

ON ELECTROMAGNETIC INDUCTION IN EINSTEIN'S UNIFIED FIELD THEORY

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A cylindrically symmetric unified field associated with translatory and uniformly variable magnetic field is considered and a particular solution of the field equations in Einstein's unified field theory (Kashyap & Upadhyay 1973) is obtained.

INTRODUCTION

In Einstein's unified field theory the composite field is given by the real non-symmetric tensor $g_{\lambda\mu}$ defined as

$$g_{\lambda\mu} = \underline{g}_{\lambda\mu} + \underset{\vee}{g}_{\lambda\mu}, \quad \dots(1)$$

where $\underline{g}_{\lambda\mu}$ is symmetric and $\underset{\vee}{g}_{\lambda\mu}$ is skew-symmetric in λ and μ . The symmetric $\underline{g}_{\lambda\mu}$ coincides with the metric tensor of Riemannian space-time and the skew-symmetric $\underset{\vee}{g}_{\lambda\mu}$ is used to interpret the electromagnetic phenomena. In case the tensor $\underset{\vee}{g}_{\lambda\mu}$ exhibits cylindrical symmetry, its symmetric and skew-symmetric parts separately remain invariant under the group of transformations

$$\text{and } \left. \begin{aligned} \bar{x}^2 &= x^2 + a, \\ \bar{x}^3 &= x^3 + b, \\ \bar{x}^2 &= -x^2, \bar{x}^3 = -x^3, \end{aligned} \right\} \dots(2)$$

where a and b are arbitrary constants. Under this group of transformations, the symmetric $\underline{g}_{\lambda\mu}$ provides the space-time metric

$$ds^2 = A(dt^2 - d\rho^2) - Bd\phi^2 - Cdz^2 \quad \dots(3)$$

in cylindrical polar coordinates (ρ, ϕ, z, t) , where

$$\left. \begin{aligned} g_{11} &= -g_{44} = -A, \\ g_{22} &= -B, g_{33} = -C, \\ g_{\lambda\mu} &= 0 \quad (\lambda \neq \mu). \end{aligned} \right\} \dots(4)$$

Also, all the six components of the skew-symmetric $g_{\lambda\mu}$ exhibiting cylindrical symmetry are, in general, different from zero. It is found that all the non-zero components of $g_{\lambda\mu}$ and $g_{\lambda\mu}$ are functions of ρ and t alone.

The physical significance of the skew-symmetric g_{23} , g_{24} and g_{34} has been given by Kashyap and Upadhyay (1973), Tiwari and Kashyap (1972) and Kashyap and Ram (1975). The unified field associated with g_{23} corresponds to an electrically charged infinite cylindrical surface with its axis at $\rho = 0$. The case of g_{24} exhibits the phenomena of magnetic induction due to current along $\rho = 0$. In the case of g_{34} , there is an electric current produced by uniform rate of change of magnetic flux in the direction of $\rho = 0$. In the present paper we discuss the unified field associated with g_{13} which, like g_{34} , again gives the phenomena of electromagnetic induction.

FIELD EQUATIONS

Here we consider the unified field as characterized by

$$\text{and } \left. \begin{aligned} g_{11} &= -g_{44} = -A, \\ g_{22} &= -B, \quad g_{33} = -C, \\ g_{13} &= \psi, \end{aligned} \right\} \dots(5)$$

all being functions of ρ and t alone. The field equations used by us are those of Einstein (1953):

$$g^{\lambda\mu, \nu} - g_{\alpha\mu} \Gamma_{\lambda\nu}^{\alpha} - g_{\lambda\alpha} \Gamma_{\nu\mu}^{\alpha} = 0, \dots(6a)$$

$$\Gamma_{\lambda\alpha}^{\alpha} = 0, \dots(6b)$$

$$R_{\lambda\mu} = 0 \dots(6c)$$

and

$$R_{[\lambda\mu, \nu]} = 0, \dots(6d)$$

where

$$\begin{aligned} R_{\lambda\mu} &= \Gamma_{\lambda\mu, \alpha}^{\alpha} - \frac{1}{2} \left(\Gamma_{\lambda\alpha, \mu}^{\alpha} + \Gamma_{\mu\alpha, \lambda}^{\alpha} \right) + \Gamma_{\lambda\mu}^{\beta} \Gamma_{\beta\alpha}^{\alpha} - \Gamma_{\lambda\alpha}^{\beta} \Gamma_{\beta\mu}^{\alpha} - \Gamma_{\lambda\alpha}^{\beta} \Gamma_{\beta\mu}^{\alpha} \\ R_{\lambda\mu} &= \Gamma_{\lambda\mu, \alpha}^{\alpha} + \Gamma_{\lambda\mu}^{\beta} \Gamma_{\beta\alpha}^{\alpha} - \Gamma_{\lambda\alpha}^{\beta} \Gamma_{\beta\mu}^{\alpha} - \Gamma_{\lambda\alpha}^{\beta} \Gamma_{\beta\mu}^{\alpha} \end{aligned} \dots(7)$$

in the usual notations. Hereafter, the differentiation with respect to ρ and t are represented throughout by the suffixes 1 and 4 respectively followed by comma (e.g., $\partial A / \partial \rho = A_{,1}$ and $\partial A / \partial t = A_{,4}$).

The field equations (6a) determine the affine connections for the unified field (5). In terms of these affine connections we obtain the components of $R_{\lambda\mu}$ and $R_{\lambda\mu}^{\vee}$ from (7). Thus, from (6b) to (6d), we have the field equations as follows :—

$$\psi \left[\frac{A_{,1}}{A} + \frac{B_{,1}}{B} - \left(\frac{A_{,1}}{A} - \frac{C_{,1}}{C} - \frac{2\psi_{,1}}{\psi} \right) \left(1 + \frac{\psi^2}{AC} \right)^{-1} \right] = 0. \quad \dots(8a)$$

$$\begin{aligned} & - K_{11} - \frac{2\psi}{A} \Gamma_{\vee 4,4}^3 + \frac{\psi}{C} \Gamma_{\vee 3,1}^1 + \frac{\psi}{A} \left(\frac{5A_{,4}}{2A} - \frac{B_{,4}}{B} - \frac{C_{,4}}{C} - \frac{2\psi_{,4}}{\psi} \right) \Gamma_{\vee 4}^3 \\ & - \frac{\psi}{C} \left(\frac{A_{,1}}{2A} - \frac{\psi_{,1}}{\psi} \right) \Gamma_{\vee 3}^1 + \frac{\psi A_{,4}}{2AC} \Gamma_{\vee 4}^1 - \left(\frac{\psi}{C} \Gamma_{\vee 3}^1 \right)^2 \\ & - 2 \left(\frac{\psi}{A} \Gamma_{\vee 4}^3 \right)^2 - \frac{2\psi^2}{AC} \Gamma_{\vee 4}^3 \Gamma_{\vee 4}^3 + 2\Gamma_{\vee 4}^3 \Gamma_{\vee 4}^3 = 0. \quad \dots(8b) \end{aligned}$$

$$- K_{22} + \frac{\psi}{2A} \left(\frac{B_{,1}}{C} \Gamma_{\vee 3}^1 + \frac{B_{,4}}{C} \Gamma_{\vee 4}^1 - \frac{B_{,4}}{A} \Gamma_{\vee 4}^3 \right) = 0. \quad \dots(8c)$$

$$\begin{aligned} & - K_{33} + \frac{2\psi}{A} \left(\Gamma_{\vee 3,1}^1 + \Gamma_{\vee 4,4}^3 \right) + \frac{\psi^2}{4A^2} \left(\frac{A_{,1}^2}{A^2} + \frac{B_{,1}^2}{B^2} \right) \\ & + \left[\frac{\psi}{A} \left(\frac{B_{,1}}{B} - \frac{3C_{,1}}{2C} + \frac{2\psi_{,1}}{\psi} \right) + \left(1 + \frac{2\psi^2}{AC} \right) \Gamma_{\vee 3}^1 \right] \Gamma_{\vee 3}^1 \\ & + \frac{\psi}{A} \left(\frac{B_{,4}}{B} - \frac{3C_{,4}}{2C} + \frac{2\psi_{,4}}{\psi} - \frac{2\psi}{C} \Gamma_{\vee 4}^3 \right) \Gamma_{\vee 4}^3 \\ & - \frac{\psi}{2A^2} \left(C_{,4} + 4\psi \Gamma_{\vee 4}^3 \right) \Gamma_{\vee 4}^3 - 2\Gamma_{\vee 4}^3 \Gamma_{\vee 4}^3 = 0 \quad \dots(8d) \end{aligned}$$

$$\begin{aligned} & - K_{44} + \frac{\psi}{AC} \left(C\Gamma_{\vee 4,4}^3 - A\Gamma_{\vee 4,4}^3 \right) - \frac{\psi}{A} \left(\frac{A_{,4}}{2A} - \frac{\psi_{,4}}{\psi} + \frac{\psi}{A} \Gamma_{\vee 4}^3 \right) \Gamma_{\vee 4}^3 \\ & + \frac{\psi}{C} \left(\frac{A_{,4}}{2A} - \frac{\psi_{,4}}{\psi} - \frac{\psi}{C} \Gamma_{\vee 4}^3 \right) \Gamma_{\vee 4}^3 \\ & - \frac{\psi A_{,1}}{2AC} \Gamma_{\vee 3}^1 + 2\Gamma_{\vee 4}^3 \Gamma_{\vee 4}^3 = 0. \quad \dots(8e) \end{aligned}$$

$$\begin{aligned} & - K_{14} - \frac{\psi}{2AC} \left(A\Gamma_{\vee 4,1}^3 + C\Gamma_{\vee 4,1}^3 - A\Gamma_{\vee 3,4}^1 \right) \\ & - \frac{\psi}{2A} \left(\frac{\psi_{,1}}{\psi} - \frac{A_{,1}}{A} + \frac{B_{,1}}{B} + \frac{C_{,1}}{C} \right) \Gamma_{\vee 4}^3 \\ & + \frac{\psi}{2C} \left(\frac{A_{,1}}{A} - \frac{\psi_{,1}}{\psi} - \frac{2\psi}{C} \Gamma_{\vee 3}^1 \right) \Gamma_{\vee 4}^3 \\ & - \left[\frac{\psi}{2C} \left(\frac{A_{,4}}{A} - \frac{\psi_{,4}}{\psi} \right) + \left(1 - \frac{\psi^2}{AC} \right) \Gamma_{\vee 4}^3 \right] \Gamma_{\vee 3}^1 = 0. \quad \dots(8f) \end{aligned}$$

and

$$R_{13,4} + R_{34,1} = 0, \quad \dots(8g)$$

where

$$\begin{aligned} K_{11} &= \frac{1}{2} \left(\frac{A_{,11}}{A} + \frac{B_{,11}}{B} + \frac{C_{,11}}{C} - \frac{A_{,44}}{A} \right) \\ &\quad - \frac{1}{4} \left[2 \left(\frac{A_{,1}}{A} \right)^2 + \left(\frac{B_{,1}}{B} \right)^2 + \left(\frac{C_{,1}}{C} \right)^2 - 2 \left(\frac{A_{,4}}{A} \right)^2 \right. \\ &\quad \left. + \frac{A_{,1}B_{,1} + A_{,4}B_{,4}}{AB} + \frac{A_{,1}C_{,1} + A_{,4}C_{,4}}{AC} \right], \\ K_{22} &= \frac{1}{2A} \left[B_{,11} - B_{,44} - \frac{1}{2} \left(\frac{B_{,1}^2 - B_{,4}^2}{B} - \frac{B_{,1}C_{,1} - B_{,4}C_{,4}}{C} \right) \right], \\ K_{33} &= \frac{1}{2A} \left[C_{,11} - C_{,44} - \frac{1}{2} \left(\frac{C_{,1}^2 - C_{,4}^2}{C} - \frac{B_{,1}C_{,1} - B_{,4}C_{,4}}{B} \right) \right], \\ K_{44} &= \frac{1}{2} \left(\frac{A_{,44}}{A} + \frac{B_{,44}}{B} + \frac{C_{,44}}{C} - \frac{A_{,11}}{A} \right) \\ &\quad - \frac{1}{4} \left[2 \left(\frac{A_{,4}}{A} \right)^2 + \left(\frac{B_{,4}}{B} \right)^2 + \left(\frac{C_{,4}}{C} \right)^2 \right. \\ &\quad \left. - 2 \left(\frac{A_{,1}}{A} \right)^2 + \frac{A_{,1}B_{,1} + A_{,4}B_{,4}}{AB} + \frac{A_{,1}C_{,1} + A_{,4}C_{,4}}{AC} \right], \\ K_{14} &= \frac{1}{2} \left(\frac{B_{,14}}{B} + \frac{C_{,14}}{C} - \frac{B_{,1}B_{,4}}{2B^2} - \frac{C_{,1}C_{,4}}{2C^2} \right. \\ &\quad \left. - \frac{A_{,1}B_{,4} + A_{,4}B_{,1}}{2AB} - \frac{A_{,1}C_{,4} + A_{,4}C_{,1}}{2AC} \right) \\ R_{13} &= \Gamma_{\sqrt{V}}^1{}_{3,1} + \Gamma_{\sqrt{V}}^4{}_{13,4} + \frac{1}{2} \left(\frac{A_{,1}}{A} + \frac{B_{,1}}{B} \right) \Gamma_{\sqrt{V}}^1{}_{13} \\ &\quad + \frac{1}{2} \left(\frac{A_{,4}}{A} + \frac{B_{,4}}{B} \right) \Gamma_{\sqrt{V}}^4{}_{13} + \frac{1}{2A} \left(A_{,4} \Gamma_{\sqrt{V}}^1{}_{34} - C_{,4} \Gamma_{\sqrt{V}}^3{}_{14} \right) \\ &\quad - \frac{4\psi}{A} \Gamma_{34}^1 \Gamma_{\sqrt{V}}^3{}_{14} + \frac{\psi}{4A} \left(\frac{A_{,1}^2}{A^2} + \frac{B_{,1}^2}{B^2} \right) \\ R_{34} &= \Gamma_{\sqrt{V}}^1{}_{34,1} + \frac{1}{2} \left(\frac{A_{,1}}{A} + \frac{B_{,1}}{B} \right) \Gamma_{\sqrt{V}}^1{}_{34} + \frac{1}{2A^3} \left(A^2 C_{,1} - \psi^2 A_{,1} \right) \Gamma_{\sqrt{V}}^3{}_{14} \\ &\quad + \frac{1}{2A} \left(A_{,1} \Gamma_{\sqrt{V}}^4{}_{13} + A_{,4} \Gamma_{\sqrt{V}}^1{}_{13} \right) \\ &\quad - \frac{3\psi}{A} \Gamma_{13}^1 \Gamma_{\sqrt{V}}^3{}_{14} + \frac{\psi}{2A} \left(\frac{A_{,14}}{A} - \frac{3A_{,1}A_{,4}}{2A^2} + \frac{A_{,1}\psi_{,4}}{A\psi} \right. \\ &\quad \left. + \frac{A_{,1}B_{,4}}{2AB} - \frac{B_{,1}B_{,4}}{2B^2} \right), \end{aligned}$$

$$\Gamma_{13}^1 = \frac{\psi C}{AC + \psi^2} \left(\frac{A_{,1}}{2A} + \frac{C_{,1}}{2C} - \frac{\psi_{,1}}{\psi} \right),$$

$$2\Gamma_{13}^4 = \frac{C^2}{AC + \psi^2} \left[\left(\frac{\psi}{C} \right)_{,4} - \frac{\psi^2}{C^2} \left(\frac{\psi}{A} \right)_{,4} \right] - \frac{\psi A_{,4}}{A^2},$$

$$2\Gamma_{14}^3 = \frac{A^2 C^2}{A^2 C^2 - \psi^4} \left[- \left(\frac{\psi}{C} \right)_{,4} + \frac{\psi^2}{C^2} \left(\frac{\psi}{A} \right)_{,4} \right]$$

and

$$\Gamma_{34}^1 = \frac{\psi}{2A} \left(\frac{\psi_{,4}}{\psi} - \frac{A_{,4}}{A} \right) + \frac{\psi^2}{A^2} \Gamma_{14}^3.$$

SOLUTION OF THE FIELD EQUATIONS

The field equations (8) are nonlinear with respect to the field variables $g_{\lambda\mu}$ and are quite complicated on account of which a rigorous solution is difficult to obtain. As a linear theory accounts, with a considerable degree of accuracy, for the description of physical phenomena it is not unreasonable to assume that the field variables are small quantities. Let us take

$$\text{and } \left. \begin{aligned} A &= 1 + \alpha, \\ B &= \rho^2 + \beta \\ C &= 1 + \gamma. \end{aligned} \right\} \dots(9)$$

The order of smallness of the field variables is as follows :

$$\text{and } \left. \begin{aligned} \psi &\equiv O(1) \\ \alpha, \beta, \gamma &\equiv O(2) \end{aligned} \right\} \dots(10)$$

Upto the second order of smallness, the field equations (8) reduce to

$$\psi_{,1} + \frac{\psi}{\rho} = 0, \dots(11a)$$

$$\alpha_{,11} - \alpha_{,44} - \frac{\alpha_{,1}}{\rho} + \frac{1}{\rho} \left(\frac{\beta}{\rho} \right)_{,11} + \gamma_{,11} = -2\psi(\psi_{,11} - \psi_{,44}) - 2\psi_{,1}^2 + \psi_{,4}^2, \dots(11b)$$

$$\left(\frac{\beta}{\rho} \right)_{,11} - \left(\frac{\beta}{\rho} \right)_{,44} + \gamma_{,1} = -2\psi\psi_{,1}, \dots(11c)$$

$$\gamma_{,11} - \gamma_{,44} + \frac{\gamma_{,1}}{\rho} = -2\psi \left(2\psi_{,11} - \psi_{,44} + \frac{\psi_{,1}}{\rho} - \frac{2\psi}{\rho^2} \right) - 2\psi_{,1}^2 + \psi_{,4}^2, \dots(11d)$$

$$\alpha_{,11} - \alpha_{,44} + \frac{\alpha_{,1}}{\rho} - \frac{\beta_{,44}}{\rho^2} - \gamma_{,44} = 2\psi_{,44} + 3\psi_{,4}^2, \dots(11e)$$

$$\frac{1}{\rho} \left(\frac{\beta}{\rho} \right)_{,14} + \gamma_{,14} - \frac{\alpha_{,4}}{\rho} = -\psi \left(\psi_{,14} - \frac{\psi_{,4}}{\rho} \right) - 2\psi_{,1} \psi_{,4} \quad \dots(11f)$$

and

$$\left(\psi_{,11} - \psi_{,44} + \frac{\psi_{,1}}{\rho} - \frac{\psi}{\rho^2} \right)_{,4} = 0 \quad \dots(11g)$$

The equation (11a), after integration, gives

$$\psi = \frac{T}{\rho}, \quad \dots(12)$$

where $T[\equiv T(t)]$ is an arbitrary function. Substituting the value of ψ from (12) in (11g), we obtain

$$T_{,44} = 0 \quad \dots(13)$$

from which we have

$$T = kt, \quad \dots(14)$$

k being constant of integration. Therefore, from (12) and (14) we have

$$\psi = k \left(\frac{t}{\rho} \right). \quad \dots(15)$$

Substituting the value of ψ in the equation (11d) we obtain

$$\gamma_{,11} - \gamma_{,44} + \frac{\gamma_{,1}}{\rho} = \frac{k^2}{\rho^2} \left(1 - \frac{4t^2}{\rho^2} \right), \quad \dots(16)$$

which provides the solution

$$\gamma = -\frac{k^2}{2} \left[(\log \rho)^2 + \frac{2t^2}{\rho^2} \right]. \quad \dots(17)$$

We omit the solution of the homogeneous part since the gravitational field is purely due to electromagnetic field. Substituting the values of ψ and γ in (11c) we have

$$\left(\frac{\beta}{\rho} \right)_{,11} - \left(\frac{\beta}{\rho} \right)_{,44} = \frac{\log \rho}{\rho}, \quad \dots(18)$$

which gives the following solution

$$\beta = \frac{k^2}{2} \rho^2 (\log \rho - 2) \log \rho. \quad \dots(19)$$

Putting the values of ψ , β and γ in the equation (11f) we obtain

$$\alpha_{,4} = 0, \quad \dots(20)$$

which suggests that α is a function ρ only. Let us put

$$\alpha = F(\rho). \quad \dots(21)$$

Substituting the values of α , β , γ and ψ so obtained in the equation (11b), we obtain

$$\left(\frac{F_{,1}}{\rho}\right)_{,1} = \frac{2k^2}{\rho^3} (1 - \log \rho), \tag{22}$$

which, after integration, provides

$$F = \frac{1}{2} k^2(\log \rho - 1) \log \rho. \tag{23}$$

Therefore, from (21) and (23) we have

$$\alpha = \frac{1}{2} k^2(\log \rho - 1) \log \rho. \tag{24}$$

DISCUSSION

The results of our present investigation resemble those obtained by Kashyap and Ram (1975) in the following respects :

(a) The density of the electric current which is in the ϕ -direction varies inversely as the square of the distance from the z -axis. Classically an electric current of this type is induced by a continuous uniform increase of magnetic flux in the z -direction.

(b) The measure of force on a test particle initially at rest along ρ -line is the same in both cases and it is given by

$$\frac{d^2\rho}{dt^2} = -\frac{k^2}{4\rho} (2 \log \rho - 1). \tag{25}$$

However, the gravitational fields produced by g_{34} and g_{13} are different. The corresponding metric potentials implying this difference are as follows :

In the case of g_{34} :

$$\left. \begin{aligned} \alpha &= \frac{1}{2} k^2(\log \rho - 2t - 1) \log \rho; \\ \beta &= \frac{1}{2} k^2\rho^2 [(\log \rho - 2t - 2) \log \rho + 2]; \\ \gamma &= \frac{1}{2} k^2(\log \rho + 2t) \log \rho. \end{aligned} \right\} \tag{26}$$

and

In the case of g_{13} :

$$\left. \begin{aligned} \alpha &= \frac{1}{2} k^2(\log \rho - 1) \log \rho; \\ \beta &= \frac{1}{2} k^2\rho^2(\log \rho - 2) \log \rho; \\ \gamma &= -\frac{1}{2} k^2 \left[(\log \rho)^2 + \frac{2t^2}{\rho^2} \right]. \end{aligned} \right\} \tag{27}$$

and

From (26) and (27) it is obvious that in the present case α and β are functions of ρ alone whereas in Kashyap and Ram (1975) they are functions of ρ and t both. It is also worth noting that α in (27) is obtained by putting $t = 0$ in (26).

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