

EFFECT OF ROTATION ON A HEATED HORIZONTAL LAYER OF DISSOCIATING FLUID

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The problem of convective instability of a layer of a dissociating fluid in the presence of rotation is discussed. It is found that overstability is not possible if the Prandtl number is greater than unity and rotation exerts a stabilizing influence on the flow. Further chemical instability occurs when the Prandtl number exceeds Schmidt number and in this case steady convection is always preferred in the marginal state.

INTRODUCTION

The classical Bénard problem i.e., the stability of horizontal layer of a viscous non-reactive liquid heated from below in a field of gravity g has been studied extensively in the literature. A convenient reference for this is the monograph by Chandrasekhar (1961). It is well known that the layer becomes unstable and convection sets in when the Rayleigh number $R_a = g\alpha\beta d^4/K\nu$, (α = coefficient of volume expansion, β = the equilibrium adverse temperature gradient, d = thickness of the layer, K = thermal diffusivity, and ν = kinematic viscosity) exceeds a critical value. Wollkind and Frisch (1971) have extended this problem to the case of a dissociating fluid heated from below or above. They have assumed that the system is in a state of chemical quasi-equilibrium so that there is only a slight departure from a dissociating fluid in chemical equilibrium. Assuming a mathematical model due to Lighthill (1960) for the dissociating fluid, these authors have shown that when the layer is heated from below, there is only a slight change in the criterion of instability as in Bénard convection mentioned above. However in the case of heating from above, a new mode of instability arises due to the dissociation of the fluid, and Wollkind and Frisch have termed them as chemical instability. Since the rate of chemical reaction is a function of the concentration and does not depend on the spatial gradient of velocity, concentration or temperature, such unusual mode of instability is not unexpected.

In this paper we propose to study the effect of a uniform rotation about a vertical axis on a horizontal layer of a dissociating fluid heated from above or below. Here we consider a fluid mixture which is chemically reactive and the reaction is simply due to dissociation as already discussed by Lighthill. Since in a non-reactive fluid rotation brings in new phenomenon like overstability when the fluid is heated from below (see Chandrasekhar 1961), it will be of interest to study the effect of rotation on Bénard convection in a reactive fluid. This is the object of the present investigation.

MATHEMATICAL FORMULATION OF THE PROBLEM

A layer of a dissociating fluid of thickness d is subjected to temperature T_0 at the bottom surface $z = 0$ and T_1 at the upper surface $z = d$. The layer is rotating with a uniform angular velocity Ω about the vertical axis z and the boundaries are assumed to be free. The equations of continuity and momentum are in a rotating frame of reference,

$$\nabla \cdot \vec{q} = 0, \tag{1}$$

$$\frac{D\vec{q}}{Dt} = - \frac{1}{\rho_0} \nabla p - g [1 - a(T - T_0) - b(\alpha - \alpha_0)] \vec{K} + \nu_0 \nabla^2 \vec{q} - 2\vec{\Omega} \times \vec{q}, \tag{2}$$

where \vec{q} is the velocity having components (u, v, w) , \vec{K} is the unit vector along z -direction and the last term is the Coriolis force arising out of rotation. The second term on the right hand side of Eq. (2) is the body force arising out of temperature and concentration variation. In deriving this term, the linearized equation of state is used as follows :—

$$\rho = \rho_0 [1 - a(T - T_0) - b(\alpha - \alpha_0)], \tag{3}$$

where T, α represent the temperature and concentration, a and b are constants and the subscript 0 refers to a standard state. The well-known Boussinesq approximation is invoked in writing down the buoyancy force term in Eq. (2).

The equations for conservation of energy and species for the dissociating fluid are (see Lighthill 1960)

$$\rho_0 \left[C_1 \frac{DT}{Dt} + C_2 \frac{D\alpha}{Dt} \right] = \lambda^0 \nabla^2 T + \rho_0 D_{12}^0 \frac{D\alpha}{Dt} \tag{4}$$

and

$$\frac{D\alpha}{Dt} + \frac{\alpha - \alpha_0}{\tau} = D_{12}^0 \nabla^2 \alpha, \tag{5}$$

where terms proportional to the thermal diffusion factor and pressure changes in the right hand side of Eqs. (4) and (5), being much smaller than the species diffusion effects are neglected.

In deriving Eq. (4), the equation of state for internal energy U is used as follows

$$U = C_1 T + C_2 \alpha \tag{6}$$

In Eq. (4), the concentration α is given by $N_A / (N_A + 2N_{A_2})$, where N_A and N_{A_2} represent the number of A atoms and A_2 molecules in the dissociation reaction $A_2 \rightleftharpoons 2A$ taking place in the fluid layer. Further λ^0 is the thermal conductivity of the binary mixture, D_{12}^0 the diffusivity of the mixture, m the mass of A , D_0 the dissociation energy per molecule, and τ the relaxation time. In Eqs. (4) and (5) ρ_0, T_0 and α_0 are constant chemical equilibrium values of ρ, T and α satisfying

$$\frac{\alpha_0^2}{1 - \alpha_0^2} = \frac{\rho_d}{\rho_0} \exp \left(- \frac{D_0}{KT_0} \right), \tag{7}$$

where ρ_d is the characteristic density and K is the Boltzmann constant. Following Wollkind and Frisch, we now make use of the fact that the perturbed system is in a state of chemical quasi-equilibrium. Hence we may replace all differential terms of α in eqs. (4) and (5) by

$$d\alpha = \varphi dT; \quad \varphi = \frac{\alpha_0(1 - \alpha_0)}{2 - \alpha_0} \cdot \frac{D_0}{KT_0^2}. \quad \dots(8)$$

Using this substitution, Eq. (4) is transformed to

$$\frac{DT}{Dt} = K_0 \nabla^2 T, \quad \dots(9)$$

where

$$K_0 = \frac{\lambda_0}{\rho_0 C_T}, \quad \lambda_0 = \lambda^0 + \frac{\varphi \rho_0 D_{12}^0 D_0}{2m}; \quad C_T = C_1 + \varphi C_2. \quad \dots(10)$$

Similarly Eq. (5) becomes

$$\frac{DT}{Dt} + M(\alpha - \alpha_0) = D_{12}^0 \nabla^2 T, \quad \dots(11)$$

where

$$M = \frac{1}{\varphi \tau}. \quad \dots(12)$$

The boundary conditions for the temperature at the free surfaces are

$$T = T_0 \text{ at } z = 0, \quad T = T_1 \text{ at } z = d. \quad \dots(13)$$

Further the normal component of the velocity and tangential components of the stress vanish at the surfaces. These give

$$w = 0, \quad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0, \quad \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \text{ at } z = 0, d. \quad \dots(14)$$

The fact that $\partial u/\partial z + \partial w/\partial x$ and $\partial v/\partial z + \partial w/\partial y$ are both zero at $z = 0$ and d follows from the vanishing of tangential stresses at the free surface.

The equilibrium state is defined by

$$\left. \begin{aligned} u = v = w = 0, \quad T = T_0 - \beta z = T_0(z), \quad \alpha = \alpha_0, \\ p = -\rho_0 g \int_0^z (1 + \alpha \beta \xi) d\xi = p_0(z), \end{aligned} \right\} \quad \dots(15)$$

where $\beta = \frac{T_0 - T_1}{d}$. To ensure the validity of the assumption of chemical quasi-equilibrium, it is necessary to assume that

$$\frac{|T_0 - T_1|}{T_0} = \frac{|\beta| d}{T_0} \ll 1. \quad \dots(16)$$

It may be noted that $\beta \geq 0$ according to the heating from below or above.

LINEARIZED STABILITY ANALYSIS

In order to investigate the stability of the solution given by Eqs. (15), we perturb this basic state and take the perturbed variables as

$$\left. \begin{aligned} \vec{q} &= \vec{q}'(u', v', w'); T = T_0(z) + T' \\ p &= p_0(z) + p'; \alpha = \alpha_0 + \alpha', \end{aligned} \right\} \quad \dots(17)$$

where \vec{q}' , T' , p' and α' are perturbation quantities. Substituting Eqs. (17) in Eqs. (1), (2), (9) and (11) and retaining only linear terms of the perturbation, we obtain

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \quad \dots(18)$$

$$\frac{\partial u'}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x} + \nu_0 \nabla^2 u' + 2 \Omega v', \quad \dots(19)$$

$$\frac{\partial v'}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial y} + \nu_0 \nabla^2 v' - 2 \Omega u', \quad \dots(20)$$

$$\frac{\partial w'}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial z} + \nu_0 \nabla^2 w' + g(aT' + b\alpha'), \quad \dots(21)$$

$$\frac{\partial T'}{\partial t} - \beta w' = K_0 \nabla^2 T' \quad \dots(22)$$

and

$$\frac{\partial T'}{\partial t} - \beta w' + M\alpha' = D_{12}^2 \nabla^2 T'. \quad \dots(23)$$

The boundary conditions are

$$\begin{aligned} w' = T' = \alpha' = \frac{\partial u'}{\partial z} + \frac{\partial v'}{\partial x} = \frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y} = 0 \\ \text{at } z = 0 \text{ and } d. \end{aligned} \quad \dots(24)$$

Following the technique of normal mode analysis, we now assume

$$\begin{aligned} [u', v', w', p', T', \alpha'] (x, y, z, t) \\ = [U, V, W, P, \theta, A] (z) \exp [i(k_x x + k_y y) + \omega t]. \end{aligned} \quad \dots(25)$$

Using Eq. (25), we get from Eqs. (18)–(23),

$$\frac{dW}{dz} + ik_x U + ik_y V = 0, \quad \dots(26)$$

$$\omega U = -\frac{ik_x}{\rho_0} P + \nu_0 \left[\frac{d^2}{dz^2} - s^2 \right] U + 2 \Omega V, \quad \dots(27)$$

$$\omega V = -\frac{ik_y}{\rho_0} P + \nu_0 \left[\frac{d^2}{dz^2} - s^2 \right] V - 2 \Omega U, \quad \dots(28)$$

$$wW = -\frac{1}{\rho_0} \frac{dP}{dz} + v \left[\frac{d^2}{dz^2} - s^2 \right] W + g(a\theta + bA), \quad \dots(29)$$

and

$$w\theta = \beta W + K_0 \left(\frac{d^2}{dz^2} - s^2 \right) \theta, \quad \dots(30)$$

$$w\theta + MA = \beta W + D_{12}^0 \left(\frac{d^2}{dz^2} - s^2 \right) \theta \quad \dots(31)$$

where

$$s^2 = k_x^2 + k_y^2. \quad \dots(32)$$

The boundary conditions (24) imply

$$W = \theta = \frac{dU}{dz} + ik_x W = \frac{dV}{dz} + ik_y W = 0 \text{ at } z = 0 \text{ and } d. \quad \dots(33)$$

It may be seen that Eqs. (32) and (33) imply that

$$W = \theta = A = \frac{d^2 W}{dz^2} = 0 \text{ at } z = 0 \text{ and } d. \quad \dots(34)$$

Multiplying Eq. (27) by ik_x , and Eq. (28) by ik_y , adding and then making use of (26) and (32), we obtain

$$\frac{s^2 P}{\rho_0} = -w \frac{dW}{dz} + v_0 \left(\frac{d^2}{dz^2} - s^2 \right) \frac{dW}{dz} - 2 \Omega \zeta, \quad \dots(35)$$

where

$$\zeta = ik_x V - ik_y U, \quad \dots(36)$$

which is the z-component of vorticity. Elimination of P from Eqs. (29) and (35) leads to

$$\begin{aligned} & \left[v_0 \left(\frac{d^2}{dz^2} - s^2 \right)^2 - w \left(\frac{d^2}{dz^2} - s^2 \right) \right] W \\ & = 2 \Omega \frac{\partial \zeta}{\partial z} + gs^2(a\theta + bA). \end{aligned} \quad \dots(37)$$

Again multiplying Eq. (27) by ik_y and Eq. (28) by $-ik_x$ and adding we get

$$w\zeta = v_0 \left(\frac{d^2}{dz^2} - s^2 \right) \zeta + 2 \Omega \frac{dW}{dz}. \quad \dots(38)$$

We now introduce the following dimensionless variables and parameters

$$\begin{aligned} \xi &= \frac{z}{d}, \quad n = sd, \quad D \equiv \frac{d}{d\xi}, \quad \sigma = \frac{wd^2}{v_0}, \quad Pr = \frac{v_0}{K_0}, \\ \gamma &= \frac{\tau v_0}{d^2} \cdot \frac{\alpha_0(1 - \alpha_0)}{(2 - \alpha_0)} \cdot \frac{D_0}{KT_0}, \quad S = \frac{v_0}{D_{12}^0}. \end{aligned} \quad \dots(39)$$

Using Eqs. (39), (30), (31), (37) and (38), we obtain

$$\frac{\beta d^2}{K_0} W + [D^2 - n^2 - \sigma P_r] \theta = 0, \quad \dots(40)$$

$$\frac{\beta d^2}{T_0 \nu_0} \cdot W + \frac{1}{T_0} [S^{-1}(D^2 - n^2) - \sigma] \theta - \gamma^{-1} \cdot A = 0, \quad \dots(41)$$

$$\begin{aligned} [(D^2 - n^2)^2 - \sigma(D^2 - n^2)] W - \frac{g a d^2 n^2}{\nu_0} \cdot \theta \\ - \frac{g b d^2 n^2}{\nu_0} \cdot A - \frac{2 \Omega d^3}{\nu_0} \cdot D\zeta = 0 \end{aligned} \quad \dots(42)$$

and

$$\frac{2 \Omega d}{\nu_0} DW + [D^2 - n^2 - \sigma] \zeta = 0. \quad \dots(43)$$

The boundary conditions (34) now take the form

$$W = \theta = D^2 W = A = 0 \text{ at } \xi = 0 \text{ and } 1. \quad \dots(44)$$

Elimination of ζ from Eqs. (42) and (43) leads to

$$\begin{aligned} \frac{2 \Omega d}{\nu_0} D^2 W + \frac{\nu_0}{2 \Omega d^3} (D^2 - n^2 - \sigma) \left[\{(D^2 - n^2)^2 - \sigma(D^2 - n^2)\} W \right. \\ \left. - \frac{g a d^2 n^2}{\nu_0} \cdot \theta - \frac{g b d^2 n^2}{\nu_0} \cdot A \right] = 0. \end{aligned} \quad \dots(45)$$

Since $W = D^2 W = \theta = A = 0$ at $\xi = 0$ and 1 , it follows from Eqs. (40) and (41) and the equations derived from them by differentiating even number of times that

$$D^2 \theta = D^4 \theta = 0 \text{ and } D^2 A = 0 \text{ at } \xi = 0 \text{ and } 1. \quad \dots(46)$$

Again since $D\zeta = 0$ at a free surface, it follows from Eq. (42) and $D^2 W = \theta = A = 0$ that

$$D^4 W = 0 \text{ at } \xi = 0 \text{ and } 1. \quad \dots(47)$$

Finally Eqs. (45) and (47) give

$$D^6 W = 0 \text{ at } \xi = 0 \text{ and } 1. \quad \dots(48)$$

Using Eqs. (46), (47) and (48) in Eqs. (40) and (41) we have

$$D^6 \theta = D^4 A = 0. \quad \dots(49)$$

Repeating this process, we may show that all even order derivatives of W , θ and A must vanish both at $\xi = 0$ and 1 if the surfaces are free.

Hence the appropriate eigen solutions are

$$[W, \theta, A](z) = [\hat{W}, \hat{\theta}, \hat{A}] \sin m\pi\xi, \quad m = 1, 2, \dots \quad \dots(50)$$

Substituting Eq. (50) in (40), (41) and (45) and eliminating \dot{W} , $\hat{\theta}$ and \hat{A} we get

$$K^2(K^2 + \sigma)^2 (K^2 + \sigma P_r) + T(K^2 - n^2) (K^2 + \sigma P_r) - Rn^2(K^2 + \sigma) (1 - \epsilon B.K^2) = 0, \quad \dots(51)$$

where

$$K^2 = m^2\pi^2 + n^2, T = \frac{4\Omega^2 d^4}{\nu_0^2}, R = \frac{g\alpha\beta d^4}{K_0\nu_0}, \epsilon = \frac{\gamma b}{aT_0},$$

$$B = \frac{1}{S} - \frac{1}{P_r}. \quad \dots(52)$$

When the marginal state is steady, $\sigma = 0$ and then Eq. (51) gives for the lowest mode $m = 1$,

$$(1 + x)^3 + T_1 - R_1 x [1 - \epsilon B_1(1 + x)] = 0, \quad \dots(53)$$

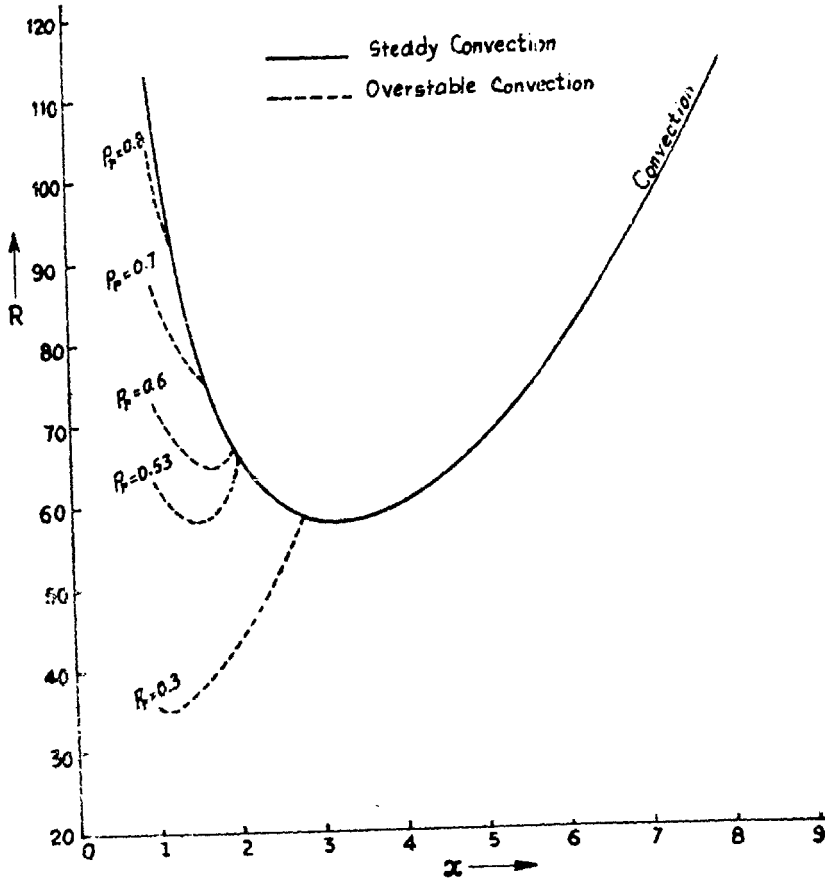


FIG. 1. Variation of Rayleigh number with square of the wave number for steady and overstable convection for different values of P_r with $T = 10,000$ and $\epsilon B = 10^{-3}$.

where

$$n^2 = \pi^2 x, R = \pi^4 R_1, T = \pi^4 T_1, B_1 = \pi^2 B. \quad \dots(54)$$

From Eq. (53), the Rayleigh number in the steady marginal state is given by

$$(R_1)_{\text{steady}} = \frac{(1+x)^3 + T_1}{x[1 - \epsilon B_1(1+x)]}. \quad \dots(55)$$

On the other hand when the marginal state is oscillatory, we take $\sigma = \pi i \sigma_1$, σ_1 being real. In this case the Rayleigh number corresponding to the lowest mode $m = 1$ is derived from the real part of Eq. (51) as

$$(R_1)_{ov} = \frac{2(1+P_r)}{x[1 - \epsilon B_1(1+x)]} \cdot \left[(1+x)^3 + \frac{P_r^2}{(1+P_r)^2} \cdot T_1 \right]. \quad \dots(56)$$

Similarly the imaginary part of Eq. (51) gives σ_1^2 , the square of the frequency as

$$\sigma_1^2 = \frac{T_1}{(1+x)} \cdot \frac{(1-P_r)}{(1+P_r)} - (1+x)^2. \quad \dots(57)$$

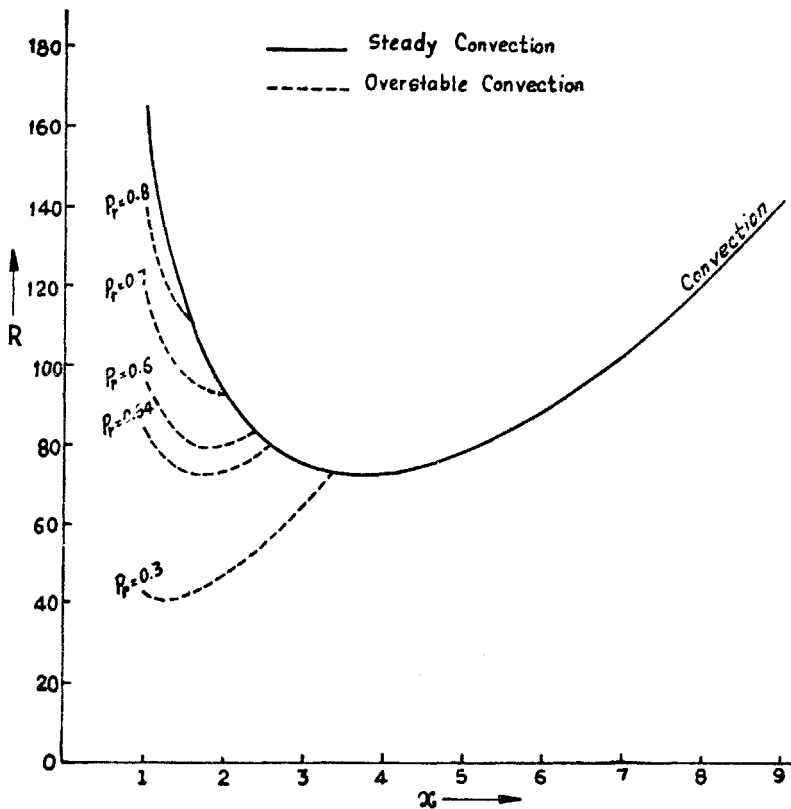


FIG. 2. Variation of Rayleigh number with square of the wave number for steady and overstable convection for different values of P_r , with $T = 15,000$ and $\epsilon B = 10^{-3}$.

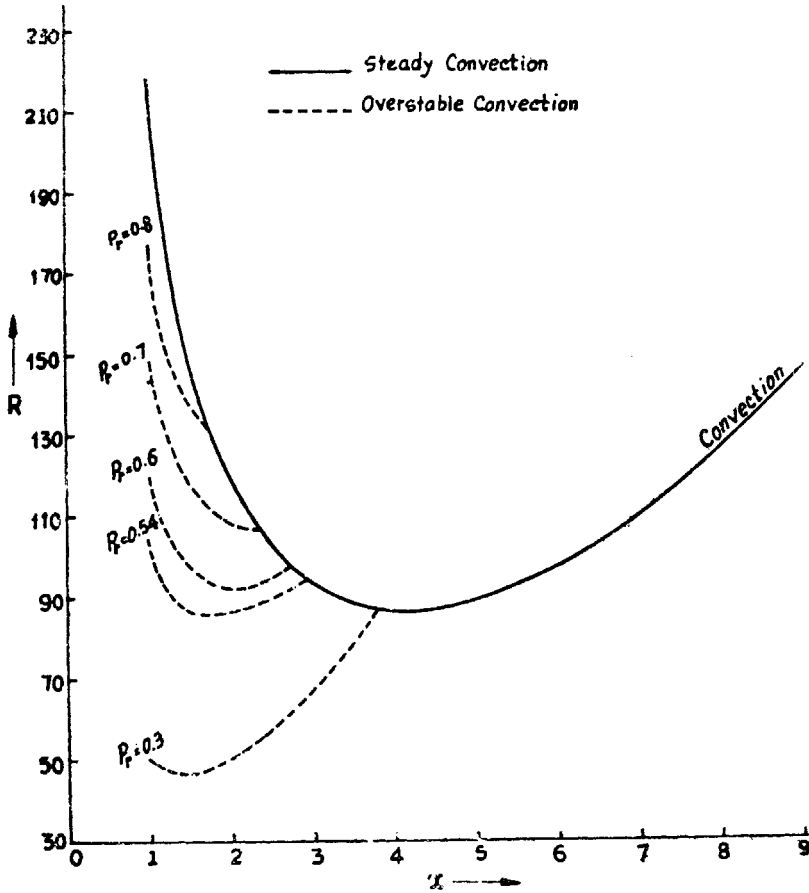


FIG 3. Variation of the Rayleigh number with square of the wave number for steady and overstable convection for different values of P_r with $T = 20,000$ and $\epsilon B = 10^{-3}$.

It is clear from Eq. (57) that real values of σ_1 are not possible if $P_r > 1$. This implies that oscillatory convection or over-stability does not occur when $P_r > 1$. It may be noted that this condition is also sufficient for non-occurrence of over-stability in a non-dissociating fluid as discussed by Chandrasekhar (1961). Now by Eqs. (52) and (54), $B_1 = \pi^2(S^{-1} - P_r^{-1})$ and this can be both positive or negative. Hence Eqs. (55) and (56) show that for a certain range of wave numbers, $(R_1)_{\text{steady}}$ and $(R_1)_{\text{ov}}$ may be negative and this corresponds to chemical instability arising out of dissociation. But from the definition of R and R_1 from Eqs. (52) and (54), it follows that R_1 can be negative only when $\beta < 0$, which corresponds to heating from above. Thus it may be concluded that chemical instability is possible only when the layer is heated from above. Of course, the positive values of R_1 given by Eqs. (55) and (56) correspond to the usual Bénard convection in the presence of rotation. For finding the critical values of Rayleigh numbers at the onset of instability as Bénard convection, we have to find the minimum values of $(R_1)_{\text{steady}}$ and $(R_1)_{\text{ov}}$ given by Eqs. (55)

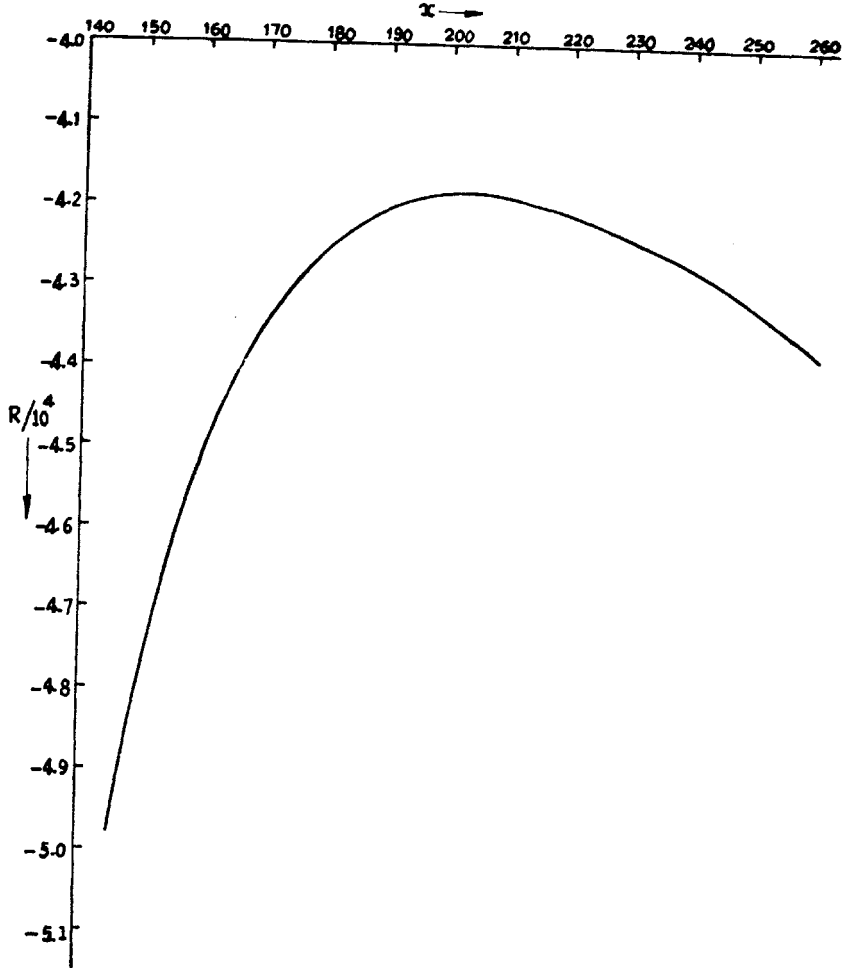


FIG. 4. Variation of Rayleigh number with square of the wave number for steady convection corresponding to chemical instability with $T = 10,000$ and $\epsilon B = 10^{-3}$.

and (56). Setting $d(R_1)_{\text{steady}}/dx = 0$ and $d(R_1)_{0v}/dx = 0$, the wave number x corresponding to the marginal state can be found.

On the other hand, for finding the critical Rayleigh number at the onset of convection as chemical instability, we have to find the maximum values of $(R_1)_{\text{steady}}$ or $(R_1)_{0v}$ in the range of wave numbers where R_1 is negative.

Variations of Rayleigh number with the wave number x for the marginally steady and oscillatory cases are evaluated from Eqs. (55) and (56) with $\epsilon B = 10^{-3}$ (for nitrogen) for several values of the rotation parameter T and Prandtl number P_r . In each case we get two distinct branches of the stability curve corresponding to the Bénard mode and the mode due to chemical instability. In Figs. 1, 2 and 3, the curves labelled convection are steady convection curves and are valid for all P_r .

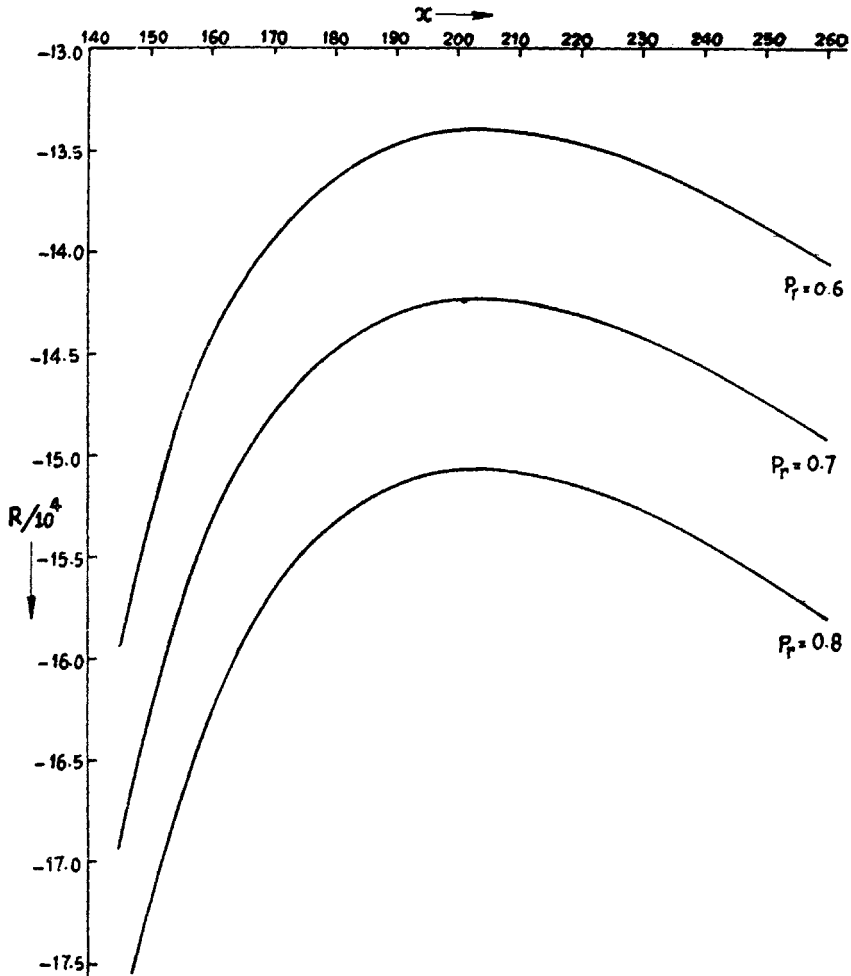


FIG. 5. Variation of Rayleigh number with square of the wave number for overstable convection corresponding to chemical instability for different values of P_r with $T = 10,000$ and $\epsilon B = 10^{-3}$.

The remaining curves are those due to overstable oscillation for values of P_r by which they are labelled. It can be seen from these figures that for a given rotation parameter, there exists a particular value of P_r for which the minimum value of the Rayleigh number R for the overstable curve is same as that for the steady convection curve. This means that for a value of P_r less than this particular value, overstable convection is preferred in the marginal state, and for a value of P_r greater than this value, steady convection will be preferred. It can also be seen from these figures that the critical Rayleigh number increases with increase in Taylor number so that rotation exerts a stabilizing influence on the flow. Further, in Figs. 4 and 5 the steady and overstable curves corresponding to chemical instability

are shown. It can be seen that for this mode of instability, steady convection is always preferred in the marginal state.

DISCUSSION

It can be seen from the figures that at the onset of chemical instability, the disturbances have shorter wave lengths than those corresponding to Bénard convection. Further equations (55) and (56) show that if $\epsilon = 0$, the values of Rayleigh number corresponding to steady and oscillatory convection revert to those for a non-reactive fluid. This is to be expected on physical grounds since from Eqs. (39) and (52), it can be seen that $\epsilon \rightarrow 0$ implies either $\tau \rightarrow 0$ or $\alpha_0 \rightarrow 0$ or 1, and these limits correspond to a non-reactive fluid since $\tau = 0$ means instantaneous chemical relaxation and $\alpha_0 = 0$ or 1 means that one is dealing with a completely diatomic or a monoatomic fluid respectively. We may further note that Eqs. (22) and (23) imply

$$\alpha' = \frac{D_{12}^0 - K_0}{M} \nabla^2 T', \quad \dots(58)$$

which by virtue of the equation of state gives

$$\delta\rho = -a\rho_0(1 - K^2\epsilon B) T', \quad \dots(59)$$

where $\delta\rho$ is the change in density caused by a small perturbation in temperature. Now for $B > 0$, the contribution to $\delta\rho$ in Eq. (59) proportional to ϵ is positive, and the net result of dissociation is to increase the density locally. This will then have a destabilizing effect on a layer heated from above.

REFERENCES

- Chandrasekhar, S. (1961). *Hydrodynamic and Hydromagnetic Stability*. Oxford.
 Lighthill, M. J. (1960). *J. Fluid. Mech.*, **8**, 161.
 Wollkind, D. J., and Frisch, H. L. (1971). *Phys. Fluids*, **14**, 13.