RESONANCE IN THE RESTRICTED PROBLEM OF THREE BODIES WITH SHORT-PERIODIC PERTURBATIONS

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The motion of an asteroid whose mean motion is approximately rational with that of Jupiter has been studied. The series occurring in the problem are expanded in powers of a small parameter ϵ which represents the ratio of the mass of Jupiter to that of the Sun. Perturbations of the osculating elements of the asteroid have also been worked out upto $O(\epsilon^{3/2})$.

INTRODUCTION

In this article, it is assumed that Jupiter moves in an unperturbed circular orbit with mass ϵ relative to that of the Sun with n' its mean motion, λ' its mean longitude and l' its mean anomaly. Therefore, let us take the orbital plane of the Jupiter as xy-plane. Further suppose that n be the mean motion of an asteroid and l its mean anomaly.

Now when the frequencies are nearly commensurable, resonance arises and the condition for which is

$$\frac{n'}{n} = \frac{p}{q} \quad \text{(Jupp 1969)},$$

where p and q are prime integers. We assume (Brouwer & Clemence (1961) that

$$| pn—qn' | \leqslant n\epsilon^{1/2} \qquad \ldots (1)$$

for a given ratio of $\frac{n}{n'}$,

EQUATIONS OF MOTION

The equations of motion of the asteroid are

$$\frac{dx}{dt} = \frac{\partial \bar{H}}{\partial \dot{x}}, \quad \frac{d\dot{x}}{dt} = -\frac{\partial \bar{H}}{\partial x},$$

$$\frac{dy}{dt} = \frac{\partial \bar{H}}{\partial \dot{y}}, \quad \frac{d\dot{y}}{dt} = -\frac{\partial \bar{H}}{dy},$$

$$\frac{dz}{dt} = \frac{\partial \bar{H}}{d\dot{z}}, \quad \frac{d\dot{z}}{dt} = -\frac{\partial \bar{H}}{\partial z},$$

where the Hamiltonian $\bar{H}=H_0+H_1$,

$$H_0 = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \mu/r,$$

$$H_1 = -\epsilon\mu\left(\frac{1}{\Delta} - \frac{xx' + yy'}{r'^3}\right)$$

where (x, y, z) are the coordinates of the asteroid,

(x', y', z') are the coordinates of Jupiter,

Δ = the distance of the asteroid from Jupiter,

r = the distance of Jupiter from the Sun,

r' = the distance of the asteroid from the Sun and

μ = the universal constant.

Let us change the variables $(x, y, z;\ \dot{x}, \dot{y}, \dot{z})$ to $(x_0, x_1, x_2;\ y_0, y_1, y_2)$ by a contact transformation where

$$\left.\begin{aligned} x_0 &= L - \frac{p}{q}H, & y_0 &= l \\ x_1 &= \frac{H}{q}, & y_1 &= pl + q\omega + q(\Omega-\lambda') \\ x_2 &= G-H, & y_2 &= \omega \end{aligned}\right] \qquad \ldots (2)$$

Here

$$L = \sqrt{\mu a}\,, \quad G = L\sqrt{1-e^2}\,, \quad H = G\cos i$$

are the Delaunay-momenta variable, ω is the argument of the perihelian, Ω the longitude of the ascending node. a and e are the semi-major axis and eccentricity of the orbit of the asteroid, and i is the inclination of the orbital plane with the reference plane.

In the new variables the equations of motion are

$$\frac{dx_i}{dt} = \frac{\partial \bar{F}}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial \bar{F}}{\partial x_i} \quad (i = 0, 1, 2)$$

where

$$\bar{F} = F_0 + F_1.$$

Here F_0 is the unperturbed Hamiltonian given by

$$F_0 = \frac{\mu}{2a} = \frac{\mu^2}{2L^2} = \frac{\mu^2}{2\,(x_0 + px_1)^2}$$

and

$$F_1 = \frac{\epsilon}{r'}\sum C_i \cos iS,$$

where

$$C_i = \left[1 - \tfrac{1}{2}\left(\frac{r}{r'}\right)^2 + \left(\frac{r}{r'}\right)^4 + \ldots\right] \text{etc.}$$

Here S is the angular distance between the asteroid and the Jupiter. Also the unit of time is so choosen such that the Gaussian constant is unity.

F_1 can be put in the form as

$$F_1 = \epsilon \, \Sigma \, C' \cos D,$$

where

$$C' = C \, e^{m_1} \sin\left(\frac{i}{2}\right)^{2m_2}$$

and $D = jl + k(\Omega - \lambda') + m\omega$ (Danby 1962).

Here j, k, m, m_1 and m_2 are integers. Also $(m-k)$ is an even integer. C is a function of a and a' (a' = radius of the circular orbit of Jupiter) of degree -1.

Since $D = jl + k(\Omega - \lambda') + m\omega$,

$$\lambda' = \omega' + \Omega + l' \text{ and } l' = n't + \epsilon',$$

the time is explicity present in the Hamiltonian F_1. We shall now make the Hamiltonian independent of t.

Now $y_1 = pl + q(\omega + \Omega - \lambda')$

or $y_1 = pl + q(\omega - \omega') - q(n't + \epsilon')$

$$\text{or } \frac{dy_1}{dt} = -qn' = -\frac{\partial \bar{F}}{\partial x_1} \text{ (say)},$$

where $\bar{F}$ is the new Hamiltonian. In order to have a Hamiltonian independent of t, we have to add this term to the Hamiltonian F_1.

Thus the equations of motion become

$$\dot{x}_i = \frac{\partial K}{\partial y_i} \; ; \; \dot{y}_i = -\frac{\partial K}{\partial x_i} \quad (i = 0, 1, 2) \qquad \ldots (3)$$

where

$$K(x, y, \epsilon) = \frac{\mu^2}{2}(x_0 + p\,x_1)^{-2} + qn'x_1 + \epsilon R \qquad \ldots (4)$$

or $K(x, y, \epsilon) = K_0(x) + K_1(x, y, \epsilon)$,

and $R(x, y, \epsilon) = \Sigma \, C_{j,k,m}(a, e, i) \cos(jl + k(\Omega - \lambda') + m\omega)$. ... (5)

In terms of the new variables (x_i, y) $(i = 0, 1, 2)$, we find that

$$\dot{R} = \sum C_{j,k,m}(a, e, i) \cos\left[\left(j-k\frac{p}{q}\right)y_0 + \frac{k}{q}y_1 + (m-k)y_2\right]$$

and $(m-k)$ is an even integer.

Short Periodic Perturbations

The elimination of the short periodic terms is achieved through well-known Von-Zeipel method. Here we assume a canonical transformation (x_i, y_i) to $(\xi_i\ \eta_i)$ $(i = 0, 1, 2)$ defined by the generating function $W(\xi_0, \xi_1, \xi_2, y_0, y_1, y_2, \epsilon)$ such that the new Hamiltonian $\varphi(\xi_0\ \xi_1\ \xi_2;\ \eta_1\ \eta_2, \epsilon)$ is free from the angular velocity η_0.

We also assume that the Hamiltonian φ and the generating function W are expanded in the power series of $\epsilon^{1/2}$,

i.e. $W = W_0 + W_{1/2} + W_1 + W_{3/2} + W_2 + \ldots\ldots\ldots$

$$\varphi = \varphi_0 + \varphi_{1/2} + \varphi_1 + \varphi_{3/2} + \varphi_2 + \ldots\ldots\ldots$$

where $W_j = O(\epsilon^j)$ and $\varphi_j = O(\epsilon^j)$.

We take $W_0 = \xi_0 y_0 + \xi_1 y_1 + \xi_2 y_2$.

Since W does not contain time explicitly

$$\varphi(\xi_0, \xi_1, \xi_2;\ \eta_1, \eta_2, \epsilon) = K(x_0, x_1, x_2;\ y_0, y_1, y_2, \epsilon)$$

and the generalized Hamilton Jacobi Eq. is

$$\varphi\left(\xi\ ; \frac{\partial W}{\partial \xi_1}, \frac{\partial W}{\partial \xi_2}, \epsilon\right) = K\left(\frac{\partial W}{\partial y}; y, \epsilon\right). \qquad \ldots (6)$$

Here ξ means ξ_0, ξ_1, ξ_2 and y means y_0, y_1, y_2.

Giacaglia (1969) has found out W and φ upto $O(\epsilon^{1/2})$ We shall calculate upto $O(\epsilon^2)$ by the same procedure.

We have

$$\left.\begin{aligned}
\varphi_0 &= K_0 \\
\varphi_{1/2} &= W_{1/2} = 0 \\
\varphi_1 &= \frac{1}{2\pi q}\int_0^{2\pi q} K_1 dy_0 \\
W_1 &= -\left(\frac{\partial K_0}{\partial \xi_0}\right)^{-1}\int (K_1 - \varphi_1)\, dy_0 \\
\varphi\frac{3}{2} &= 0 \\
W\frac{3}{2} &= -\left(\frac{\partial K_0}{\partial \xi_1}\right)\left(\frac{\partial K_0}{\partial \xi}\right)^{-1}\int (K_1 - \varphi_1)\, dy_0 \\
\varphi_2 &= \overline{K_2}
\end{aligned}\right\} \qquad \ldots (7)$$

and
$$K_2 = \frac{\partial W_1}{\partial y_0}\frac{\partial K_1}{\partial \xi_0} + \frac{\partial W_1}{\partial y_1}\frac{\partial K_1}{\partial \xi_1} + \frac{\partial W_1}{\partial y_2}\cdot\frac{\partial K_1}{\partial \xi_2} + \frac{\partial W_{3/2}}{\partial y_1}\cdot\frac{\partial K_0}{\partial \xi_1} + \tfrac{1}{2}\times$$
$$\left(\frac{\partial W_1}{\partial y_0}\right)^2\frac{\partial^2 K_0}{\partial \xi_0^2} + \tfrac{1}{2}\left(\frac{\partial W_1}{\partial y_1}\right)^2\frac{\partial^2 K_0}{\partial \xi_1^2} + \frac{\partial W_1}{\partial y_0}\cdot\frac{\partial W_1}{\partial y_1}\frac{\partial^2 K_0}{\partial \xi_0 \partial \xi_1} - \frac{\partial W_1}{\partial \xi_1}\times$$
$$\frac{\partial \varphi_1}{\partial y_1} - \frac{\partial W_1}{\partial \xi_2}\frac{\partial \varphi_1}{\partial y_2}$$

where $\overline{K_2}$ = average value of K_2.

$$\therefore W_2 = -\left(\frac{\partial K_0}{\partial \xi_0}\right)^{-1}\int(\overline{K_2} - \varphi_2)\, dy_0. \qquad \dots (8)$$

Thus we have found out the two series

$$\varphi(\xi_i\ \eta_1, \eta_2, \epsilon) = \varphi_0 + \varphi_1 + \varphi_2 + \dots$$
$$W(\xi; y; \varepsilon) = \xi y + W_1 + W_{3/2} + W_2 + \dots$$

Suppose these series converge to some well-defined functions and are expanded for all values of x, y, in some domain of the phase-space. If we take the new Hamiltonian up to order $O(\epsilon^2)$ then it is given by

$$\varphi^2 = \varphi_0 + \varphi_{1/2} + \varphi_1 + \varphi_{3/2} + \varphi_2,$$

or $\varphi^2 = \varphi_0 + \varphi_1 + \varphi_2,$

where
$$\varphi_0 = \frac{\mu^2}{2}(\xi_0 + p\ \xi_1)^{-2} + q\, n'\xi_1,$$
$$\varphi_1 = \epsilon \sum C_{jp,\, jq,\, m}\,(a^* e^* i^*) \cos\left[\, j\eta_1 + \left(m - j\ q\right)\eta_2\right],$$
$$\varphi_2 = \overline{K_2}$$

and the asterisks denote the averaged value of the corresponding variables. If φ is evaluated upto $O(\epsilon^n)$

then
$$\dot{\xi}_0 = \frac{\partial \varphi^{(n)}}{\partial \eta_0} = O(\epsilon^{\,n-1/2})$$

and within that error we have

$$\xi_0 = L^* - \frac{p}{q}\, H^* = \text{constant}$$

This is a quasi-integral as ξ_0 will be an integral for $n \to \infty$ only if the process converges. The short-periodic perturbations are given by the generating function W in an implicit form as

$$\left.\begin{aligned} x_j &= \xi_j + \frac{\partial W_1}{\partial y_j} + \frac{\partial W_{3/2}}{\partial y_j} + \frac{\partial W_2}{\partial y_j} + \cdots = \xi_j + \epsilon\, \Delta x_j \\ \eta_j &= y_j + \frac{\partial W_1}{\partial \xi_j} + \frac{\partial W_{3/2}}{\partial \xi_j} + \frac{\partial W_2}{\partial \xi_j} + \cdots = y_j - \epsilon\, \Delta y_j \end{aligned}\right\} \quad \ldots (9)$$

$$\eta_j = y_j - \epsilon \Delta y_j,$$

where Δx_j and Δy_j indicates short-periodic terms. The amplitude of the short-periodic perturbations is therefore limited by a quantity of $O(\epsilon)$.

Elimination of the Critical Argument

Now we shall discuss the case when $pn = qn'$. At this point the motion is stationary and the critical argument is η_1 as when $| pn - qn' | \leqslant n\epsilon^{1/2}$ the argument η_1 is only effected. Our effort now is to reduce the degree of freedom by removing the critical argument η_1.

Consider the transformations

$$(\xi_0, \xi_1, \xi_2; \eta_1, \eta_2, \epsilon) \rightarrow (X_0, X_1 X_2; Y_1, Y_2, \epsilon)$$

through the generating function S such that the new Hamiltonian F is independent of Y_1.

$$\text{and } \xi_j = \frac{\partial S}{\partial \eta_j}; \; Y_j = \frac{\partial S}{\partial X_j} \quad (j=0,1,2) \qquad \ldots (10)$$

We again suppose that the generating function S and the Hamiltonian can be developed in the power series of $\epsilon^{1/2}$. Suppose

$$S(X, \eta_j, \epsilon) = X.\, \eta_j + \Delta S (X; \eta_1, \eta_2, \epsilon)$$

where
$$\Delta S = S_{1/2} + S_1 + S_{3/2} + \ldots,$$

$$F = F_0 = F_{\frac{1}{2}} + F_1 + F_{3/2} + F_2 + \ldots$$

$$S_j = O(\epsilon^j) \text{ ; and } F_j = O(\epsilon^j).$$

With Giacaglia (1969) we shall determine the two series F and S by successive approximations. Giacaglia has taken the terms up to $O(\epsilon^{3/2})$ but we shall include higher order terms as well.

The stationary motion will exist for the mean motion of the orbit and it will correspond to exact mean resonance, i.e., at the stationary solution we have

$$\left.\begin{aligned} \dot{\xi}_1 &= \frac{\partial \varphi}{\partial \eta_1} = 0, \\ \dot{\eta}_1 &= -\frac{\partial \varphi}{\partial \xi_1} = 0, \\ \text{and } pn^{**} &- qn' = 0. \end{aligned}\right\} \quad \ldots (11)$$

from which we determine the values of ξ_0, ξ_1 and η_1 when ξ_2 and η_2 are known. Proceeding as in Giacaglia (1969) we have

$$F_0\,(X_0, X_1) = \varphi_0\,(X_0, X_1) \qquad \ldots (12)$$

and in this case there is no perturbation in the osculating elements.

(*a*) *Approximation of* $O(\epsilon^{1/2})$

By considering the approximation up to $O(\epsilon^{1/2})$ we get

$$F_{1/2} = 0. \qquad \ldots (13)$$

(*b*) *Approximation of* $O(\epsilon)$

By considering the successive approximation up to order of ϵ we get

$$F_1 = \epsilon\,\Sigma C_{jp,jq,m}\,(a^{**}, e^{**}, i^{**}) \qquad \ldots (14)$$

and $S_{1/2}$ is given by

$$\frac{\partial S_{1/2}}{\partial \eta_1} = \frac{L^{**}}{3p^2 n^{**}}\left[-(qn' - pn^{**}) \pm \left\{(qn' - pn^{**})^2 - \frac{6p^2\,n^{**}}{L^{**}}\,U_1\right\}^{1/2}\right] \qquad \ldots (15)$$

where

$$U_1\,(X; \eta_1, \eta_2, \epsilon) = \varphi_1\,(X; \eta_1, \eta_2, \epsilon) - \varphi_1\,(X; \eta_1^{\circ}\,(X; \eta_2, \epsilon), \eta_2\, \epsilon) \qquad \ldots (16)$$

and η°_1 is the point where φ_1 has a minimum value and it can be obtained from the equation

$$\left|\frac{\partial \varphi_1}{\partial \eta_1}\right|_{\eta_1 = \eta^{\circ}_1} = 0.$$

From Eq. (15) we see that $S_{1/2}$ is equal to zero at the stationary solution. But in general the motion will be circular, asymptotic or libration in η_1 if

$$\frac{6p^2\,n^{**}}{L^{**}}\,U_1 \lesseqgtr (qn' - pn^{**})^2 \text{ provided } \eta_1 \text{ is taken maximum).}$$

U_1 is minimum at $\eta_1 = \eta^{\circ}_1$ (the libration centre) where it is zero and maximum at the end point of oscillation.

The amplitude of vibrations is given by the equation

$$U_1 = \frac{L^{**}}{3p^2 n^{**}}\,(qn' - pn^{**})^2$$

and is obtained as

$$\eta_1 = \bar{\eta_1}\,(X_0, X_1\,X_2, \eta_2, \epsilon)$$

which is of order ϵ in this case.

Also the parameters of the trajectory are given by the following equations :—

$$\left.\begin{aligned}
&a^{**} = \text{const.,}\\
&\text{where } a^{**} = \left(\frac{p^2\mu}{q^2n'}\right)^{1/3}\\
&(1-e^{**2})^{1/2}\cos i^{**} = \text{const,}\\
&X_2 = \frac{\partial F^{(1)}}{\partial y_2} = H(X_0, X_1, X_2, y_2, \epsilon),\\
&\dot{Y}_2 = -\frac{\partial F^{(1)}}{\partial x_2} K(X_0, X_1, X_2, Y_2, \epsilon),\\
&\dot{Y}_0 = -\frac{\partial F^{(1)}}{\partial x_0} = n^{**} - F(X_0, X_1, \epsilon, t),\\
&\dot{Y}_1 = -\frac{\partial F^{(1)}}{\partial X_1} = pn^{**} - qn' - G(x_0, x_1, \epsilon, t),
\end{aligned}\right\} \quad \dots (17)$$

where

$$F^{(1)} = F_0 + F_1.$$

The period of Y_0 is $\frac{2\pi}{n^{**}}$ which is short. But of Y_1 is long which is given by $\frac{2\pi}{pn^{**}-qn'}$, and of X_2 and Y_2 are very long which are given by $2\pi/n^{**}\epsilon$.

(*c*) *Approximation of* $O(\epsilon^{3/2})$

When we take the approximation upto $O(\epsilon^{3/2})$ we have

$$F_{3/2} = P_{3/2}(X; \eta^\circ_1(X, \eta_2, \epsilon), \eta_2, \epsilon)$$

$$\text{or} \quad F_{3/2} = \left|\frac{\partial \varphi_1}{\partial X_2}\frac{\partial S_{1/2}}{\partial \eta_2} - \frac{\partial F_1}{\partial \eta_2}\frac{\partial S_{1/2}}{\partial X_2}\right|_{\eta_1 = \eta^0_1}. \quad \dots (18)$$

Here η°_1 is given by

$$\left|\frac{\partial \varphi^{(3/2)}}{\partial \eta_1}\right|_{\eta_1 = \eta^\circ_1} = 0 \text{ or } \left|\frac{\partial \varphi}{\partial \eta_1}\right|_{\eta_1 = \eta^\circ_1} = 0.$$

Hence in this case the location of libration centre is not changed. And S_1 is given by the following equation

$$\frac{\partial S^{(1)}}{\partial \eta_1} - \frac{L^{**}}{3p^2n^{**}}\left[-(qn' - pn^{**}) \pm \left\{(qn' - pn^{**})^2 - \frac{6p^2n^{**}}{L^{**}}(U_1 + U_{3/2})\right\}^{1/2}\right] \quad \dots (19)$$

where $S^{(1)} = S_0 + S_1$

$$U_{3/2}(X, \eta_1, \eta_2, \epsilon) = P_{3/2}(X, \eta_1, \eta_2, \epsilon) - F_{3/2}(X, \eta_1, \eta_2, \epsilon),$$

or $U_{3/2}(X, \eta_1, \eta_2, \epsilon) = P_{3/2}(X, \eta_1, \eta_2, \epsilon) - P_{3/2}(X, \eta_1, \eta_2, \epsilon)$ ... (20)

where

$$P_{3/2}(X; \eta, \epsilon) = \frac{1}{6}\frac{\partial^3 \varphi_0}{\partial X_1^3}\left(\frac{\partial S_{1/2}}{\partial \eta_1}\right)^3 + \frac{\partial \varphi_1}{\partial X_1}\frac{\partial S_{1/2}}{\partial \eta_1} + \frac{\partial \varphi_1}{\partial X_2}\frac{\partial S_{1/2}}{\partial \eta_2} - \frac{\partial F_1}{\partial \eta_2}\frac{\partial S_{1/2}}{\partial X_2}$$

obviously at the stationary solution $S_1 = 0$ and it can be determined from the equations (15) and (19). And in general the motion will be circular, asymptotic or libration in η_1 if

$$\frac{6p^2 n^{**}}{L^{**}}(U_1 + U_{3/2}) \lesseqgtr (qn' - pn^{**})^2$$

provided η_1 is taken maximum. And the amplitude of vibrations is given by the equation

$$U_1 + U_{3/2} = \frac{L^{**}}{6p^2 n^{**}}(qn' - pn^{**})^2$$

and can be found out as

$$\eta_1 = \eta'_1(X_0\, X_1, X_2, \eta_2, \epsilon^{3/2}).$$

Also the parameters of the trajectory are given by the equations

$$\left.\begin{aligned}
\dot{a}^{**} &= \left(\frac{p^2\mu}{q^2 n'}\right)^{1/3} = \text{const.}\\
(l - e^{**2})^{1/2} \cos i^{**} &= \text{const.}\\
\dot{X}_2 &= \frac{\partial F^{(3/2)}}{\partial Y_2} = H'(X_0, X_1, X_2, Y_2, \epsilon^{3/2})\\
\dot{Y}_2 &= -\frac{\partial F^{(3/2)}}{\partial X_2} = K'({}_0, X_1, X_2, Y_2, \epsilon^{3/2})\\
\dot{Y}_0 &= -\frac{\partial F^{(3/2)}}{\partial x_0} = n^{**} - F'(X_0, X_1, X_2, Y_2, \epsilon^{3/2}\, t)\\
\dot{Y}_1 &= -\frac{\partial F^{(3/2)}}{\partial X_1} = pn^{**} - qn' - G(X_0, X_1, X_2, Y_2, \epsilon^{3/2}, t)
\end{aligned}\right\} \quad \text{... (21)}$$

where $F^{(3/2)} = F_0 + F_1 + F_{3/2}$

The period of Y_0 is $\frac{2\pi}{n^{**}}$ which is short. But of Y_1 is long and is given by $\frac{2\pi}{pn^{**} - qn'}$ and of Y_2 and X_2 are very long which are given by $2\pi/n^{**}\epsilon^{3/2}$.

(*d*) *Approximations of* $O(\epsilon^2)$

Up to this approximation the new Hamiltonian is

$$F^{(2)} = F_0 + F_1 + F_{3/2} + F_2; \qquad \ldots (22)$$

where F_0, F_1 and $F_{3/2}$, are given by Eqs. (12), (14) and (18) and F_2 can be obtained from the equation

$$F_2 = P_2\,(X, \eta^{\circ}_1\,(X, \eta_2, \epsilon), \eta_2, \epsilon) \qquad \ldots (23)$$

$$\text{where } P_2\;(X, \eta_1, \eta_2\,\epsilon) = \varphi_2 + \frac{\partial \varphi_1}{\partial X_2}\frac{\partial S_1}{\partial \eta_2} + \frac{\partial \varphi_1}{\partial X_1}\frac{\partial S_1}{\partial \eta_1} + \tfrac{1}{2}\frac{\partial^2 \varphi_0}{\partial X_1^2}\left(\frac{\partial S_1}{\partial \eta_1}\right)^2$$

$$+ \tfrac{1}{2}\left(\frac{\partial S_{1/2}}{\partial \eta_1}\right)^2 \frac{\partial^2 \varphi_1}{\partial X_1^2} + \tfrac{1}{2}\left(\frac{\partial S_{1/2}}{\partial \eta_2}\right)^2 \frac{\partial^2 \varphi_1}{\partial X_2^2}$$

$$+ \frac{\partial S_{1/2}}{\partial \eta_1}\frac{\partial S_{1/2}}{\partial \eta_2}\frac{\partial^2 \varphi_0}{\partial X_1 \partial X_2} + \tfrac{1}{2}\left(\frac{\partial S_{1/2}}{\partial \eta_1}\right)^2 \frac{\partial S_1}{\partial \eta_1}\frac{\partial^3 \varphi_0}{\partial X_1^3}$$

$$+ \frac{1}{24}\left(\frac{\partial_{1/} S_2}{\partial \eta_1}\right)^4 \frac{\partial^4 \varphi_0}{\partial X_2} - \frac{\partial S_{1/2}}{\partial X_2}\frac{\partial F_{1/2}}{\partial \eta_2} - \tfrac{1}{2}\left(\frac{\partial_{/2} S_1}{\partial \eta_2^2}\right)\frac{\partial^2 F_2}{\partial \eta_2^2}$$

and η°_1 is given by the equation

$$\left|\frac{\partial \varphi^{(2)}}{\partial \eta_1}\right|_{\eta_1 = \eta^{\circ}_1} = 0,$$

where $\varphi^{(2)} = \varphi_0 + \varphi_1 + \varphi_2$..

In this case the location of libration centre (η_1^0) is changed by a small quantity of $O(\epsilon^{1/2})$.; and $S_{3/2}$ is given by the equation

$$\frac{\partial S^{(3/2)}}{\partial \eta_1} = \frac{L^{**}}{3p^2 n^{**}}\left[-(qn' - pn^{**}) \pm \left\{(qn' - pn^{**})^2 - \frac{6p^2 n^{**}}{L^{**}}\,(U_1 + U_{3/2} + U_2)\right\}^{1/2}\right] \qquad \ldots (24)$$

Hence $S_{3/2}$ can be obtained from Eqs. (15), (19) and (24); and obviously our requirement that $S_{3/2}$ should be zero at the stationary solution holds good. But in general it is real. Also the motion will be circular, asymptotic or libration in η_1 if

$$\frac{6p^2 n^{**}}{L^{**}}\,(U_1 + U_{3/2} + U_2) \leqslant (qn' - pn^{**})^2$$

(provided η_1 is taken maximum).

The amplitude of vibrations is given by the equation

$$(U_1 + U_{3/2} + U_2) = \frac{L^{**}}{6p^2 n^{**}}\,(qn' - pn^{**})^2,$$

and can be found out as

$$\eta_1 = \eta_1'' (X_0, X_1, X_2; \eta_2, \epsilon)$$

which is of order ϵ^2.

Also the parameters of the trajectory in this case are given by the following equations

$$\left.\begin{aligned}
a^{**} &= \left(\frac{p^2\mu}{q^2 n'}\right)^{1/3} = \text{const.} \\
(1-e^{**2})^{1/2} \cos i^{**} &= \text{const.} \\
\dot{X}_2 &= \frac{\partial F^{(2)}}{\partial Y_2} = H'' (X_0, X_1, X_2 ; Y_2, \epsilon), \\
\dot{Y}_0 &= -\frac{\partial F^{(2)}}{\partial X_0} = n^{**} - F'' (X_0, X_1, \epsilon, t), \\
\dot{Y}_1 &= -\frac{\partial F^{(2)}}{\partial X_1} = pn^{**} - qn' - G''(X_0, X_1 . \epsilon, t), \\
\dot{Y}_2 &= -\frac{\partial F^{(2)}}{\partial X_2} = K'' (X_0, X_1, X_2; Y_2, \epsilon),
\end{aligned}\right\} \quad \ldots (25)$$

where $F^{(2)}$ is given by Eq. (22) and the time period of Y_0 is $2\pi/n^{**}$ which is short. But of Y_1 is given by $2\pi/pn^{**}-qn'$ which is long; and of X_2 and Y_2 are very long is this case which are given by $2\pi/n^{**} \epsilon^2$.

Giacaglia (1969) has determined perturbations of the osculating elements up to order $(\epsilon^{1/2})$ where as we shall be finding out the above perturbations upto order (ϵ^2).

PERTURBATIONS IN THE OSCULATING ELEMENTS UP TO $O(\epsilon^{3/2})$

Up to order ϵ we have the transformations

$$\left.\begin{aligned}
\xi_j &= X_j + \frac{\partial S_{1/2}}{\partial \eta_j} + \frac{\partial S_1}{\partial \eta_j} \\
\eta_j &= Y_j + \frac{\partial S_{1/2}}{\partial X_j} + \frac{\partial S_1}{\partial X_j}
\end{aligned}\right\} \quad j = 0,1,2. \quad \ldots (26)$$

In order to solve the above six equations we required six constants of integrations which are given by Eq. (21).

First we shall find out perturbations in Delaunay variables and from that we shall deduce the variations in the osculating elements taking terms up to $O(\epsilon)$.

We have from Eq. (2)

$$L = x_0 + px_1 \qquad l = y_0$$

$$G = x_0 + px_1 - x_2 \qquad \omega = y_1$$

$$H = qx_1 \qquad \Omega-\lambda' = \frac{1}{q}\, y_1 - \frac{p}{q}\, y_0 - y_2.$$

Also $\quad L^* = \xi_0 + p\xi_1 = X_0 + \dfrac{\partial S^{(1)}}{\partial\eta_0} + pX_1 + p\,\dfrac{\partial S^{(1)}}{\partial\ _1}.$

But $\quad \dfrac{\partial S_{1/2}}{\partial\eta_0} = 0,\ \dfrac{\partial S_1}{\partial\eta_0} = 0$ i.e., $\dfrac{\partial S^{(1)}}{\partial\eta_0} = 0.$

Therefore $\quad L^* = X_0 + p\bar{X}_1 + p\,\dfrac{\partial S^{(1)}}{\partial\eta_1}$

$$\left.\begin{aligned} \text{or}\quad L^* &= L^{**} + p\,\frac{\partial S^{(1)}}{\partial\eta_1}. \\ \text{Similarly}\quad G^* &= G^{**} + p\,\frac{\partial S^{(1)}}{\partial\eta_1} = \frac{\partial S^{(1)}}{\partial\eta_2}, \\ H^* &= H^{**} + q\,\frac{\partial S^{(1)}}{\partial\eta_1} \end{aligned}\right\} \qquad \dots (27)$$

The variation of mean semi-major axis is given by

$$a^* = \frac{L^{*2}}{\mu} = 1/\mu\left[L^{**} + p\,\frac{\partial S^{(1)}}{\partial\eta_1}\right]^2.$$

Putting the value of $\dfrac{\partial S^{(1)}}{\partial\eta_1}$ from Eq. (19) we get

$$a^* = a_0^* + \Delta\, a^* \qquad .. (28)$$

where

$$a_0^* = a^{**}\left(\frac{5}{3} - \frac{2}{3}\,\frac{qn'}{pn^{**}}\right) \qquad \dots (29)$$

$$\Delta a^* = \frac{2}{3}\, a^{**}\left[\left(1 - \frac{qn'}{pn^{**}}\right) - \frac{6}{n^{**}L^{**}}\,(U_1 + U_{3/2})\right]^{1/2}. \qquad \dots (30)$$

For exact resonance

$$a^* = a_0^* = a^{**}.$$

In general the maximum variation from the mean value a_0^* is given by putting $\eta_1 = \eta_1^0$ in Eq. (30)

$$\text{i.e.,}\quad (\Delta a^*)_{max} = \frac{2}{3}\, a^{**}\left|\, 1 - \frac{qn'}{pn^{**}}\right|.$$

Similarly if we put

$$C^* = (1 - e^{*2})^{1/2} \cos i^*.$$

Then

$$H^* = L^* \,(1-e^{*2})^{1/2} \cos i^* = C^* \, L^*.$$

By making use of Eq. (27), we get

$$C^* = C^{**} + \frac{1}{3p}\,(q-pc^*)\left[\left(1-\frac{qn'}{pn^{**}}\right) \pm \left\{\left(1-\frac{qn'}{pn^{**}}\right)^2 - \frac{6(U_1+U_{3/2})}{n^{**}L^{**}}\right\}^{1/2}\right] \quad \ldots (31)$$

For stable stationary solution i.e., for exact resonance we have

$$C^* = C^{**} = \text{constant}.$$

By the numerical integration of Eq. (31), we can find the constant of integration C^*.

Similarly we can show that

$$\left.\begin{aligned} l^* &= l^{**} - \frac{\partial S^{(1)}}{\partial L^{**}} - \frac{\partial S^{(1)}}{\partial G^{**}}. \\ \omega^* &= \omega^{**} + \frac{\partial S^{(1)}}{\partial G^{**}} \\ \Omega^* &= \Omega^{**} + \frac{\partial S^{(1)}}{\partial G^{**}} - \frac{\partial S^{(1)}}{\partial H^{**}}. \end{aligned}\right\} \quad \ldots (32)$$

Since $S_{1/2}$, S_1, i.e., $S^{(1)} = S_{1/2} + S_1$ is known from Eq. (19). We can find out the perturbations of all the osculating elements.

Perturbations in the Osculating Elements upto $O(\epsilon^{3/2})$

Vp to order $\epsilon^{3/2}$ we have the transformation

$$\left.\begin{aligned} \xi_j &= X_j + \frac{\partial S_{1/2}}{\partial \eta_j} + \frac{\partial S_1}{\partial \eta_j} + \frac{\partial S_{3/2}}{\partial \eta_j} \\ \eta_j &= Y_j + \frac{\partial S_{1/2}}{\partial X_j} + \frac{\partial S_1}{\partial X_j} + \frac{\partial S_{3/2}}{\partial X_j} \end{aligned}\right\} \quad \ldots (33)$$

In order to solve the above equations we require six constants of integration which are given by Eq. (25).

Again we shall first find out the perturabations in Delaunay variables and from that we shall deduce the variations in the osculating elements taking terms upto $O(\epsilon^{3/2})$.

We have $L^* = \xi_0 + p\xi_1 = X_0 + \frac{\partial S_{1/2}}{\partial \eta_0} + \frac{\partial S_1}{\partial \eta_0} + \frac{\partial S_{3/2}}{\partial \eta_0} + p\left(\frac{\partial S_{1/2}}{\partial \eta_1} + \frac{\partial S_1}{\partial \eta_1} + \frac{\partial S_{3/2}}{\partial \eta_1}\right)$

or $\quad L^* = X_0 + pX_1 + \dfrac{\partial S^{(3/2)}}{\partial \eta_0} + p\,\dfrac{\partial S^{(3/2)}}{\partial \eta_1}.$

But $\quad \dfrac{\partial S_{1/2}}{\partial \eta_0} = 0,\ \dfrac{\partial S_1}{\partial \eta_0} = 0$ and $\dfrac{\partial S_{3/2}}{\partial \eta_0} =$ i.e., $\dfrac{\partial S^{(3/2)}}{\partial \eta_0} = 0.$

Here $\quad S^{(3/2)} = S_{1/2} + S_1 + S_{3/2}.$

Therefore $\quad L^* = X_0 + pX_1 + p\,\dfrac{\partial S^{(3/2)}}{\partial \eta_1}$

$$\left.\begin{aligned} \text{Or} \quad L^* &= L^{**} + p\,\frac{\partial S^{(3/2)}}{\partial \eta_1}. \\ \text{Similarly} \quad o^* &= G^{**} + p\,\frac{\partial S^{(3/2)}}{\partial \eta_1} + \frac{\partial S^{(3/2)}}{\partial \eta_2} \\ \text{and} \quad H^* &= H^{**} + q\,\frac{\partial S^{(3/2)}}{\partial \eta_1}. \end{aligned}\right\} \quad \dots (34)$$

The variation of mean semi-major axis is given by

$$a^* = \frac{L^{*2}}{\mu} = \frac{1}{\mu}\left[L^{**} + p\,\frac{\partial S^{(3/2)}}{\partial \eta_1}\right]^2.$$

Putting the value of $\dfrac{\partial S^{(3/2)}}{\partial \eta_1}$ from Eq. (24) we get

$$a^* = a_0^* \pm \Delta\, a^*, \quad \dots (35)$$

where a_0^* is given by Eq. (29) and $\Delta\, a^*$ is given by

$$\Delta\, a^* = \frac{2}{3}\, a^{**}\left[\left(1 - \frac{qn'}{pn^{**}}\right)^2 - \frac{6}{n^{**}L^{**}}\,(U_1 + U_{3/2} + U_2)\right]^{1/2} \quad (36)$$

For exact resonance i.e. $\eta_1 = \eta_1^0$, $U_1 = 0$, $U_{3/2} = 0$, $U_2 = 0$ and $pn^{**} = qn'$.

We have

$$a^* = a_0^* = a^{**}.$$

In general the maximum variation from the mean value a_0^* is given by putting $\eta_1 = \eta_1^0$ in Eq. (36).

i.e.. $\quad (\Delta\, a^*)_{max} = \dfrac{2}{3}\, a^{**}\left|1 - \dfrac{qn'}{pn^{**}}\right|$

Similarly if we put

$$C^* = (1-e^{*2})^{1/2} \cos i^*$$

Therefore

$$H^* = L^* (1-e^{*2})^{1/2} \cos i^* = C^* L^*$$

By making use of equations (34), we will get

$$C^* = C^{**} + \frac{1}{3p}(q-pc^*)\left[\left(1-\frac{qn'}{pn^{**}}\right) \pm \left\{\left(1-\frac{qn'}{pn^{**}}\right)^2 - \frac{6}{n^{**}L^{**}}(U_1+U_{3/2}+U_2)\right\}^{1/2}\right] \quad \dots (37)$$

For stable stationary solution i.e., for exact resonance we have

$$C^* = C^{**} = \text{constant.}$$

By the numerical integration of Eq. (37) we can find the constant of integration C^*.

Similarly we can show that

$$\left.\begin{aligned} l^* &= l^{**} - \frac{\partial S^{(3/2)}}{\partial L^{**}} - \frac{\partial S^{(3/2)}}{\partial G^{**}}. \\ \omega^* &= \omega^{**} + \frac{\partial S^{(3/2)}}{\partial G^{**}}. \\ \Omega^* &= \Omega^{**} + \frac{\partial S^{(32)}}{\partial G^{**}} - \frac{\partial S^{(3/2)}}{\partial H^{**}}. \end{aligned}\right\} \quad \dots (38)$$

$S_{1/2}$, S_1 and $S_{3/2}$ can be obtained from Eqs. (15), (19) and (24) respectively. After finding these we can find all the perturbations in the osculating elements. And in evaluating $S_{1/2}$, S_1 and $S_{3/2}$ the lower limits hould be chosen to be any value in the circulation case, (because we need in general that $S_{1/2}$, S_1 and $S_{3/2}$ to be real the least of the two roots (η_1^0) of

$$(qn'-pn^{**})^2 - \frac{6p^2n^{**}}{L^{**}}(U_1+U_{3/2}+U_2) = 0$$

When we solve the Hamilton Jacobi equation by successive approximation it does not give the unique values of S and F. But if we agree, then we can solve these equations upto any order of approximation. In higher orders we see that it is very difficult to solve the equations numerically. Also it is difficult to remove periodic terms in higher orders. And if $| pn-qn' | \geqslant n\epsilon^{1/2}$, resonance will not arise because if this condition holds good then $pn-qn'$ is no longer a small quantity, and then it will not give the long period perturbations.

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