MAGNETOHYDRODYNAMIC CONVECTION WITH A SPATIAL HEAT SOURCE SUBJECT TO ROTATION

by S. P. BHATTACHARYYA *and* S. K. JAIN, *Department of Mathematics, Indian Institute of Technology, Powai, Bombay-400 076*

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The stability of an infinite horizontal layer of electrically conducting incompressible fluid which loses heat throughout its volume at a constant rate is discussed when the fluid is in a state of uniform rotation and in the presence of a magnetic field. The value of the critical Rayleigh number R is found to decrease with an increase in the rate of heat loss showing that the layer becomes less stable. It is observed that the destabilizing effect of the heat source parameter $\bar{Q}$ is more prominent for small values of magnetic field parameter and the Taylor number.

INTRODUCTION

Ray and Scorer (1963) have discussed the stability of a horizontal layer of viscous fluid which loses heat throughout its volume with a constant temperature conditions on the bounding surfaces. Two different cases have been studied by these authors. In the first case the heat loss is assumed to be constant throughout the volume while it is taken to be periodic in the second case. It has been found that in each case the critical Rayleigh number R decreases as the rate of heat loss increases. But the aspect ratio a increases with increasing heat source parameter Q when the boundaries are free and decreases with increasing Q when the boundaries are rigid. No reason has been assigned for this unexpected behaviour of the aspect ratio 'a' with the change in the boundary conditions. Moreover, the results obtained by Ray and Scorer have been shown to be erroneous by Watson (1968). Watson has reconsidered the same problem under the condition when the heat loss in the whole volume is assumed to be constant. Bhattacharyya and Jain (1975) have studied the stability of a horizontal layer which loses heat throughout its volume at a constant rate in the presence of a vertical magnetic field.

The problem of hydromagnetic stability of a hot rotating layer of fluid seems to be of special significance in Astrophysics and in a study of power generation by MHD. The problem is also useful in geophysics, particularly in geothermal regions. We have considered the effect of heat loss on the critical Rayleigh number R when the fluid is in a state of uniform rotation and subjected to a uniform magnetic field with free boundaries.

Goldstein and Graham (1969) have devised a method of obtaining boundaries with negligible shear by considering a liquid with high viscosity supported by a liquid with considerably lower viscosity. They have measured the value of the critical Rayleigh number in such a layer by heating it from below and found it to be in reasonable agreement with the analytical results obtained by Rayleigh (1916). Thus the results obtained by using free boundaries can be interpreted appropriately.

Basic Equations

Let us consider a horizontal layer of electrically conducting incompressible fluid of infinite extent in a state of uniform rotation with an angular velocity $\vec{\Omega}$ and subject to a uniform magnetic fieid $\vec{H}$. We take X_1 axis and X_2 axis on the lower boundary and X_3 axis vertically upwards. The fluid is assumed to be confined between the boundaries $X_3 = 0$ and $X_3 = d$ which are kept at constant temperatures T_0 and T_1 respectively.

Applying Boussinesq approximation and the perturbation method, the linearized equations during steady convection in non-dimensional form are (cf. Chandrasekhar 1961; Watson 1968)

$$(D^2 - a^2)\,\theta = -\frac{P_1}{R}\,[R + Q\,(1 - 2Z)]\,W, \qquad \ldots(1)$$

$$(D^2 - a^2)\,h_3 = -P_2 DW, \qquad \ldots(3)$$

$$(D^2 - a^2)\,\psi_3 = -P_2 D\Phi_3, \qquad \ldots(3)$$

$$(D^2 - a^2)\,\Phi_3 = -\sqrt{T}\,DW - \frac{Q_1}{P_2}\,D\psi_3 \qquad \ldots(4)$$

and

$$(D^2 - a^2)^2\,W = \frac{R}{P_1}\,a^2\theta - \frac{Q_1}{P_2}\,D(D^2 - a^2)\,h_3 + \sqrt{T}\,D\,\Phi_3. \qquad \ldots(5)$$

Here P_1, P_2, T, R, Q, Q_1 are the Prandtl number (ν/k), the magnetic Prandtl number (ν/η), the Taylor number $\left(\frac{4\,\Omega^2}{\nu^2}d^4\right)$, the Rayleigh number $\left(\frac{g\alpha\,\Delta\,Td^3}{\nu k}\right)$, the heat source parameter $\left(\frac{g\alpha q d^5}{2k^2\,\nu}\right)$ and the magnetic field parameter $\left(\frac{\mu H^2 d^2}{4\,\pi\rho\nu\eta}\right)$ respectively; a is the wave number of the disturbance and $D = \frac{d}{dZ}$. $\vec{\Phi} = \text{Curl}\,\vec{u}$ and $\vec{\psi} = \text{Curl}\,\vec{h}$ where $\vec{u}$ and $\vec{h}$ are the perturbations in the velocity and the magnetic field respectively.

Boundary conditions — We consider the case when both the boundaries are free and the medium adjoining the fluid is non-conducting. Thus the boundary conditions are :

$$W = 0 = D^2 W \quad \text{at } Z = 0 \text{ and } 1$$

$$\psi_3 = 0 \quad \text{at } Z = 0 \text{ and } 1.$$

The thermal boundary condition T (mean temperature) $=$ constant implies that $\theta = 0$ on both the surfaces.

Critical Value of the Rayleigh Number

Eliminating θ and h_3 from Eqs. (1), (2), (4) and (5) and introducing the new independent variable $\overline{Z}$, where $Z = \frac{\overline{Z}}{\pi},$ we get

$$[\{(D^2 - a_1^2)^2 - \overline{Q}_1 D^2\}^2 + T_1 D^2 (D^2 - a_1^2)] (D^2 - a_1^2) W$$

$$= - [(D^2 - a_1^2)^2 - \overline{Q}_1 D^2] \left[\frac{R}{\pi^4} + \overline{Q}\left(1 - \frac{2Z}{\pi}\right)\right] a_1^2 W, \quad ..(6)$$

where

$$a_1^2 = \frac{a^2}{\pi^2}, \overline{Q}_1 = \frac{Q}{\pi^2}, T_1 = \frac{T}{\pi^4}, \overline{Q} = \frac{Q}{\pi^4} \text{ and } D \equiv \frac{d}{d\overline{Z}}. \quad ..(7)$$

We substitute

$$W = \sum_{n=1}^{\infty} A_n \sin(n\overline{Z}) \text{ in Eq. (6)} \quad ..(8)$$

and obtain

$$\sum_{n=1}^{\infty} \left(n^2 + a_1^2\right)\left[\left\{\left(n^2 + a_1^2\right)^2 + \overline{Q}_1 n^2\right\}^2 + n^2\left(n^2 + a_1^2\right) T_1\right] A_n \sin(n\overline{Z})$$

$$= a_1^2 \sum_{n=1}^{\infty} \left\{\left(n^2 + a_1^2\right)^2 + \overline{Q}_1 n^2\right\} \left\{\frac{R}{\pi^4} + \overline{Q}\left(1 - \frac{2Z}{\pi}\right)\right\} A_n \sin(n\overline{Z}) \quad (9)$$

Multiplying both sides of Eq. (9) by $\sin(mZ)$ and integrating from 0 to π, we get

$$\left[\left(m^2 + a_1^2\right)\left[\left\{\left(m^2 + a_1^2\right)^2 + \overline{Q}_1 m^2\right\}^2 + m^2\left(m^2 + a_1^2\right) T_1 - \frac{R}{\pi^4} a_1^2 \left\{\left(m^2 + a_1^2\right)^2 + \overline{Q}_1 m^2\right\}\right]\right] A_m$$

$$+ \frac{4\overline{Q} a_1^2}{\pi^2} \sum_{m \neq n} \frac{2mn[(-1)^{m+n} - 1]}{(n^2 - m^2)^2} \left[\left(n^2 + a_1^2\right)^2 + \overline{Q}_1 n^2\right] A_n = 0, \quad ..(10)$$

Equation (10) gives a set of homogeneous linear equations in the constants A_n, and for the non-trivial solution of the system, the determinant of coefficients must vanish. Solving this determinant equation obtained from Eq. (10) by including successively more rows and columns, the critical value of the Rayleigh number can be evaluated.

We have solved the determinant, so obtained for the critical value of the Rayleigh number 'R' with respect to the wave number a for different values of $\bar{Q}_1$, Q and T_1.

Discussion

In Fig. 1 we have presented the curves for ln R drawn against $\bar{Q}_1$ ($>$ 100) for different values of $\bar{Q}$, the heat source parameter when the Taylor number $T_1 = 1000$. It is found that ln R decreases with increasing $\bar{Q}$, showing that the heat source parameter destabilizes the system. The destablizing effect of $\bar{Q}$ is less pronounced for higher values of the magnetic field parameter.

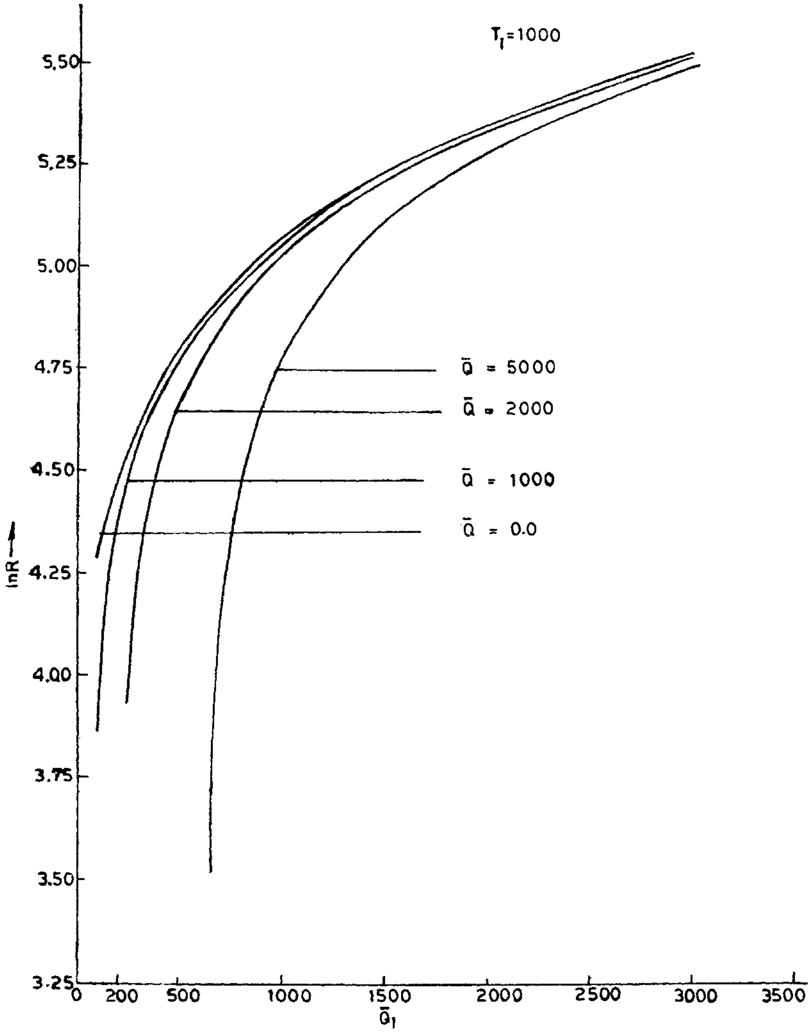

FIG. 1. The variation of the critical Rayleigh number R for the onset of cellular convection as a function of $\bar{Q}_1$ ($>$ 100) for various values of $\bar{Q}$ when $T_1 = 1000$.

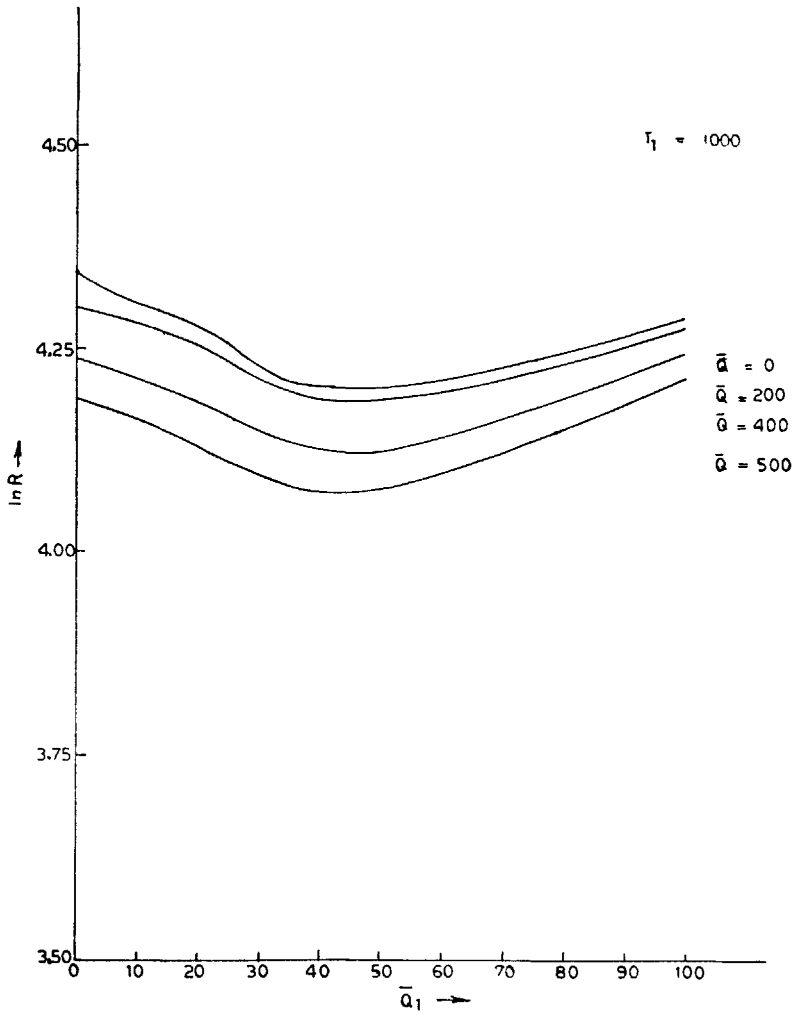

FIG. 2. The variation of the critical Rayleigh number R for the onset of cellular convection as a function of $\overline{Q_1}$ ($\leqslant$ 100) for various values of Q when T_1= 1000.

In Fig. 2 the curves for ln R are plotted against $\overline{Q_1}$ ($0 \leqslant Q_1 \leqslant 100$) for certain values of $\overline{Q}$ with T_1 remaining the same. These curves show that there is a range of the magnetic field strength parameter $\overline{Q_1}$ when the system becomes more unstable with increasing values of $\overline{Q_1}$. This effect is found to exist even in the case when the heat source parameter $\overline{Q}$ is zero (Chandrasekhar 1961). But with increasing values of $\overline{Q}$, the destabilizing effect becomes more pronounced. Due to this result we have investigated the range $31 \leqslant \overline{Q_1} \leqslant 35$ with more details. The corresponding curves for ln R against $\overline{Q_1}$ for the same value of T_1 are given in Fig. 3, when $\overline{Q} = 994.98$, 995.00 and 995.10. It is observed that when $\overline{Q} = 995$, the system is unstable if $\overline{Q_1}$ lies between 33 and 33.5. For larger values of $\overline{Q}$ this critical range of $\overline{Q_1}$ increases.

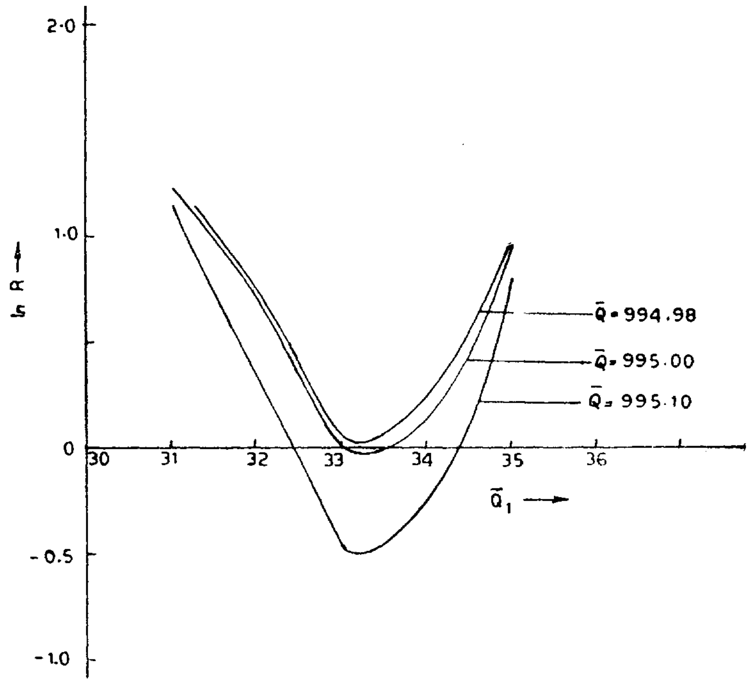

FIG. 3. The variation of the critical Rayleigh number R for the onset of cellular convection as a function of $\overline{Q}_1$ for various values of Q (994.8, 995.00, 995.10) when T_1 = 1000.

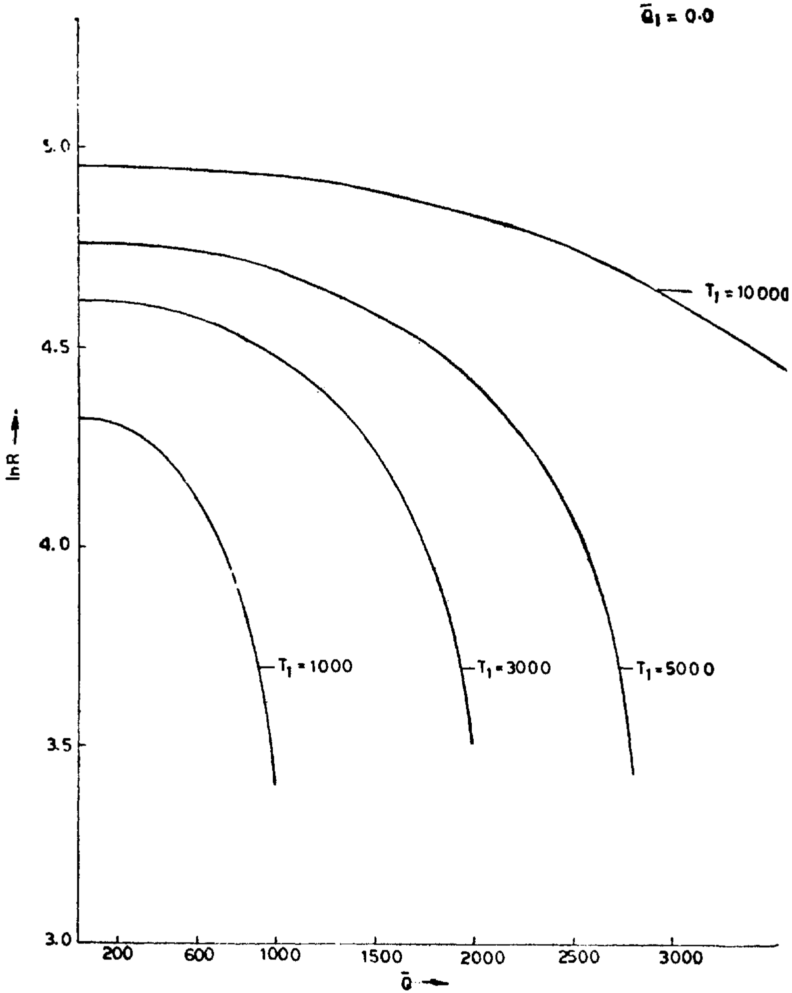

FIG. 4. The variation of the critical Rayleigh number R for the onset of cellular convection as a function of $\bar{Q}$ for various values of T_1 when $\bar{Q}_1$=0.0.

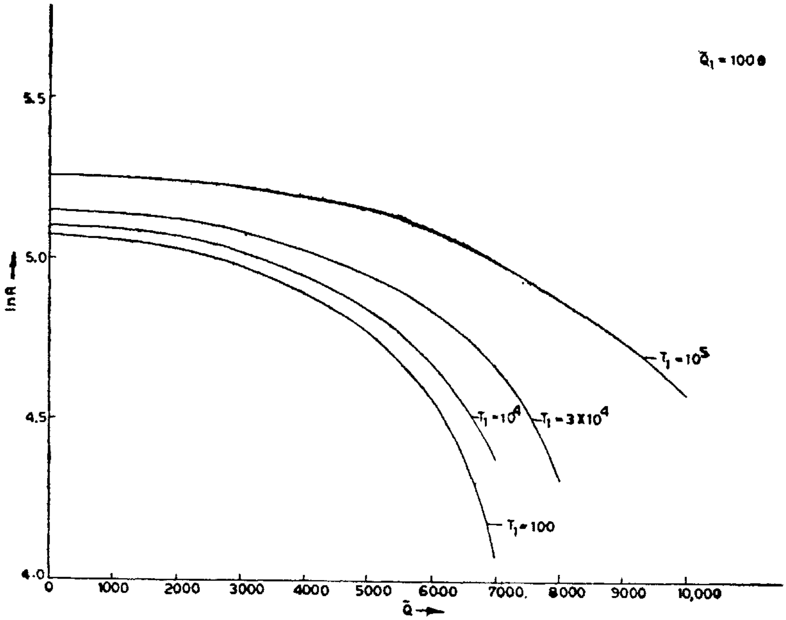

FIG. 5. The variation of the critical Rayleigh number R for the onset of cellular convection as a function of $\overline{Q}$ for various values of T_1 when $\overline{Q_1} = 1000.0$.

To investigate the effect of $\overline{Q}$ on R for different values of T_1, we have presented the curves for two particular cases.

(*a*) $\overline{Q_1} = 0$, i.e., in absence of the magnetic field (cf. Fig. 4).

and

(*b*) $\overline{Q_1} = 1000$ [cf. Fig. 5].

From both of these figures it is found that the Taylor number stabilizes the flow and its effect is more prominent for larger values of $\overline{Q}$.

Thus, the magnetic field has a stabilizing effect on the rotating system, except for a certain range of the values of $\overline{Q_1}$ and if the heat source parameter is sufficiently large the system becomes unstable for any difference of temperature between the bounding surfaces. Rotation of the system is found to have a stabilising effect.

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