EFFECT OF SOLID PARTICLES ON THE DECAY OF HOMOGENEOUS ISOTROPIC TURBULENCE

by C. SYAMALA RAO, *Department of Chemical Engineering, Birla Institute of Technology and Science, Pilani, Rajasthan*

(*Received 12 June 1975*)

Starting with the equations of motion of a gas-solid suspension, the dynamic equation for the change in the double velocity correlation tensor was developed to study the behaviour of small solid particles on the decay of the intensity of turbulence in a homogeneous isotropic flow field. It was shown that during the initial period, the solid particles cause the absolute energy to fall to a very low level more rapidly in a non-linear manner than that of the single-phase system where the decay is of linear nature. The final period was also characterized by a similar process where during later stages the particles would settle out from the suspension.

NOMENCLATURE

A_p = projected area of solid particle

C_d = drag coefficient

d_p = particle size

f = spatial longitudinal velocity correlation coefficient

f_0'' = $(\partial^2 f/\partial r^2)_{r=0}$

h = spatial triple velocity correlation coefficient

K_i = first-order correlation tensor

k = spatial triple velocity correlation coefficient

p = instantaneous static pressure; p', turbulent fluctuation of static pressure

$Q_{i,j}$ = second-order velocity-correlation tensor

R = $\epsilon t^2/(\mu_{app}/\rho_g)$

r = distance between two points

$S_{i,kj}$ or $S_{ik,j}$ = third-order correlation tensor

t = time variable

U_g = instantaneous gas velocity; $\bar{U}_g$, time-mean value; U_g', turbulence component; $\overline{U_g'^2}$, mean square turbulence-velocity component

U_p = instantaneous particle velocity; $\bar{U}_p$, time-mean value; U_p', turbulence component

V_p = volume of individual particle

x_i = Eulerian Cartesian coordinates

α = local volume fraction of solids, $\bar{\alpha}$, time-mean value; α', solids concentration fluctuating component

ϵ = Energy dissipation by turbulence

ρ_g = density of gas

ρ_p = density of solid particle

μ_{app} = apparent viscosity of the suspension

ξ_i = Eulerian Cartesian coordinates

γ_g = Gas kinematic viscosity

Introduction

In research on air-pollution control, metallized propellant combustion in rockets and nuclear reactor coolant applications, the study of the turbulent behaviour of two-phase gas-solids suspensions is of considerable importance. With the recent progress made in understanding the mechanism of turbulence through statistical theory, interest has turned to analytical approaches aimed at a better understanding of this physically complex flow phenomenon.

In real fluids the viscous stresses in turbulent motions will cause the kinetic energy to dissipate into heat. For maintaining the turbulent motions a continuous supply of energy is needed through some external effects, otherwise the turbulent motions will decay in the course of time. A considerable amount of work has been done in the past on the problem of decay of homogeneous isotropic turbulence, the foundation having been laid by Taylor (1935) and by Kármán and Howarth (1938). A summary of the theoretical and experimental work done in this field is given in the books by Batchelor (1960), Hinze (1959) and Townsend (1956). The behaviour of solid particles in a turbulent stream depends on the concentration of the solid particles and on the intensity and scale of turbulence in the fluid. An interesting problem to investigate is how the intensity of turbulence changes during decay in the presence of solid particles. The present paper is an attempt to deduce the form of the intensity of gas-phase turbulence during decay of a homogeneous isotropic flow field containing suspended particles.

Equations of Motion

The equation of motion for a homogeneous gas-solids flow mixture with slip between the two phases can be written by neglecting the gravity and electrostatic forces as (Rao 1970)

$$(1-\alpha)\,\rho_g\,\frac{\partial U_{gi}}{\partial t}+(1-\alpha)\,\rho_g U_{gk}\,\frac{\partial U_{gi}}{\partial x_k}+\frac{d}{dt_p}\,(\alpha\,\rho_p\,U_{pi})$$
$$=-\frac{\partial p}{\partial x_i}+\mu_{app}\,\frac{\partial^2 U_{gi}}{\partial x_k \partial x_k}, \qquad \ldots(1)$$

where $\frac{d}{dt_p}$ is the substantial derivative which refers to the coordinate system moving with the particles. In the above equation, the Einstein's summation notation is employed. The terms on the L. H. S. of the equation represent the rate of change of momentum of the gas and particle phases. The first term on the R. H. S. is the total static pressure gradient in the suspension and last term is the viscous force expressed in terms of the velocity gradients. The apparent viscosity (μ_{app}) (Zuber 1964) was introduced to take into account any mutual interactions and collisions between the particles. The momentum balance equation for the particulate phase can be written as (Rao 1970)

$$\frac{d}{dt_p}(\alpha \rho_p U_{pi}) = \alpha \Sigma F_i, \qquad \text{...(2)}$$

where ΣF_i represents the sum of the interaction forces due to the fluid per unit volume of the individual particles. Here

$$\sum F_i = -\frac{\partial p}{\partial x_i} + B\rho_g \frac{d}{dt_p}(U_{gi} - U_{pi}) + \tfrac{1}{2} C_d \rho_g \frac{A_p}{V_p}(U_{gi} - U_{pi})^2. \qquad \text{...(3)}$$

In equation (3), the first, second and the last terms on the R. H. S. are the force due to the pressure gradient in the fluid, force to accelerate the apparent mass of the particle relative to the fluid, and the drag force respectively. Other interaction forces such as the "Basset" term (very small compared to viscous drag), the collision force between the particles (particles are assumed to be of uniform size), and the magnus force ($\rho_p/\rho_g >> 1$) (Hinze 1972) are neglected. It is further assumed that the particles are small compared to the smallest scale of the turbulence and that the relative motion between the fluid and the particles is small such that the drag resistance is of the Stoke's form. Therefore for spherical particles the drag force becomes

$$\tfrac{1}{2} C_d \rho_g \frac{A_p}{V_p}(U_{gi} - U_{pi})^2 = 3\,\pi \mu_{app} \frac{d_p}{V_p}(U_{gi} - U_{pi}). \qquad \text{...(4)}$$

Combining of equations (1), (2), (3) and (4) results in

$$\rho_g \frac{\partial U_{gi}}{\partial t} + \rho_g U_{gk} \frac{\partial U_{gi}}{\partial x_k} + B\alpha\rho_g \frac{d}{dt_p}(U_{gi} - U_{pi}) = -\frac{\partial p}{\partial x_i} + \mu_{app} \frac{\partial^2 U_{gi}}{\partial x_k\, \partial x_k} - 3\,\pi\mu_{app} \frac{d_p}{V_p} \alpha(U_{gi} - U_{pi}), \qquad \text{...(5)}$$

where the volume fraction of the solid particles (α) in the suspension in small enough to take $(1 - \alpha) \approx 1$. The fraction 'B' for a single particle in non-ideal flow was shown by Lumley (1957) to be equal to $\frac{1}{2}$. However, in multiparticles, flow B may differ from $\frac{1}{2}$.

Since the particle velocity is smaller than the fluid velocity and the term $B\alpha << 1$, we have

$$\rho_g \frac{\partial U_{gi}}{\partial t} >> B\alpha\rho_g \frac{\partial U_{pi}}{\partial t} \quad \text{and}$$

$$\rho_g U_{gk} \frac{\partial U_{gi}}{\partial x_k} >> B\alpha\rho_g U_{pk} \frac{\partial U_{pi}}{\partial x_k}$$

With the above considerations, equation (5) simplifies to

$$\rho_g \frac{\partial U_{gi}}{\partial t} + \rho_g U_{gk} \frac{\partial U_{gi}}{\partial x_k} = - \frac{\partial p}{\partial x_i} + \mu_{app} \frac{\partial^2 U_{gi}}{\partial x_k \partial x_k} - 3\alpha\pi\, \mu_{app} \frac{d_p}{V_p} (U_{gi} - U_p). \qquad ...(6)$$

Correlation Equation of Gas-Solids Suspension

Since the decay of turbulent intensity can be described by a change in the double velocity correlation tensor $Q_{i,j}$ with time, the equation of motion has to be transformed in such a way as to obtain other equations containing the double velocity correlation. If the instantaneous gas and solid velocities, solids concentration, and pressure are expressed as the superposition of a mean value and a fluctuating component, and further if the mean motion is uniform and steady throughout the region, the equations of motion (6) for the points A and B separated by the vector $\vec{r}$ are

$$\rho_g \frac{\partial U'_{giA}}{\partial t} + \rho_g (\bar{U}_{gk} + U'_{gk})_A \frac{\partial U'_{giA}}{\partial x_{kA}} = - \frac{\partial p'_A}{\partial x_{iA}} + \mu_{app} \frac{\partial^2 U'_{giA}}{\partial x_{kA}\, \partial x_{kA}} - \phi\Big[\bar{\alpha}_A \Big\{(\bar{U}_{gi} - \bar{U}_{pi})_A + (U'_{gi} - U'_{pi})_A \Big\} + \alpha'_A \Big\{ (\bar{U}_{gi} - \bar{U}_{pi})_A + (U'_{gi} - U'_{pi})_A \Big\}\Big] \qquad ...(7)$$

and

$$\rho_g \frac{\partial U'_{gjB}}{\partial t} + \rho_g (\bar{U}_{gk} + U'_{gk})_B \frac{\partial U'_{gjB}}{\partial x_{kB}} = - \frac{\partial p'_B}{\partial x_{iB}} + \mu_{app} \frac{\partial^2 U'_{gjB}}{\partial x_{kB}\, \partial x_{kB}} - \phi \Big[\bar{\alpha}_B\Big\{ (\bar{U}_{gj} - \bar{U}_{pj})_B + (U'_{gj} - U'_{pj})_B \Big\} + \alpha'_B \Big\{ (\bar{U}_{gj} - \bar{U}_{pj})_B + (U'_{gj} - U'_{pj})_B \Big\}\Big]. \qquad ...(8)$$

where $\phi = 3\, \pi\mu_{app} \dfrac{d_p}{V_p}$.

Multiplying equation (7) by U'_{gjB} and equation (8) by U'_{giA}, and making use of the fact that for an incompressible fluid

$$U'_{gi} \frac{\partial U'_{gk}}{\partial x_k} = 0 \text{ and } U'_{gj} \frac{\partial U'_{gk}}{\partial x_k} = 0$$

We obtain after adding the two equations together and taking the time average

$$\rho_g \frac{\partial}{\partial t} \overline{U'_{giA} U'_{gjB}} + \rho_g \bar{U}_{gk} \left[\left(\frac{\partial}{\partial x_k} \right)_A + \left(\frac{\partial}{\partial x_k} \right)_B \right] \overline{U'_{giA} U'_{gjB}}$$

$$+ \rho_g \left[\left(\frac{\partial}{\partial x_k} \right)_A \overline{U'_{giA} U'_{gkA} U'_{gjB}} + \left(\frac{\partial}{\partial x_k} \right)_B \overline{U'_{giA} U'_{gkB} U'_{gjB}} \right]$$

$$= - \left[\left(\frac{\partial}{\partial x_i} \right)_A \overline{p'_A U'_{gjB}} + \left(\frac{\partial}{\partial x_j} \right)_B \overline{p'_B U'_{giA}} \right]$$

$$+ \mu_{app} \left[\left(\frac{\partial^2}{\partial x_k \partial x_k} \right)_A + \left(\frac{\partial^2}{\partial x_k \partial x_k} \right)_B \right] \overline{U'_{giA} U'_{gjB}}$$

$$- 2 \bar{\alpha} \phi \overline{U'_{giA} U'_{giB}} . \qquad \ldots(9)$$

In arriving at equation (9), an assumption was made that the density of the solid particles is so much higher than the fluid density that the particle concentration and velocity fluctuations at one point are not effected by the fluid velocity fluctuations at the other point, and hence there is no correlation between them.

If we take point A as the origin and write

$$\xi_k = x_{kB} - x_{kA}$$

and introducing the first, second and third-order correlations in the usual notations as

$$K_{i,p} = \overline{U'_{giA} p'_B},$$

$$K_{p,j} = \overline{p'_A U'_{gjB}},$$

$$Q_{i \cdot j} = \overline{U'_{giA} U'_{gjB}},$$

$$S_{ik,j} = \overline{U'_{giA} U'_{gkA} U'_{gjB}}$$

and

$$S_{i,kj} = \overline{U'_{giA} U'_{gkB} U'_{gjB}},$$

then the dynamic equation (9) becomes

$$\frac{\partial Q_{i,j}}{\partial t} - \frac{\partial}{\partial \xi_k} S_{ik,j} + \frac{\partial}{\partial \xi_k} S_{i,kj} = - \frac{1}{\rho_g} \left[- \frac{\partial}{\partial \xi_i} K_{p,j} + \frac{\partial}{\partial \xi_j} K_{i,p} \right]$$

$$+ 2 \frac{\mu_{pp}}{\rho_g} \frac{\partial^2 Q_{i,j}}{\partial \xi_k \partial \xi_k} - \bar{2} \alpha \phi Q_{i,j}. \qquad \ldots(10)$$

Isotropic Turbulence : Decay Relations

Equation (10) forms the starting point for studying the dynamic behaviour of turbulent flow containing suspended particles. For isotropic flow certain simplifications may be effected which would make the above equations amenable to theoretical solutions. For homogeneous isotropic turbulence of an incompressible fluid

$$K_{p,j} = K_{i,p} = 0$$

and from the condition of invariance under reflections with respect to point A, we have

$$S_{i,kj} = -S_{kj,i}.$$

The differential equation then simplifies to

$$\frac{\partial Q_{i,j}}{\partial t.} - S_{i,j} = 2\,\frac{\mu_{app}}{\rho_g}\,\frac{\partial^2 Q_{i,j}}{\partial \xi_k\,\partial \xi_k} - 2\,\bar{\alpha}\phi Q_{i,j}, \qquad \ldots(11)$$

where the second-order tensor

$$S_{i,j} = \frac{\partial}{\partial \xi_k}\,S_{ik,j} + \frac{\partial}{\partial \xi_k}\,S_{kj,i}.$$

By applying a contraction

$$\frac{\partial}{\partial t}\,Q_{i,i} - S_{i,i} = 2\,\frac{\mu_{app}}{\rho_g}\,\frac{\partial^2 Q_{i,i}}{\partial \xi_k\,\partial \xi_k} - 2\bar{\alpha}\phi Q_{i,i}. \qquad \ldots(12)$$

The above equation will now be solved for the two cases : (1) the final period of decay, where the viscosity effects become predominant, so that the inertial term, $S_{i,i}$, which may be considered to originate from the interaction of the eddies, may be neglected; and (2) the initial period of decay, where the inertial forces predominate.

FINAL PERIOD

By neglecting $S_{i,i}$, equation (12) may now be written as

$$\frac{\partial}{\partial t}\,Q_{i,i}(r,t) = 2\,\frac{\mu_{app}}{\rho_g}\,\frac{1}{r^2}\,\frac{\partial}{\partial r}\left[r^2\,\frac{\partial}{\partial r}\,Q_{i,i}(r,t)\right] - 2\bar{\alpha}\phi Q_{i,i}(r,t), \qquad \ldots(13)$$

where r is the distance between two points.

Following the method of Agostini and Bass (1950), the solution of equation (13) is given as

$$Q_{i,i}(r,t) = \frac{\sqrt{8\,\frac{\mu_{app}}{\rho_g}}}{r}\,e^{-2\bar{\alpha}\phi t}\,e^{-\frac{r^2}{8\frac{\mu_{app}}{\rho_g}}t}\,\sum_{1}^{\infty}\frac{A_n}{t_n}\,H_{2n-1}\left(\frac{r}{\sqrt{8\,\frac{\mu_{app}}{\rho_g}t}}\right), \qquad \ldots(14)$$

where n is an integer and H is the Hermitian polynomial. The constants A_n are so chosen such that the series converges and that $Q_{i,i}(o, t) = 3\,\overline{U_g'^2}$, and that the Loitsianskii's (1945) integral

$$\int_0^\infty dr\, r^2\, Q_{i,i}(r, t) = 0$$

is satisfied. Hence, the solution becomes

$$Q_{i,i}(r, t) = -\frac{4A_2}{t^{5/2}}\, e^{-2\bar{\alpha}\phi t} \left[3 - \frac{r^2}{4\left(\frac{\mu_{\alpha pp}}{\rho_g}\right)t} \right] e^{-\frac{r^2}{8\left(\frac{\mu_{\alpha pp}}{\rho_g}\right)t}} \qquad \ldots(15)$$

Therefore,

$$Q_{i,i}(o, t) = -12\, A_2\, t^{-5/2}\, e^{-2\bar{\alpha}\phi t} = 3\, \overline{U_g'^2}$$

$$\text{or } \overline{U_g'^2} = -4\, A_2\, t^{-5/2}\, e^{-2\bar{\alpha}\phi t}.$$

Finally, the decay law for the turbulence intensity of a fluid containing suspended particles can be written as

$$\overline{U_g'^2} = \text{const. } t^{-5/2}\, e^{-2\bar{\alpha}\phi t}. \qquad \ldots(16)$$

Equation (16) differs from that of the single-phase system by the last term which gives the effect of solid particles on the intensity of the fluid. As can be seen that this term which consists of the concentration and size of the solid particles causes the intensity to decay at much faster rate than that of the single-phase system.

INITIAL PERIOD

By inserting the inertial term $S_{i,i}$ into equation (13), we have

$$\frac{\partial}{\partial t} Q_{i,i} - S_{i,i} = 2\,\frac{\mu_{\alpha pp}}{\rho_g}\,\frac{1}{r^2}\,\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r} Q_{i,i} \right] - 2\,\bar{\alpha}\phi Q^i{}_{,i}. \qquad \ldots(17)$$

By introducing expressions (Hinze 1959)

$$Q_{i,i} = \left(\frac{\overline{U_g'^2}}{r^2}\right) \frac{\partial}{\partial r}\left(r^3 f \right)$$

and

$$S_{i,i} = \left(\overline{U_g'^2}\right)^{3/2} \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^3 \frac{\partial k}{\partial r} + 4\, r^2 k \right)$$

and integrating equation (17) with respect to r, keeping in mind that $k = -2h$. gives

$$\frac{\partial}{\partial t}\left(\overline{U_g'^2}\, f\right) + 2\left(\overline{U_g'^2}\right)^{3/2}\left[\frac{\partial h}{\partial r} + \frac{4}{r}\, h\right] =$$
$$2\,\frac{\mu_{app}}{\rho_g}\,\overline{U_g'^2}\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\,\frac{\partial f}{\partial r}\right) - \left(\overline{U_g'^2}\, f\right) 2\,\bar{\alpha}\phi. \qquad \ldots(18)$$

Equation (18) is the famous Kármán-Howarth equation extended to include the effect of solid particles. f and h are the spatial longitudinal and triple velocity correlation coefficients respectively. The corresponding Taylor's equation for the decay of mean kinetic energy can be obtained by putting $r = 0$. It follows that

$$\frac{d\,\overline{U_g'^2}}{dt} = 10\,\frac{\mu_{app}}{\rho_g}\, f_0''\,\overline{U_g'^2} - 2\,\bar{\alpha}\phi\,\overline{U_g'^2}\,\cdot \qquad \ldots(19)$$

The dissipation of energy for isotropic turbulence may be written as

$$\epsilon = 15\,\frac{\mu_{app}}{\rho_g}\overline{\left(\frac{\partial U'_{g1}}{\partial x_1}\right)^2} = -15\,\frac{\mu_{app}}{\rho_g}\,\overline{U_g'^2}\, f_0''. \qquad \ldots(20)$$

Substituting equation (20) into equation (19) gives

$$\frac{d\,\overline{U_g'^2}}{dt} = -\frac{2}{3}\,\epsilon - 2\,\bar{\alpha}\phi\,\overline{U_g'^2}\,\cdot \qquad \ldots(21)$$

In the initial period, the decay of total energy in single-phase flow is determined mainly by that of the energy-containing eddies. This is where the intertial forces predominate and the conditions $\varepsilon t^2/\gamma_g =$ const. should apply. Similarly for a gas containing small solid particles with particular average concentration $(\bar{\alpha})$, it will be assumed that $\epsilon t^2/(\mu_{app}/\rho_g) =$ const. That is

$$\epsilon = \mathbf{R}\left(\frac{\mu_{app}}{\rho_g}\right)t^{-2}.$$

Therefore, equation (21) becomes

$$\frac{d\,\overline{U_g'^2}}{dt} + 2\,\bar{\alpha}\phi\,\overline{U_g'^2} = -\frac{2}{3}\,\mathbf{R}\left(\frac{\mu_{app}}{\rho_g}\right)t^{-2}. \qquad \ldots(22)$$

The solution of the above equation can be written as

$$\overline{U_g'^2} = -\frac{2}{3}\,\mathbf{R}\,\frac{\mu_{app}}{\rho_g}\left[-\frac{1}{t} + 2\,\bar{\alpha}\phi\, e^{-2\bar{\alpha}\phi t}\,(\ln t + 2\,\bar{\alpha}\phi\, t)\right]. \qquad \ldots(23)$$

With the integration constant taken to be equal to zero, equation (23) shows the effect of solid particles on the intensity of turbulence during the initial period of decay. Here, the solid particles cause the energy to decay in a non-linear manner at a much faster rate than that of the single-phase system where the decay is of linear nature.

The equations (16) and (23) are valid only for the Stoke's flow regime (for small relative motion between the particles and the fluid stream) and any extrapolation could lead to errors. The particles would cause extra viscous dissipation and since the turbulence is decaying, there are no external effects to replenish this viscous dissipation and the particles could no longer be suspended by the low density stream, and hence they will settle out during later stages of decay.

Since large amount of work has been done to understand the mechanism of turbulence decay in single-phase flow that some direct experimental evidence of the decay of turbulence when solids are present would be of great value in determining the flow behaviour of the suspension.

Conclusions

The effect of solid particles in a homogeneous isotropic flow field is to cause the energy of the fluid turbulence to decay more rapidly. At present there are virtually no experimental measurements available to check against the theoretical relations. Thus, a better understanding of the true nature of flow mechanism of decay of a gas-solids suspension calls for a much needed experimental study.

References

Agostini, L., and Bass, J. (1950). Les theories de la turbulence. *Publ. Sci. et tech. du ministere de l' air, No. 237, Paris.*

Batchelor, G. K. (1960). *The Theory of Homogeneous Turbulence.* Cambridge University Press, London.

Hinze, J. O. (1959). *Turbulence.* McGraw-Hill Co., Inc., New York.

——— (1972). Turbulent fluid and particle interaction. *Prog. in Heat and Mass Transfer,* Vol. VI. Pergamon Press, Oxford. pp. 433.

Karman. T. Von, and Howarth, L. (1938). On the statistical theory of isotropic turbulence. *Proc. R. Soc.,* A, **164,** 192.

Loitsianskii, L. G. (1945). Some basic laws of isotropic turbulent flow. *NACA Tech. Memo No.,* 1079, Washington.

Lumley, J. L. (1957). Some problems connected with the motion of small particles in turbulent fluid. Ph. D. dissertation, The Johns Hopkins Univ., Baltimore.

Rao, C. S. (1970). Studies in two-phase gas-solids flow. Ph.D. dissertation, Univ. Houston, Houston.

Taylor, G. I. (1935). Statistical theory of turbulence. *Proc. R. Soc.,* **A, 151,** 421

Townsend, A. A. (1956). *The Structure of Turbulent Shear Flow.* Cambridge University Press, London.

Zuber, N. (1964). On the dispersed two-phase flow in the laminar flow regime. *Chem. Engng. Sci.,* **19,** 897.