TORSIONAL WAVE PROPAGATION IN A FINITE PIEZOELECTRIC CYLINDRICAL SHELL

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Torsional wave motion in a finite hollow right circular cylinder of piezoelectric material that belongs to (622) crystal class is investigated. Two cases are discussed by considering time-dependent electrical boundary condition and time-dependent mechanical boundary condition separately. The roots of the frequency equation for β-quartz are computed.

INTRODUCTION

Applications of piezoelectric materials in various devices such as electric wave filters, fluid loaded transducers for sonar and ultrasonic cleaning, phonograph cartridges, force transducers, high voltage ignition, air-loaded transducers, displacement generators, piezoelectric transformers, piezoelectric pumps, surface wave filters etc., are well known. Finite piezoelectric cylinders, plates and discs are used for many devices. Torsional vibrations of an infinite circular cylinder of piezoelectric material that belongs to (622) crystal class are discussed by Paul (1962). Ice and β-quartz belong to the (622) crystal class. In the present paper, we consider the torsional wave propagation in a finite piezoelectric hollow cylinder of the same class when either electrical or mechanical boundary condition is time-dependent. Liu and Chang (1965) have considered a similar problem in which the elastic cylindrical shell is isotropic. They have applied the technique due to Mindlin and Goodman (1950) to tackle time dependency in the boundary condition. In the case of piezoelectric medium, the presence of charge equation of electrostatics creates considerable difficulty when we proceed to apply the same method. We extend the Mindlin and Goodman's technique to investigate the torsional wave motion in the piezoelectric medium subjected to time-dependent boundary conditions. We illustrate the procedure in the first case by considering time-dependent electrical boundary condition and in the second case by considering time-dependent mechanical boundary condition. Numerical results for the calculation of the roots of the frequency equation are performed on the digital computer IBM 370/155. The first few roots are listed in Table I.

BASIC EQUATIONS

We refer to the cylindrical polar coordinate system (r, θ, z) such that the axis of the circular cylindrical shell lies along the z-axis. As the piezoelectric circular cylindrical shell is subjected to torsional wave motion, the displacement $v(r, z, t)$ occurs along the cross-radial direction and is independent of θ. The electric potential function $\phi(r, z, t)$ is also assumed to be independent of θ. The appropriate piezoelectric relations for the (622) crystal class are (Paul & Rao 1969)

$$T_{\theta z} = c_{44}\, v_{,z} + e_{14}\, \phi_{,r},\ T_{r\theta} = c_{66}\, (v_{,r} - v/r),$$
$$D_r = e_{14}\, v_{,z} - \epsilon_{11}\, \phi_{,r},\ D_z = - \epsilon_{33}\, \phi_{,z}, \qquad ..(1)$$

where $T_{r\theta}$ and $T_{\theta z}$ denote the shearing stresses, D_r and D_z denote the dielectric displacements. c_{44} and c_{66} are the elastic, e_{14} is the piezoelectric and ϵ_{11} and ϵ_{33} are the dielectric constants. Comma followed by the subscripts indicates the partial derivatives with respect to those variables.

In the absence of the body force the equation of motion and the charge equation of electrostatics, known as Gauss's equation are

$$c_{44}\, v_{,zz} + c_{66}\, \nabla^2 v + e_{14}\, \phi_{,rz} = s v_{,tt};$$
$$e_{14}\, [v_{,rz} + (v_{,z}/r)] - \epsilon_{11}\, [\phi_{,rr} + (\phi_{,r}/r)] - \epsilon_{33}\, \phi_{,zz} = 0; \qquad ..(2)$$

where $\nabla^2 \equiv (\partial^2/\partial r^2) + r^{-1}(\partial/\partial r) - r^{-2}$ and s is the density of the material.

Formulation of the Problem — Case I

The ends of the cylindrical shell lying along the planes $z = 0$, and $z = L$ (where L is the length of the shell) are coated with electrodes that are shorted. The ends are kept stress free. Hence

$$\phi(r, o, t) = 0 = \phi(r, L, t),\ T_{\theta z}\, (r, o, t) = 0 = T_{0z}\, (r, L, t). \qquad ..(3)$$

In the present case, both the curved surfaces are kept free from tractions while an electric potential $p(z, t)$ is applied on one of the lateral surfaces. To be specific, the potential is prescribed on the outer curved surface and the inner curved surface is coated with electrodes that are shorted. Hence

$$\phi(r_2, z, t) = p(z, t),\ \phi(r_1, z, t) = 0,$$
$$T_{r\theta}(r_i, z, t) = 0;\ i = 1, 2, \qquad ..(4)$$

where r_1 and r_2 are the inner and outer radii respectively.

The equations (1) to (4) determine the problem of the first case.

Frequency Equation

We seek for the solution of the system of Eqs. (2) in the form

$$v = \sum_{l=1}^{\infty} V_l\, (r, t) \cos\, (\alpha_l z),$$

$$\phi = \sum_{l=1}^{\infty} [E_l\, (r, t) + K_l\, (r, t)] \sin\, (\alpha_l\, z), \qquad ..(5)$$

where $\alpha_1 = 1\pi/L$. In the sequel, we determine the function K_l so as to get rid of the time dependency in the boundary conditions (4). We observe that the solution (5) satisfies the end conditions (3). We substitute the result of the Eqs. (5) for v and ϕ into the Eq. (2) and rearrange as follows :

$$\alpha_l^2\, c_{44}\, V_l - c_{66}\, \nabla^2 V_l - e_{14}\, \alpha_l\, E_{l,r} + sV_{l,tt} = e_{14}\, \alpha_l\, K_{l,r};$$

$$\alpha_l\, e_{14}\, [V_{l,r} + (V_l/r)] + \epsilon_{11}\, [E_{l,rr} + (E_{l,r}/r)] - \epsilon_{33}\alpha_l^2\, E_l$$

$$= -\epsilon_{11}\, [K_{l,rr} + (K_{l,r}/r)] + \epsilon_{33}\alpha_l^2\, K_l. \quad ..(6)$$

For convenience we represent the function $p(z, t)$ as

$$p(z, t) = \sum_{l=1}^{\infty} P_l(t) \sin(\alpha_l z) \quad ..(7)$$

With the help of the Eqs. (1), (5) and (7) the boundary conditions (4) may be reduced to

$$E_l(r_2, t) = P_l(t) - K_l(r_2, t);\ E_l(r_1, t) = -K_l(r_1, t);$$

$$[V_{l,r}]_{r=r_i} - r_i^{-1} V_l(r_i, t) = 0;\ i = 1, 2. \quad ..(8)$$

We choose the function K_l such that the right hand sides of the second equation in Eq. (6) and the Eqs. (8) vanish. Hence K_l is uniquely found as the solution of the boundary value problem

$$\epsilon_{11}\, [K_{l,rr} + (K_{l,r}/r)] - \epsilon_{33}\, \alpha_1^2\, K_l = 0\ ;$$

$$\text{and } K_l(r_2, t) = P_l(t);\ K_l(r_1, t) = 0 \quad ..(9)$$

For simplicity if we put

$$K_l = k_l(r)\, P_l(t) \quad ..(10)$$

then the Eqs. (9) yield

$$k_l'' + \left(k_l'/r\right) - \alpha^2 k_l = 0;\ k_l(r_2) = 1;\ k_l(r_1) = 0; \quad ..(11)$$

where

$$\alpha^2 = \epsilon_{33}\, \alpha_1^2/\epsilon_{11},$$

and the prime denotes the ordinary derivative with respect to r. Solving the boundary value problem (11) we obtain

$$k_l = [I_0(\alpha r_1)\, K_0\,(\alpha r) - K_0(\alpha r_1)\, I_0(\alpha r)]\,[I_0\,(\alpha r_1)\, K_0\,(\alpha r_2) - I_0\,(\alpha r_2)\, K_0\,(\alpha r_1)], \quad ..(12)$$

where $I_0(r)$ and $K_0(r)$ denote the modified vessel functions of order zero. With the help of the Eqs. (9), the system of Eqs. (6) reduces to the system

$$\alpha_l^2\, c_{44}\, V_l - c_{66}\, \nabla^2 V_l - e_{14}\, \alpha_l\, E_{l,r} + s V_{l,tt} = e_{14}\, \alpha_l\, K_{l,r};$$

$$\alpha_l\, e_{14}\, [V_{l,r} +) V_l/r)] + \epsilon_{11}\, [E_{l,rr} + (E_{l,r}/r)] - \epsilon_{33}\, \alpha_l^2\, E_l = 0. \quad ..(13)$$

The Eqs. (10) and (12) determine the function K_l and hence the right hand side of the first equation in Eqs. (13) is known. From the Eqs. (8) and (9) we arrive at the homogeneous boundary conditions

$$E_l\,(r_i,\, t) = 0;\; [V_{l,\,r}\,]_{r=r_i}\; r_i^{-1}\, V_l\,(r_i,\, t) = 0, \quad i = 1, 2. \quad ..(14)$$

If we drop the right hand side of the first equation in Eq. (13) then we obtain a completely homogeneous system of differential equations.

This homogeneous system can be reduced by the transformations

$$V_l\,(r,\, t) = \sum_{n=1}^{\infty} \exp\,(i w_{ln}\, t)\; V_{ln}\,(r),$$

$$E_l\,(r,\, t) = \sum_{n=1}^{\infty} \exp\,(i w_{ln}\, t)\; E_{ln}\,(r), \quad ..(15)$$

to the following system

$$-\, \alpha_l^2\, c_{44}\, V_{ln} + c_{66}\Big[\, V''_{ln} + \Big(\, V'_{ln}/r\, \Big) - (V_{ln}/r^2)\, \Big] + s w_{ln}^2\, V_{ln} + e_{14}\alpha_l\, E'_{ln} = 0,$$

$$\alpha_l\, e_{14}\Big[\, V'_{ln} + (V_{ln}/r)\, \Big] + \epsilon_{11}\Big[\, (E''_{ln} + E'_{ln}/r)\Big] - \epsilon_{33}\, \alpha_l^2\, E_{ln} = 0. \quad ...(16)$$

Using the transformations (15), the boundary conditions (14) become

$$\Big[\, V'_{ln}\Big]_{r=r_i} - r_i^{-1}\, V_{ln}\,(r_i) = 0;\; E_{ln}\,(r_i) = 0 \quad i = 1, 2. \quad ..(17)$$

Let β_1^2 and β_2^2 be the roots of the following auxiliary equation, corresponding to the system of simultaneous differential Eqs. (16) :

$$\bar{\epsilon}_{11}\,\bar{c}_{66}\,\beta^4 + \left(K_{14}^2 + \bar{c}_{66} + \bar{\epsilon}_{11}\, W_{ln} \right) \alpha_l^2\, \beta^2 + \alpha_l^4\, W_{ln} = 0$$

where $\bar{\epsilon}_{11} = \epsilon_{11}/\epsilon_{33}$; $\bar{c}_{66} = c_{66}/c_{44}$; $K_{14}^2 = e_{14}^2/(\epsilon_{33}\, c_{44})$ ..(18)

and $W_{ln} = 1 - \left(s w_{ln}^2 / \left(c_{44}\, \alpha_1^2 \right) \right)$

Then the solutions of the system (16), called the characteristic functions, are obtained as

$$V_{ln} = \sum_{i=1}^{2} [A_{lni}\, J_1\,(\beta_i\, r) + B_{lni}\, Y_1\,(\beta_i\, r)], \qquad ..(19)$$

$$E_{ln} = -\, c_{44}/(\alpha_l\, e_{14}) \sum_{i=1}^{2} d_{lni}\, [A_{lni}\, J_0\,(\beta_i\, r) + B_{lni}\, Y_0\,(\beta_i\, r)],$$

where

$$d_{lni} = \beta_i\, [\bar{c}_{66} + (\alpha_l/\beta_i)^2\, W_{ln}];\; i = 1,\, 2, \qquad ..(20)$$

and A_{lni} and B_{lni} are arbitrary constants. $J_n(r)$ and $Y_n(r)$ denote the bessel functions of order n.

We substitute the solutions (19) into the boundary conditions (17) and obtain a system of linear homogeneous equations in the unknown A_{lni} and B_{lni}. Hence elimination of the unknown constants leads to the frequency equation

$$|\, a_{ij} \,| = 0;\; i, j = 1, 2, 3, 4. \qquad ..(21)$$

The elements in the first two columns of the determinant may be expressed as

$$a_{ij} = \beta_j\, J_2\,(\beta_j\, r_i);\; i = 1,\, 2,$$

$$a_{3j} = d_{lnj}\, J_0\,(\beta_j\, r_1),\; a_{4j} = d_{lnj}\, J_0\,(\beta_j\, r_2);\; j = 1,2,3,4. \qquad ..(22)$$

The elements in the remaining two columns may be obtained from the corresponding elements of the first two columns on just replacing the bessel functions of the first kind by the second kind.

Numerical Results

For numerical calculations, we consider the hollow cylinder of β-quartz whose size is determined by $r_2/r_1 = 6$; $r_1/L = 0.1$

The physical constants are taken from (Paul & Rao 1969) and we obtain

$$\bar{c}_{66} = 1.400552;\; \bar{\epsilon}_{11} = 0.955642;\quad K_{14}^2 = 0.002933.$$

For these constants, the roots of the Eq. (18) considered as quadratic in β^2 are always real for any real value of the quantity $L W_n c_1^{-1}$ where $c_1 = (c_{44}/s)$.

When a root, say, β_1^2 is negative, then the bessel function with the argument $\beta_1 r$, occurring in the Eq. (19) is replaced by the corresponding modified bessel function with the argument $|\beta_1 r|$. Hence the elements of the determinant in the Eq. (21) are changed accordingly. The roots LW_{ln}/c_1 of the frequency Eq. (21) are calculated numerically by programming on IBM 370/155 digital computer. The first few roots are listed in the Table I. We have adopted the following iterative procedure. For a fixed value of l we calculate the determinant in the Eq. (21), by giving a fixed but small increment to the unknown value LW_{ln}/c_1, commencing with initial value zero, till a change of sign occurs. Then the bisection method is employed to locate the root correct to a chosen number of decimal places. With this root as an initial value the procedure is repeated to find the next root.

TABLE I

Values of $LW_{ln}c_1^{-1}$

n \ l	1	2	3	4	5
1	3.1416	11.9958	13.9013	12.2191	14.6376
2	10.6907	18.1633	19.4744	16.1970	18.7393
3	17.3290	24.8005	25.7763	21.1739	23.1768
4	24.1962	31.7187	32.4874	27.0832	28.6763
5	31.2485	38.8034	39.4342	33.5338	34.8331

THE COMPLETE SOLUTION

It is shown in the Appendix that the homogeneous solutions $V_{ln}(r)$ satisfy the orthogonality conditions, viz.

$$\int_{r_1}^{r_2} r\, V_{lm}\, V_{ln}\, dr = 0, \text{ if } m \neq n. \quad ..(23)$$

For future reference, we put

$$\theta_{ln} = \int_{r_1}^{r_2} r\, V_{ln}^2\, dr \neq 0. \quad ..(24)$$

We proceed to solve the nonhomogeneous system (13). We apply the transformations

$$V_l(r, t) = \sum_{n=1}^{\infty} T_{ln}(t)\, V_{ln}(r), \quad E_l(r, t) = \sum_{n=1}^{\infty} T_{ln}(t)\, E_{ln}(r), \quad ..(25)$$

to the Eq. (13) and observe that the second transformed equation in Eq. (13) vanishes identically in view of the seocnd equation in Eqs. (16) while the first equation in Eqs. (13) transforms as follows :

$$\sum_{n=1}^{\infty} \Big[\Big(- \alpha_l^2\, c_{44}\, V_{ln} + c_{66}\Big(V''_{ln} + r^{-1} V'_{ln} - r^{-2} V_{ln} \Big)$$

$$+ e_{14}\, \alpha_l\, E'_{ln}\Big)\, T_{ln} - s\, V_{ln}\, \ddot{T}_{ln}\Big] = - e_{14}\, \alpha_l\, k'_l\, P_l \qquad ..(26)$$

where the dot over T_{ln} denotes the ordinary derivative with respect to time t.

Using the first equation in Eqs. (16), the above equation reduces to

$$s \sum_{n=1}^{\infty} V_{ln}\Big(\ddot{T}_{ln} + w_{ln}^2\, T_{ln}\Big) = e_{14}\, \alpha_l\, k'_l\, P_l. \qquad ..(27)$$

The orthogonal property of the functions V_{ln} expressed in the Eqs. (23) and (24), enables us to obtain generalized, Fourier type series expansion for k'_l as

$$k'_l = s \sum_{n=1}^{\infty} b_{ln}\, V_{ln}\,(r) \qquad ..(28)$$

where

$$s\;\; b_{ln} = \theta_{ln}^{-1} \int_{r_1}^{r_2} r\, V_{ln}\, k'_l\, dr; \quad n = 1, 2, \ldots \qquad ..(29)$$

We substitute Eq. (23) into Eq. (27) and equate the coefficients of the n-th terms on both sides :

$$\ddot{T}_{ln} + w_{ln}^2\, T_{ln} = q_{ln}\,(t), \qquad ..(30)$$

where

$$q_{ln}\,(t) = e_{14}\, \alpha_l\, b_{ln}\, P_l\,(t) \qquad ..(31)$$

Solving Eq. (30), we obtain

$$T_{ln} = M_{ln1} \cos\,(w_{ln}\, t) + M_{ln2} \sin\,(w_{ln}\, t)$$

$$+ w_{ln}^{-1} \int_0^t q_{ln}\,(x) \sin\,[w_{ln}\,(t\text{-}x)]\, dx, \qquad ..(32)$$

where M_{lni}, for $i = 1, 2$, are arbitrary constants. With the help of Eqs. (5), (10), (25), (28) and (32), we arrive at the required solutions as

$$v = \sum_{l,n=1}^{\infty} [M_{ln} \cos (w_{ln}\, t) + M_{ln_2} \sin (w_{ln}\, t) +$$

$$w_{ln}^{-1} \int_0^t q_{ln} (x) \sin [w_{ln} (t\text{-}x)]\, dx]\ V_{ln} \cos (\alpha_l\, z),$$

$$\phi = \sum_{l,n=1}^{\infty} [c_{ln}\, P_l\, V_{ln} + \{M_{ln_1} \cos (w_{ln}\, t) + M_{ln_2} \sin (W_{ln}\, t) \qquad ..(33)$$

$$+ w_{ln}^{-1} \int_0^t q_{ln} (x) \sin [w_{ln} (t\text{-}x)]\, dx\}\ E_{ln}] \sin (\alpha_1\, z),$$

where we obtained the Fourier type series expansion for k-(r), viz.,

$$k_l = \sum_{n=1}^{\infty} c_{ln}\, V_{ln}$$

$$c_{ln} = \theta_{ln}^{-1} \int_{r_1}^{r_2} r\, V_{ln}\, k_1\, dr. \qquad ..(34)$$

with the help of Eqs. (23) and (24).

Initial Conditions

The initial conditions, in general, may be stated as

$$[v]_{t=0} = g_1 (r, z);\ [v,\, t]_{t=0} = g_2 (r, z). \qquad ..(35)$$

For convenience, we assume the following infinite series expansion :

$$g_i(r,z) = \sum_{l=1}^{\infty} R_{li} (r) \cos (\alpha_l\, z);\quad i = 1,\, 2, \qquad ..(36)$$

whose coefficients are obtained as

$$R_{li} = (2/L) \int_0^L g_i (r, z) \cos (\alpha_l\, z)\, dz;\quad i = 1,\, 2. \qquad (37)$$

We make use of Eq. (33), (35) and (36) to obtain

$$R_{li} = \sum_{n=1}^{\infty} M_{lni}\, V_{ln}\, w_{ln}^{i-1}\ ,\quad i = 1,\, 2. \qquad ..(38)$$

If we apply the orthogonality conditions (23) and (24) to the equations (38), then

$$w_{ln}^{i-1} M_{lni} = \theta_{ln}^{-1} \int_{r_1}^{r_2} r R_{li} V_{ln} \, dr; \quad i = 1, 2 \qquad ..(39)$$

Eq. (39) determine the unknown constants in the final solutions (33). The expressions for the shearing stressses and dielectric displacements are simply obtained by substituting Eq. (33) into (1). Hence the boundary value problem of the first case is solved.

FORMULATION AND SOLUTION OF THE PROBLEM—CASE II

The cylindrical shell is subjected to the same end conditions as expressed in Eq. (3). Both the lateral surfaces are coated with the electrodes that are shorted. We prescribe the shearing force $c_{66}p(z, t)$ on one of the curved surfaces, say, on the exterior lateral surface. Hence.

$$\phi(r_i, z, t) = 0 \; ; \; i = 1, 2$$

$$T_{r\theta}(r_2, z, t) = c_{66} \, p(z, t), \; T_{r\theta} \, (r_1, z, t) = 0. \qquad ..(40)$$

In the present case we assume that

$$p(z, t) = \sum_{l=1}^{\infty} P_l(t) \cos (\alpha_l z). \qquad ..(41)$$

Eqs. (1) to (3) and (40) specify the boundary value problem of the second case. In case, the piezoelectric cylindrical shell, subjected to the time-dependent traction, executes torsional vibrations, we establish that we may have to decompose both the electric potential and the displacement as follows

$$v = \sum_{l=1}^{\infty} [V_l \, (r, t) + G_l \, (r, t)] \cos (\alpha_l \, z), \qquad ..(42)$$

$$\phi = \sum_{l=1}^{\infty} [E_l \, (r, t) + K_l \, (r, t)] \sin (\alpha_l \, z).$$

We substitute Eqs. (42) into (2) and rearrange to obtain

$$- \alpha_l^2 \, c_{44} \, V_l + c_{66} \, \Delta^2 \, V_l + e_{14} \, \alpha_l \, E_{l,\,r} - s V_{l,tt}$$

$$= s G_{l,tt} + \alpha_l^2 \, c_{44} \, G_l - c_{66} \, \Delta^2 \, G_l - e_{14} \, \alpha_l \, K_{l,\,r};$$

$$-\alpha_l \, e_{14} \, [V_{l,\,r} + (V_l/r)] - \epsilon_{11} \, [E_{l,rr} + (E_{l,r}/r)] + \epsilon_{33} \, \alpha_l^2 \, E_l \qquad ..(43)$$

$$= \alpha_l \, e_{14} \, [G_{l,\,r} + (G_l/r)] + \epsilon_{11} \, [K_{l,rr} + r^{-1} K_{l,\,r}] - \epsilon_{33} \, \alpha_l^2 \, K_l.$$

With the help of Eqs. (1), (41) and (42) the boundary conditions (40) reduce to

$$V_{l,r} - (V_l/r) = P_l - G_{l,r} + (G_l/r);\ if\ r = r_2$$

$$= -G_{l,r} + G_l/r);\ if\ r = r_1 \qquad ..(44)$$

$$E_l\ (r_i,\ t) = -K_l\ (r_i,\ t)\ ;\ i = 1,\ 2.$$

As before we proceed to choose the functions G_l and K_l such that the right hand sides of the second equation in Eq. (43) and Eq. (44) vanish. Hence we obtain

$$\alpha_l\ e_{14}\ [G_l + (G_l/r)] + \epsilon_{11}\ [K_{l,\ rr} + K_{l,\ r}/r)] - \epsilon_{33}\ \alpha_l^2\ K_l = 0 \qquad ..(45)$$

with the boundary conditions

$$G_{l,r} - (G_l/r) = P_l;\ \text{if}\ r = r_2 \qquad ...\ (46)$$

$$= 0;\ \text{if}\ r = r_1$$

$$K_l(r_i,\ t) = 0;\ i = 1,\ 2.$$

As another differential equation involving G_l and K_l is needed, for simpluicity, we construct the following one from the right hand side of Eq. (43) :

$$\nabla^2 G_l - a^2\ G_l = 0; \qquad ...\ (47)$$

where $a^2 = \alpha_l^2\ /\bar{c}_{66}$

If we write

$$G_l = P_l(t) g_l(r)\ ;\ K_l = P_l(t) k_l(r) \qquad ...\ (48)$$

then the above equations reduce to the system of simultaneous differential equation

$$g_l'' + \left(g_l'/r \right) - (a^2 + r^{-2})\, g_l = 0,$$

$$\alpha_l\ e_{14}\ \left[g_l' + (g_l/r) \right] + \epsilon_{11}\left[k_l'' + \left(k_l'/r \right)\right] - \epsilon_{33}\ \alpha_l^2\ k_l = 0 \qquad ...\ (49)$$

with the boundary conditions

$$g_l' - (g_l/r) = 1;\ \text{if}\ r = r_2$$

$$= 0;\ \text{if}\ r = r_1$$

$$k_l\ (r_i) = 0;\ i = 1,\ 2. \qquad ...\ (50)$$

The solution of the first equation in Eq. (49) that satisfies the first two boundary conditions in Eq. (50) is given by

$$g_l = AI_1(ar) + BK_1(ar), \qquad ...\ (51)$$

where $I_1(r)$ and $K_1(r)$ are the modified bessel functions of order one. The constants A and B are given by

$$A = N_1/(N_1 M_2 - N_2 M_1) \; ; \; B = M_1/(N_1 M_2 - M_1 N_2) \qquad \dots (52)$$

where

$$M_i = \left[I_1'(ar) \right]_{r=r_i} - r_i^{-1} I_1 (ar_i), \; N_i = \left[K_1'(ar) \right]_{r=r_i} - r_i^{-1} K_1 (ar_i), \; i = 1, 2.$$

Substitution of Eq. (51) into the second equation in Eqs. (49) yields

$$k_l'' + \left(k_l'/r \right) - \alpha^2 k_l = f(r) \qquad \dots (53)$$

where we have put

$$f(r) = (-\alpha_l \, e_{14}/\epsilon_{11}) \left[\left(I_1'(ar) + r^{-1} I_1(ar) \right) A + \left(K_1'(ar) + r^{-1} K_1(ar) \right) B \right]$$

and $\alpha^2 = a_l^2/\bar{\epsilon}_{11}$. We note that k_l in the solution of the non-homogeneous Eq. (53) with the homogeneous last boundary conditions in Eqs. (50). Hence we apply the well known green's function technique to obtain the solution. The details of the precedure may be found in the standard texts like (Churchill 1958).

If $G(r, x)$ is the green's function corresponding to the above said boundary value problem then the function k_l is given by

$$k_l = \int_{r_1}^{r_2} x \, f(x) \, G(r, x) \, dx \qquad \dots (54)$$

With the help of Eqs. (48), (51) and (54), we determine the functions G_1 and K_1 and hence the right hand side of the second equation in Eqs. (43) is known. The system of equations (43) and (44) reduce to the system

$$-\alpha_l^2 \, c_{44} V_l + c_{66} \nabla^2 V_l + e_{14} \alpha_l E_{l,r} - s V_{l,tt} = s G_{l,tt} - e_{14} \alpha_l K_{l,r}; \qquad \dots (55)$$

$$\alpha_l \, e_{14} [V_{l,r} + (V_l/r)] + \epsilon_{11} [E_{l,rr} + (E_{l,r}/r)] - \epsilon_{33} \alpha_l^2 E_l = 0,$$

with the homogeneous boundary conditions

$$[V_{l,r}]_{r=r_i} - r_i^{-1} V_l(r_i, t) = 0 = E_l(r_i, t) \; ; \; i = 1, 2. \qquad \dots (56)$$

The remaining procedure is the same as in the previous case and Eq. (15) through Eq. (39) repeat with the following exceptions.

In place of Eq. (27), we have

$$s \sum_{n=1}^{\infty} V_{ln} [\ddot{T}_{ln} + w^2_{ln} T_{ln}] = - sg_l \ddot{P}_l + \alpha_l e_{14} k'_l P_l \quad \ldots (57)$$

Hence we require to expand g_l also with the help of Eqs. (23) and (24)

$$g_l = \sum_{n=1}^{\infty} a_{ln} V_{ln} \quad \ldots (58)$$

$$a_{ln} = \theta^{-1}_{ln} \int_{r_1}^{r_2} r \, g_l V_{ln} \, dr$$

In place of Eq. (31), we have

$$q_{ln} = \alpha_l e_{14} b_{ln} P_l - a_{ln} \ddot{P}_l \quad \ldots (59)$$

Instead of the first equation in Eqs. (33), we have

$$v = \sum_{n=1}^{\infty} \Big[a_{ln} P_l + M_{ln1} \cos (w_{ln} t) + M_{ln2} \sin (w_{ln} t) +$$

$$+ w^{-1}_{ln} \int_{o}^{t} q_{ln}(x) \sin [w_{ln} (t-x)] \, dx \Big] V_{ln} \cos (\alpha_l z) \quad \ldots (60)$$

Hence the problem of the present case is solved.

Appendix

We derive the orthogonality condition of the homogeneous solutions as follows. We rewrite the first equation in Eqs. (16) as

$$\left(c_{44} \alpha_1^2 - s w^2_{ln} \right) V_{ln} = c_{66} \Big[V''_{ln} + \left(V'_{ln}/r \right) - V_{ln}/r^2) \Big] + e_{14} \alpha_1 E'_{ln}$$

We multiply this result throughout with V_{lm}, where $m \neq n$, and obtain

$$\left(c_{44} \alpha_1^2 - s w^2_{ln} \right) V_{lm} V_{ln} = c_{66} \Big[V''_{ln} V_{lm} - r^{-1} V'_{ln} V_{lm} -$$

$$- r^{-2} V_{lm} V_{ln} \Big] + \alpha_l e_{14} E'_{ln} V_{lm} \quad \ldots (A.1)$$

We substract from the above result another one obtained by interchanging the indices m and n in (A.q) and have

$$s\left(w_{lm}^2 - w_{ln}^2 \right) V_{lm} \; V_{ln} = c_{66} \left[\left(V_{ln}^{''} \, V_{lm} - V_{lm}^{''} \, V_{ln} \right) + \right.$$

$$\left. + r^{-1} \left(V_{ln}^{'} \, V_{lm} - V_{lm}^{'} \, V_{ln} \right) \right] + \alpha_1 \, e_{14} \left(E_{ln}^{'} \, V_{lm} - E_{lm}^{'} \, V_{ln} \right) \quad \ldots \text{(A.2)}$$

Next we multiply the second equation in Eqs. (16) throughout with E_{lm} and substract from this a similar one got by interchanging the indicee m and n, to obtain

$$\alpha_l e_{14} \left[\left(V_{ln}^{'} \, E_{lm} - V_{lm}^{'} \, E_{ln} \right) + r^{-1} \left(V_{ln} \, E_{lm} - V_{lm} \, E_{ln} \right] + \right.$$

$$+ \epsilon_{11} \left[\left(E_{ln}^{''} E_{lm} - E_{lm}^{''} E_{ln} \right) + r^{-1} \left(E_{ln}^{'} E_{lm} - E_{lm}^{'} \, E_{ln} \right) \right] = 0 \quad \ldots \text{(A.3)}$$

We substract the Eq. (A.3) from Eq. (A.2), then multiply the result with r, and integrate both sides of the equation with respect to r in the interval (r_1, r_2) as

$$s\left(w_{lm}^2 - w_{ln}^2 \right) \int_{r_1}^{r_2} r \, V_{lm} \, V_{ln} \, dr = \int_{r_1}^{r_2} c_{66} \left[r \left(V_{ln}^{''} \, V_{lm} - V_{lm}^{''} \, V_{ln} \right) + \right.$$

$$\left. + \left(V_{ln}^{'} \, V_{lm} - V_{lm}^{'} \, V_{lm} \right) \right] + \alpha_l \; e_{14} \left[r \left(E_{ln}^{'} \, V_{lm} - E_{ln}^{'} \right) - \right.$$

$$\left. - \left(V_{ln} \;\; E_{lm} - V_{lm} \, E_{ln} \right) \right] - \epsilon_{11} \left[r \left(E_{ln}^{''} \;\; E_{lm} - E_{lm}^{''} \; E_{ln} \right) + \right.$$

$$\left. + \left(E_{ln}^{'} \, E_{lm} - E_{lm}^{'} \, E_{ln} \right) \right] dr \qquad \ldots \text{(A.4)}$$

We observe that the integrand on the right hand side of (A.4) is the derivative W' where

$$W = r \left[c_{66} \left(V_{ln}^{'} \, V_{lm} - V_{lm}^{'} \, V_{ln} \right) - \epsilon_{11} \left(E_{ln}^{'} \;\; E_{lm} - \right. \right.$$

$$\left. \left. - E_{lm}^{'} \;\; E_{ln} \right) + \alpha_l \, e_{14} \left(V_{lm} \;\; E_{ln} - V_{ln} \, E_{lm} \right) \right] \ldots \text{(A.5)}$$

Thus the right hand side of Eq. (A.4) is $[W]_{r_1}^{r_2}$ which vanishes in view of the boundary conditions in (17).

Hence we have established the orthogonal property of the characteristic functions, expressed in Eq. (23).

REFERENCES

Churchill, R. V. (1958). *Operational Mathematics*. McGraw-Hill Book Co., pp. 260–263.

Liu, C. K., and Chang, C. H. (1965). Wave propagation in hollow elastic cylinders with end constraints. *Proc. ninth midwest. Mech. Conf. 'Dev. Mech.'*, Vol. **3**, part-2, John Wiley & Sons, pp. 203–214.

Mindlin, R. D., and Goodman, L. E. (1950). Beam vibrations with time-dependent boundary conditions. *J. appl. Mech.*, **17**; *Trans, ASME,* **72**, 377–380.

Paul, H. S. (1962). Torsional vibration of a circular cylinder of piezoelectric β-quartz. *Archiwum Mech. stosow.*, **14**, 127–134.

Paul, H. S., and Rao, B. S. (1969). Propagation of torsional disturbance in an infinite solid of piezoelectric β-quartz. *Proc. Camb. phil. Soc. math. phys. Sci.*, **66**, 205–213.