CYLINDRICAL WAVE SOLUTIONS OF A SCALAR–TENSOR THEORY OF GRAVITATION IN A LYRA MANIFOLD

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The cylindrical wave solutions for the Einstein-Rosen metric of a scalar-tensor theory of gravitation proposed by Sen and Dunn have been obtained. A method has been given by which one can construct, under certain conditions, solutions of Sen and Dunn's scalar-tensor theory from known solutions of the empty space field equations of Einstein's theory of gravitation. It has also been pointed out that one of the solutions of Sen and Dunn's scalar-tensor theory is also non-singular in the sense of Bonnor.

INTRODUCTION

Brans and Dicke (1961) have formulated a scalar-tensor theory of gravitation in which the tensor field is identified with the metric tensor of a Riemannian geometry and the scalar field is alien to the geometry. Recently Sen and Dunn (1971) proposed a new scalar-tensor theory of gravitation in a modified Riemannian manifold in which both the scalar and tensor fields have intrinsic geometrical significance. The scalar field in this theory is characterized by the function $x^0 = x^0\,(x^i)$, where x^i are the coordinates in the four-dimensional Lyra manifold and the tensor field is identified with the metric tensor g_{ij} of the manifold.

The field equations given by Sen and Dunn (1971) for the combined scalar and tensor fields are

$$R_{ij} - \tfrac{1}{2}\, g_{ij}\, R = -\, 8\, \pi\, G\, (x^0)^{-2}\, T_{ij} + \omega\, (x^{0-2}\, (x^{0}{}_{,i}\, x^0{}_{,j} - \tfrac{1}{2}\, G_{ij}\, x^0{}_{,k}\, x^{0,k}), \quad \ldots \quad (1)$$

where $\omega = 3/2$, T_{ij} is the energy-momentum tensor of the field, R_{ij} the Ricci tensor and R the usual Riemannian curvature scalar. Here comma and semicolon denote partial and covariant differentiations respectively. It was pointed out that these equations are identical with Brans-Dicke equations viz.,

$$R_{ij} - \tfrac{1}{2}\, g_{ij}\, R - \omega\, \Phi^{-2}\, \Phi_{,i}\, \Phi_{,j} - \tfrac{1}{2}\, g_{ij}\, \Phi_{,k}\, \Phi_{,k}), = -8\pi\, \Phi^{-1}\, T_{ij} +$$
$$\Phi^{-1}\, (\Phi_{,i;j} - g_{ij}\, \Box\Phi \quad \ldots \quad (2)$$

if the scalar function Φ satisfies the condition

$$\Phi_{,ij} - g_{ij}\, \Box\Phi = 0 \quad \ldots \quad (3)$$

and the Brans-Dicke constant $\omega = 3/2$. It should be added that the gravitational "constant" must be redefined as well. Sen and Dunn (1971) gave only a series type soluton to the static vacuum field equations of the scalar-tensor theory in a Lyra manifold. Halford (1972) has obtained a closed-form exact solution and has shown that the present theory predicts the same effects within the observational limits as Einstein's theory. Furthermore Reddy (1973) has shown that an analogue of Birkhoff's theorem of general relativity exists in this theory also. Recently the present author (1974) has obtained static plane symmetric solutions of this theory. In this paper we have investigated cylindrical wave solutions of these field equations for the Einstein -Rosen metric (1937, 54) and have given a method by which one can construct, under certain conditions, solutions of Sen and Dunn's scalar-tensor theory from known solutions of the empty space field equations of Einstein's theory of gravitation.

Solution of field Equations

This section deals with the general solution to the vacuum field equations of the scalar-tensor theory in a Lyra manifold in the cylindrically symmetric case with Einstein-Rosen metric. The field equations in the matter-free region are

$$R_{ij}-\tfrac{1}{2}\, g_{ij}\, R=\omega\,(x^0)^{-2}\,(x^0{}_{,i}\, x^0{}_{,j}-\tfrac{1}{2}\, g_{ij}\, x^0{}_{,k}\, x^{0,k}), \qquad \dots \quad (4)$$

where $\omega = 3/2$.

We consider the cylindrically symmetric Einstein-Rosen metric.

$$ds^2=e^{\eta-\delta}\, dt^2-dr^2-r^2e^{-\delta}\, d\phi^2-e^{\delta}dz^2 \qquad \dots \quad (5)$$

where δ and η are functions of r and t only and r, ϕ, z, t correspond respectively to x^1, x^2, x^3, x^4 coordinates. The cylindrical symmetry assumed obviously implies that the scalar field x^0 also shares the same symmetry. As a consequence we may note that $x^0{}_{,2} = x^0{}_{,3} = 0$, i.e.(x^0 is a function of r and t only.

With the metric (5) the field equations (4) reduce to the following equations :

$$\delta_{11}-\delta_{44}+(\delta_1 \mid r)=0; \qquad \dots \quad (6)$$

$$\eta_1=\frac{1}{2}\, r\left(\delta_1^2+\delta_4^2\right)-r\left(h_1^2+h_4^2\right); \qquad \dots \quad (7)$$

$$\eta_4=r\,\delta_1\,\delta_4-2r\,h_1\,h_4;\ \text{and} \qquad \dots \quad (8)$$

$$\eta_{11}-\eta_{44}+\tfrac{1}{2}\,(\delta^2{}_1-\delta^2{}_4)=h^2{}_1-h^2{}_4 \qquad \dots \quad (9)$$

where we have put

$$x^0=e^h \mid \sqrt{w}. \qquad \dots \quad (10)$$

The lower suffixes 1 and 4 after an unknown function denote partial differentiation with respect to r and t respectively. It can be easily verified that the condition of integrability for eqs (7) and (8) viz., $\eta_{14} = \eta_{41}$ is satisfield only when h satisfies

$$h_{11}-h_{44}+(h_1 \mid rem)=0 \qquad \ldots (11)$$

As Eq. (9) is obtainable from Eqs. (6), (7), (8) and (11) we shall consider the solution of these four equations only.

Eqs. (6) and (11) are Euclidean wave equations in cylindrical coordinates whose solution can be obtained by well-known methods (Coulson 1955). Eq. (6) determines δ while Eq. (11) determines the form of h, i.e., of x^0, both representing waves. Eqs. (7) and (8), on inspection, show that η consists of two parts, one depending on δ and the other depending on h or x^0 (the scalar field). Thus in the scalar-tensor theory of Sen and Dunn the space-time metric (5) depends on the tensor field g_{ij} as well as the scalar field x^0.

There are, of course, infinite number of possible combinations of δ and h that can be used to obtain a solution of Sen and Dunn's scalar-tensor theory. However, if we confine ourselves only to the case where δ and h have the same functional form so that we may write

$$\delta=ah \qquad \ldots (12)$$

where a is a constant. In view of Eq. 12, Eq. 6 reduces to Eq. 11 which determines the form of h. Use of Eqs. 12 in Eqs. 7 and 8 gives

$$\eta_1 = \frac{1}{2}k^2 r \,(h_1^2 + h_4^2 \Big) \qquad \ldots (13)$$

$$\eta_4 = k^2 r \,(h_1 h_4) \qquad \ldots (14)$$

where

$$k^2=a^2-2. \qquad \ldots (15)$$

Making the substitution

$$h=\Psi/k \text{ i.e., } x^0=e^{\Psi/k\sqrt{-}} \qquad \ldots (16)$$

in Eqs. 11, 13 and 14 we have

$$\Psi_{11}-\Psi_{44}+\Psi_1 \mid r = 0, \qquad \ldots (17)$$

$$\eta_1 = \frac{1}{2}r\,(\Psi_2^1 + \Psi_4^2). \qquad \ldots (18)$$

$$\eta_4=r\,\Psi_1\,\Psi_4. \qquad \ldots (19)$$

The condition of integrability for the Eqs. 12 and 19 viz., $\eta_{14}=\eta_{41}$ is satisfied by virtue of Eq. (17). Hence whenever Ψ is known from Eq. (17), h is determined from Eq. (16), η from Eqs. (18) and 19 and δ from Eqs. (12) and (16) in terms of Ψ being given by

$$\delta = (a \mid k)\,\Psi \qquad \ldots (20)$$

Eq. 17 is known to have a solution of the form,

$$[A\, J_0\,(k_1 r) + B\, Y_0\, k_1 r)] \cos k_1 t + \epsilon$$

where A, B, k_1, ϵ are constants and $J_0\,(k_1 r)$ and $y_0\,(k_1 r)$ are Bessel's functions of order zero and of the first and second kind respectively. Since $y_0\;(k_1 r) \rightarrow -\infty$ as $r \rightarrow 0$, we take $B = 0$ and a typical solution of Eq. (17) assumes the form

$$A\, J_0\,(k_1 r \cos k_1 t) + \epsilon. \qquad \ldots (21)$$

The general solution of Eq. (17) can obviously be obtained by superposing terms of the form (21) and from that the solution corresponding to any desired situation can be obtained by a proper choice of the arbitrary constants.

Consider now the space-time metric

$$ds^2 = e^{\eta-\Psi}\,(dt^2 - dr^2) - r^2\, e^{-\Psi} d\phi^2 - e^{\Psi}\, dz^2 \qquad \ldots (22)$$

where Ψ and η are functions of r and t only. The field equations for empty space time in Einstein's theory corresponding to Eq. (22) reduce to Eqs. (17), (18) and (19) only. Hence of the five equations (16), (17), (18), (19) and (20) taken along with the space-time metric (5) representing an empty space solution of Sen and Dunn's scalar-tensor theory, three equations viz., (17), (18) and (19) taken along with the metric (22) represent an empty space-time in Einstein's theory of gravitation. Thus, we have established the following result :

"For every solution of Eqs. (17), (18) *and* (19) *corresponding to the metric* (22) *which represents and empty space-time in Einstein's theory, we have a solution given by Eqs.* (5), (16) *and* (20), ψ *and* η *remaining the same, which represents an empty space time in the scalar-tensor theory of Sen and Dunn."*

A Non-Singular Solution of Sen and Dunn's Theory

Einstein and Rosen (1937) and Rosen (1954) obtained solutions of wave equation of the type (17) which lead to particular cases of the metric (22) corresponding to progressive or stationary gravitational waves. These solutions contain singularity along the axis of z, presumably representing the source of the waves. Later Bonnor (1957) obtained a non-singular solution of type (17) by adapting the procedure used by Synge (1956). Bonnor (1957) has shown that Eqs. (17) to (19) have non-singular solutions given by

$$\Psi = 2\sqrt{2}\, C[(u + \sqrt{u^2} + v^2)/(u + {}^2v^2)]^{\frac{1}{2}} \qquad \ldots (23)$$

$$\eta = -[\{2\, C^2\, r^2\, (u^9 - v^2)\}\,/(u^2 + v^2)^2] + (C^2/l^2)\,[\{(r^2 - t^2 - l^2) \mid \sqrt{u^2 \pm v^2}\} + 1] \qquad \ldots (24)$$

where

$$u = r^2 - t^2 + l^2,\;\; v = 2lt; \qquad \ldots (25)$$

c and l being arbitrary constants. Hence corresponding to a non-singular solution of empty space field equations of Einstein's theory given by Eqs. (22), (23), (24) and (25) we have solution of Sen and Dunn's scalar-tensor theory given by Eqs. (5), (16), (20), (23), (24) and (25). This is non-singular in the sense of Bonnor (1957).

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