} u^k = 0. \qquad ..(10)$$

Also $u^a \underset{u}{\pounds} R_{ab} = 0 \Rightarrow$

$$(A - B)_{;k} u^k u_b + (A - B) \dot{u}_b = 0. \qquad ..(11)$$

Using $(A - B)_{;k} u^k = 0$ in Eq. (11) we obtain $(A-B) \dot{u}_b = 0$. But since

$$A - B \neq 0,$$

$$U_b = 0, \qquad ..(12)$$

which is the required condition.

Theorem 2 — *In relativistic thermodynamical non-Zeldovich fluid if Q^a is harmonic and $p_{;k} u^k = 0$,* then

$$\underset{u}{\pounds} R_{ab} = 0 \Rightarrow \underset{u}{\pounds} g_{ab} = 0.$$

PROOF : We have

$$\underset{u}{\pounds} R_{ab} = R_{ab;k} u^k + R_{kb} u^k_{;a} + R_{ak} u^k_{;b}$$

$$= [A u_a u_b - B g_{ab} - Q_a u_b - Q_b u_a]_{;k} u^k$$

$$+ [A u_k u - B g_{kb} - Q_k u_b - Q_b u_k] u^k_{;a} + [A u_a u_k - B g_{ak}$$

$$- Q_a u_k - Q_k u_a] u^k_{;b} .$$

Since Q^a is harmonic this equation reduces to

$$\underset{u}{\pounds} R_{ab} = T_1 + T_2 - T_3$$

where

$$T_1 \underset{=}{\text{def}} A_{;k} u_a u_b u^k - B_{;k} g_{ab} u^k$$

$$T_2 \underset{=}{\text{def}} \dot{u}_a (A u_b - Q_b) + (A u_a - Q_a) \dot{u}_b$$

$$T_3 \underset{=}{\text{def}} B (u_{a;b} + u_{b;a}).$$

In order to prove $\underset{u}{\pounds} R_{ab} = 0 \Rightarrow \underset{u}{\pounds} g_{ab} = 0$, we have to prove that T_1 and T_2 vanish. Eqs. (10) and (12) show that T_1 and T_2 will vanish.

$$\underset{u}{\pounds} R_{ab} = 0\ ,\quad T_1 = 0\ ,\quad T_2 = 0$$

together imply that

$$B(\lambda u_{a;b} + \lambda u_{b;a}) = 0.$$

Since the fluid is non-zeldovich that is $\rho \neq p$, therefore

$$B \neq 0 \Rightarrow U_{a;b} + U_{b;a} = 0$$

which is the required result.

Remark : In relativistic thermodynamical non-zeldovich fluid for harmonic heat flux vector and uniform pressure Ricci-collineation along flow vector implies that the world lines of material particles are geodesic. Ricci collineation implies motion along flow vector in the four dimensional space time V_4 with the above mentioned assumptions.

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