GRADIENTS BEHIND THREE DIMENSIONAL PSEUDO-STATIONARY SHOCK—WAVES IN RADIATING GASES

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Gradients of flow variables and vorticity have been obtained when radiative heat transfer term has been taken into account. Since vorticity is independent of energy equation we arrive at a very interesting result that for an optically thin gas vorticity is unaffected and remains the same as in pseudo-stationary flows.

INTRODUCTION

At 10^4K° the radiation heat transfer is of the same order of magnitude as that by convection. Hence we can include the radiation-heat-transfer term in gas flow neglecting the radiation pressure and radiation energy. Radiation-heat-transfer, expressed in integral form when included in the gas flows converts the equation of motion into a very complicated form for which general solution has not been obtained so far and we have to solve them by approximations.

Heaslet and Baldwin (1963) have given numerical solutions of the problems. Dhyani (1973) has considerared the case of a two-dimensional straight-wedge in hypersonic flows for an attached shock wave. Similar problem has been tackled by Zhigulev *et al.* (1963) using the method of small perturbations.

Pant and Mishra (1964) have determined the gradients of flow variable and vorticity in three-dimensional steady gas flows. Kanwal (1958 *a*, *b*, *c*) has considered the same case for an unsteady, curved shock and pseudo-stationary curved shock waves. Pant and Mishra (1965) have tackled the same problem for three dimensional stationary and pseudo-stationary curved shock waves in conducting gases. Dhyani (1973) has obtained the gradients of flow variables, curvature of stream line, and vorticity components for an optically thick gas.

In this paper we have obtained the vorticity and gradients of the flow parameters for a three-dimensional pseudo-stationary shock wave for an optically thin gas, taking into account the radiative heat transfer term.

EQUATIONS OF MOTION

Let the shock configuration in three-dimensional pseudo-stationary flow be given by

$$x_i = t\, a_i\,(y^\alpha), \qquad \dots \quad (1)$$

where x^is are rectangular cartesian coordinates and y^αs are Gaussian coordinates of the shock surface. In this case i ranges from 1 to 3 and α for I and II.

Neglecting viscosity and heat conductivity the equations of motion are given by

$$\rho \partial_i U_i + U_i \partial_i \rho + 3\rho = 0, \qquad \dots (2)$$

$$\rho(U_j \partial_j U_i + U_i) + \partial_i p = 0 \qquad \dots (3)$$

and

$$\rho U_i U_j \partial_j U_i - \rho C^2 \partial_i U_i = Q\,(\gamma - 1) - \rho U^2 + 3\rho C^2, \qquad \dots (4)$$

where $U_i = u_i - a_i$ which is the relative velocity of the fluid with respect to the shock surface, p is the pressure, ρ represents the density and $Q = 4\,\rho\sigma K_p T^4$ representing radiation heat transfer, σ is the Stefan-Boltzmann constant for radiative flux and K_p is the Plank mean absorption coefficient.

Radiative heat transfer expressed in integral form complicates the equations of motion. Therefore, for simplicity we consider the case of optically thin gas and assume that 'the gas is completely absorbed into the shock layer' i.e., there is no emission of radiative flux. For this particular case the Rankine-Hugonict jump conditions remain the same as those for a pseudostationary case, hence we have

$$[U^i] = -\frac{\delta}{1+\delta} U_{1n} X^i, \qquad \dots (5)$$

$$[p] = \frac{\delta}{1+\delta} \rho_i U_{1n}^2 \qquad \dots (6)$$

and

$$[\rho] = \rho_i \delta, \qquad \dots (7)$$

where [ ] denote the difference of values on the two sides of the shock surface of the quantity enclosed, X^i are the components of the unit normal to the shock surface directed from the region in front to the region behind the shock surface, δ is the shock strength and

$$U_{1n} = U_{1_i} X^i. \qquad \dots (8)$$

For a perfect gas

$$\delta = \frac{(2M_{1n}^2 - 1)}{2+(\gamma-1)M_{1n}^2}, \qquad \dots (9)$$

where

$$M_{1n}^2 = \frac{U_{1n}^2}{C^2{}_1}$$

Results of Differential Geometry

In this section certain properties of the surfaces will be summarized. Taking the lines of curvature as the Gaussian coordinate curves on the shock surface the following relation holds good.

$$a_{\alpha\beta} \overset{\text{def}}{=} \partial_\alpha x^i \, \partial_\beta x^i, \qquad \dots \quad (10)$$

where $a_{\alpha\beta}$ are the components of the first fundamental form of the surface and ∂_α denotes the partial derivatives with respect to y^α.

The components $b_{\alpha\beta}$ of the second fundamental form of the surface are given by

$$b_{\alpha\beta} \overset{\text{aer}}{=} \tfrac{1}{2}\, \epsilon^{\gamma\delta} \epsilon_{ijk} \, (\partial_\alpha x^i)_{,\,\beta} \, \partial_\gamma x^j \, \partial_\delta x^k, \qquad \dots \quad (11)$$

where (,) denotes the covariant derivative with respect to y^α.

$$\epsilon^{11} = \epsilon^{22} = 0,$$

$$\epsilon^{12} = \frac{1}{\sqrt{a}}, \; \epsilon^{21} = -\frac{1}{\sqrt{a}}, \; a = \det \|a_{\alpha\beta}\|,$$

and ϵ_{ijk} is an isotropic tensor having the values

(1) $+1$ when i, j, k is an even permutation of 1, 2, 3,

(2) -1 when i, j, k is an odd permutation of 1, 2, 3,

(3) 0 when i, j, k are not all different.

As the Gaussian coordinate curves are the lines of curvature,

$$a_{12} = b_{12} = a_{21} = b_{21} = 0.$$

If the shock surface is real, $a \neq 0$. In that case a symmetric tensor $a^{\alpha\beta}$ may be defined, such that

$$a^{\alpha\beta} a_{\beta\gamma} = \delta^\alpha_\gamma \, ,$$

$a^{\alpha\beta}$ is then the cofactor of $a_{\alpha\beta}$ in a divided by a. Hence

$$a^{12} = a^{21} = 0, \; a^{11} = \frac{a_{22}}{a}, \quad a^{22} = \frac{a_{11}}{a}.$$

The normal curvatures in the directions of coordinate curves are

$$k_\alpha = \frac{b_{\alpha\alpha}}{a_{\alpha\alpha}} \qquad \dots \quad (12)$$

(α, not summed).

Weingarten's formula for the derivative of the unit normal is

$$\partial_\alpha X^i = -\, a^{\beta\gamma} \; b_{\beta\alpha} \;\; \partial_\gamma x^i. \qquad \dots \quad (13)$$

New Coordinate System

At a point x^i behind the shock surface let ds be the elementary are length along a stream line. Then the components of the unit tangent vector to the stream-line are given by

$$\frac{\partial a^i}{\partial s} = \frac{U^i}{U}, \quad \ldots (14)$$

where

$$U^2 = U_i\, U^i.$$

The equation of any surface, passing through a point x^i, at a distance s along a pseudo-stream line from the shock surface, is given by

$$x^i = ta^i(y^\alpha, s) \quad \ldots (15)$$

with the initial conditions

$$a^i\,(y^\alpha, 0) = a^i(y^\alpha)$$

and

$$X^i(y^\alpha, 0) = X^i(y^\alpha)$$

Now

$$\partial a^i = \frac{\partial a^i}{\partial s}\,\partial s + \frac{\partial a^i}{\partial y^\alpha}\,\partial y^\alpha$$

or

$$\frac{\partial a^i}{\partial a^j} = \delta^i_j = \frac{U^i}{U}\,\partial_j s + \partial_\alpha a^i \partial_j y^\alpha \quad \ldots (16)$$

and

$$\frac{\partial f}{\partial a^j} = \frac{\partial f}{\partial s}\,\partial_j\, s + \partial_\alpha f\, \partial_j y^\alpha, \quad \ldots (17)$$

where f is any funtion of a^i.

Now we have (Pant & Mishra 1965)

$$X^j = \frac{U_n}{U}\,\partial_j s, \quad \ldots (18)$$

$$\partial_j y^\alpha = \frac{t}{U_n}\,\epsilon^{\alpha\beta}\,\epsilon_{ijk}\,U^i\,\partial_\beta x^k, \quad \ldots (19)$$

$$U^j\,\partial_j\,y^\alpha = 0, \quad \ldots (20)$$

$$X^j \partial_j y^\alpha = -\frac{t}{U_n} V^\alpha, \qquad \ldots (21)$$

and

$$\partial_\gamma x^j \partial_j y^\alpha = t\delta^\alpha_\gamma, \qquad \ldots (22)$$

where

$$V^\alpha = V_\beta a_{\alpha\beta} = U_i \partial_\alpha x^i. \qquad \ldots (23)$$

Proceeding in a manner similar to Pant and Mishra (1965) Eq. (2) to (4) reduce to

$$U\frac{\partial \rho}{\partial s} + \rho \frac{UX_i}{U_n} \frac{\partial U_i}{\partial s} + \rho\partial_\alpha U_i \partial_i y^\alpha + 3\rho = 0, \qquad \ldots (24)$$

and

$$\rho U \frac{\partial U_i}{\partial s} + X^i \frac{U}{U_n} \frac{\partial p}{\partial s} + \partial_\alpha p \partial_i y^\alpha + \rho U_i = 0 \qquad \ldots (25)$$

$$\rho U U_i \frac{\partial U_i}{\partial s} - \rho C^2 X^i \frac{U}{U_n} \frac{\partial U_i}{\partial s} - \rho C^2 \partial_\alpha U_i \partial_i y^\alpha + \rho U^2 - 3\rho C^2 = Q(\gamma - 1). \qquad \ldots (26)$$

Multiplying Eq. (25) by U_i we get

$$\rho U U_i \frac{\partial U_i}{\partial s} + U \frac{\partial p}{\partial s} + \rho U^2 = 0. \qquad \ldots (27)$$

Multiplying Eq. (24) by C^2 we get

$$UC^2 \frac{\partial \rho}{\partial s} + \rho C^2 \frac{UX_i}{U_n} \frac{\partial U_i}{\partial s} \rho C^2 \partial_\alpha U_i \partial_i y^\alpha + 3\rho C^2 = 0. \qquad \ldots (28)$$

Adding Eq. (26) and (28) we get

$$\rho U U_i \frac{\partial U_i}{\partial s} + \rho U^2 + C^2 U \frac{\partial \rho}{\partial s} = Q(\gamma - 1). \qquad \ldots (29)$$

Subtracting Eq. (29) from Eq. (27) we get

$$U\frac{\partial p}{\partial s} = UC^2 \frac{\partial \rho}{\partial s} - Q(\gamma - 1). \qquad \ldots (30)$$

This connects the variations of pressure and density along a pseudo-stream line behid the shock and shows that the flow behind is no more isentropic.

Determination of Gradients

For gradients of velocity, pressure and density at any point P we have

$$\partial_j U_i = \frac{\partial U_i}{\partial s} \partial_j s + \partial_\alpha U_i \partial_j y^\alpha. \qquad \ldots (31)$$

$$\partial_j p = \frac{\partial p}{\partial s} \partial_j s + \partial_\alpha p \partial_j y^\alpha \qquad \dots (32)$$

and

$$\partial_j \rho = \frac{\partial \rho}{\partial s} \partial_j s + \partial_\alpha \rho \partial_j y^\alpha. \qquad \dots (33)$$

Multiplying Eq. (25) by $\frac{X^i}{U_n}$, we get

$$\frac{\rho U X^i}{U_n} \frac{\partial U_i}{\partial s} + X^i X^i \frac{U}{U_n^2} \frac{\partial p}{\partial s} + \frac{X^i}{U_n} \partial_\alpha p \partial_i y^\alpha + \frac{X^i \rho U_i}{U_n} = 0$$

or

$$\frac{\rho U X^i}{U_n} \frac{\partial U_i}{\partial s} + \frac{U}{U_n^2} \frac{\partial p}{\partial s} + \frac{X^i}{U_n} \partial_\alpha p \partial_i y^\alpha + \rho = 0. \qquad \dots (34)$$

Using Eq. (30) in Eq. (34), we get

$$\frac{\rho U X^i}{U_n} \frac{\partial U_i}{\partial s} + \frac{U}{M_n^2} \frac{\partial p}{\theta s} - \frac{Q(\gamma - 1)}{U_n^2} + \frac{X^i}{U_n} \partial_\alpha \rho \partial_i y^\alpha + \rho = 0. \qquad \dots (35)$$

Subtracting Eq. (35) from Eq. (24) we get

$$U \frac{\partial \rho}{\partial s} = \frac{M_n^2}{\left(M_n^2 - 1\right)} \frac{X^i}{U^u} \partial_\alpha p \partial_i y^\alpha - \frac{M_n^2 \rho}{\left(M_n^2 - 1\right)} \partial_\alpha U_i \partial_i y^\alpha$$

$$- \frac{M_n^2}{\left(M_n^2 - 1\right)} \left[2 \rho + \frac{Q(\gamma - 1)}{U_n^2} \right]. \qquad \dots (36)$$

Also from Eq. (30)

$$U \frac{\partial p}{\partial s} = C^2 U \frac{\partial p}{\partial s} - Q (\gamma - 1).$$

Thus with the help of Eqs. (36) and (30) we get

$$U \frac{\partial p}{\partial s} = \frac{C^2 M_n^2}{\left(M_n^2 - 1\right)} \frac{X^i}{U_n} \partial_\alpha p \partial_i y^\alpha - \frac{C^2 M_n^2 \rho}{\left(M_n^2 - 1\right)} \partial_\alpha U_i \partial_i y^\alpha$$

$$- \frac{C^2 M_n^2}{\left(M_n^2 - 1\right)} \left[2\rho + \frac{Q(\gamma - 1)}{U_n^2} \right] - Q(\gamma - 1). \qquad \dots (37)$$

Substituting the value of $U \frac{\partial p}{\partial s}$ from Eq. (37) in Eq. (25), we get

$$\rho U \frac{\partial U_i}{\partial s} + \frac{M_n^2}{\left(M_n^2 - 1\right)} \partial_\alpha p \partial_i y^\alpha - \frac{\rho U_n X^i}{\left(M_n^2 - 1\right)} \partial U_i \partial_i y^\alpha$$

$$- \frac{X^i U_n}{\left(M_n^2 - 1\right)} \left[2\rho + \frac{Q(\gamma - 1)}{U_n^2} \right] - \frac{X^i}{U_n} Q(\gamma - 1) + \rho U_i = 0$$

or

$$\frac{\partial U_i}{\partial s} = \frac{U_n X^i}{U\left(M_n^2 - 1\right)} \partial_\alpha U_k \partial_k y^\alpha - \frac{M_n^2}{\rho U\left(M_n^2 - 1\right)} \partial_\alpha p \partial_i y^\alpha$$

$$+ \frac{X^i U_n}{\rho U\left(M_n^2 - 1\right)} \left[2\rho + \frac{Q(\gamma - 1}{U_n^2} \right] + \frac{X^i}{\rho U U_n} Q(\gamma - 1) - \frac{U_i}{U}. \qquad \dots (38)$$

Substituting the values of $\frac{\partial U_i}{\partial s}$, $\frac{\partial \rho}{\partial s}$ and $\frac{\partial p}{\partial s}$ in Eqs. (31), (33) and (32) respectively and using $\partial_j s$ from Eq. (18) we have the following relations :

$$\partial_j U_i = \frac{X^i X^j}{\left(M_n^2 - 1\right)} \left[\partial_\alpha U_k \partial_k y^\alpha + \left\{ 2 + \frac{Q(\gamma - 1)}{\rho U_n^2} \right\} + \frac{\left(M_n^2 - 1\right)}{\rho U_n^2} Q(\gamma - 1) \right]$$

$$- \frac{X^j}{U_n} \left\{ \frac{M_n^2}{\rho U_n \left(M_n^2 - 1\right)} \partial_\alpha p \partial_i y^\alpha + U_i \right\} + \partial_\alpha U_i \beta_j y^\alpha \; ; \qquad \dots (39)$$

$$\partial_j \rho = \frac{M_n^2 X^j}{U_n\left(M_n^2 - 1\right)} \left[\frac{X^i}{U_n} \partial_\alpha p \partial_i y^\alpha - \rho \partial_i y^\alpha \partial_\alpha U^i \right] - \frac{M_n^2 X^j}{U_n\left(M_n^2 - 1\right)}$$

$$\times \left[2\rho + \frac{Q(\gamma - 1)}{U_n^2} \right] + \partial_\alpha \rho \partial_j y^\alpha ; \qquad \ldots (40)$$

and

$$\partial_j p = \frac{C^2 M_n^2 X^j}{U_n \left(M_n^2 - 1 \right)} \left\{ \frac{X^i}{U_n} \partial_\alpha p \partial_i y^\alpha - \rho \partial_i y^\alpha \partial_\alpha U^i \right\} + \partial_\alpha p \partial_j y^\alpha$$

$$- \frac{C^2 M_n^2 X^j}{U_n \left(M_n^2 - 1 \right)} \left[2\rho + \frac{Q(\gamma - 1)}{U_n^2} \right] - Q(\gamma - 1) \frac{X^j}{U_n}. \qquad \ldots (41)$$

Equations (39), (40) and (41) represent the gradients of velocity, density and pressure respectively behind a shock at a point P.

VORTICITY

Vorticity will be given by

$$w^i = t^{-} \epsilon^{ijk} \partial_j U_k. \qquad \ldots (42)$$

We know (Kanwal 1958*a*) that the energy equation and equation of state are not required for determination of the vorticity jump across a shock of given strength in ideal gases in this case. We see that the vorticity generated by the shock remains the same as in the case of pseudo-stationary flows.

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