RESONANCE IN THE RESTRICTED PROBLEM OF THREE BODIES WITH SHORT PERIOD PERTURBATIONS IN ELLIPTIC CASE

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This is a generalization of the authors' paper "Resonance in the restricted problem of three bodies with short periodic perturbations." Here, the motion of an asteroid moving in the gravitational field of Jupiter is considerd. Previously it was supposed that Jupiter is moving in a circular orbit around the Sun. But here in this paper, we have taken the orbit of Jupiter as elliptic. The series occurring in the problem are expanded in powers of small parameter ε, which represents the ratio of the mass of Jupiter with that of Sun. Also the perturbations in the osculating elements are obtained upto $O(\varepsilon^{3/2})$.

EQUATIONS OF MOTION

Let us suppose that Jupiter moves in an unperturbed elliptic orbit with Sun at one of its focii. Take the orbital plane of Jupiter as (x, y) plane. Let e' be the eccentricity of the Jupiter, a' its semi-major axis, λ' its mean longitude, l' its mean anomaly and n' its mean motion, and that of asteroid, the corresponding elements are denoted by e, a, λ, l and n.

The equations of motion of the asteroid with negligible mass are :

$$\frac{dx}{dt} = \frac{\partial H}{\partial \dot{x}}, \quad \frac{d\dot{x}}{dt} = -\frac{\partial H}{\partial x},$$

$$\frac{dy}{dt} = \frac{\partial H}{\partial \dot{y}}, \quad \frac{d\dot{y}}{dt} = -\frac{\partial H}{\partial y},$$

$$\frac{dz}{dt} = \frac{\partial H}{\partial \dot{z}}, \quad \frac{d\dot{z}}{dt} = -\frac{\partial H}{\partial z},$$

where the Hamiltonian

$$H = H_0 + H_1,$$

$$H_0 = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{\mu}{r}$$

and

$$H_1 = \varepsilon\mu\left(\frac{1}{\Delta} - \frac{xx' + yy'}{r'^3}\right).$$

In these equations (x, y, z) are the co-ordinates of an asteroid, $(x'\ y', o)$ the co-ordinates of Jupiter, Δ is the Jupiter-asteroid distance r' the Sun-Jupiter distance, r the Sun-asteroid distance and μ the universal constant.

Let us change the variables

$$(x, y, z; \dot{x}, \dot{y}, \dot{z}) \rightarrow (L, G, H, l, g, \tilde{h})$$

defined by the canonical transformations :

$$l = -\frac{\partial W'}{\partial L}, \quad g = -\frac{\partial W'}{\partial G}, \quad \tilde{h} = -\frac{\partial W'}{\partial H},$$

$$\dot{x} = \frac{\partial W'}{\partial x}, \quad \dot{y} = -\frac{\partial W'}{\partial y}, \quad \dot{z} = \frac{\partial W'}{\partial z}.$$

Here W' is a generating function. And L, G, H, l, g and $\tilde{h}$ are the Delaunay variables given by

$$L = \sqrt{\mu a}, \quad G = L\sqrt{1-e^2}, \quad H = G \cos i,$$

$$l = l, \quad g = \omega, \quad \tilde{h} = \Omega,$$

where i is the inclination of the orbital plane of the asteroid with the reference plane, ω the argument of the perihelion from the ascending node, and Ω the longitude of the ascending node.

The equations of motion become

$$\frac{dL}{dt} = \frac{\partial \tilde{F}}{\partial l}, \quad \frac{dl}{dt} = -\frac{\partial \tilde{F}}{\partial L},$$

$$\frac{dG}{dt} = \frac{\partial \tilde{F}}{\partial g}, \quad \frac{dg}{dt} = -\frac{\partial F}{\partial G},$$

and

$$\frac{dH}{dt} = \frac{\partial \tilde{F}}{\partial \tilde{h}}, \quad \frac{d\tilde{h}}{dt} = -\frac{\partial \tilde{F}}{\partial H}$$

with

$$\tilde{F} = \frac{\mu^2}{2L^2} + F_1, \quad F_1 = \epsilon k^2 \left[\frac{1}{\triangle} - \frac{xx^1 + yy^1}{r'^3}\right],$$

where k^2 is the Gaussian constant.

Assuming with Brouwer and Clemence (1961) the equations of motion can be written as

$$\left.\begin{aligned} \frac{dL}{dt} &= \frac{\partial F}{\partial l}, \quad \frac{dl}{dt} = -\frac{\partial F}{\partial L}, \\ \frac{dG}{dt} &= \frac{\partial F}{\partial g}, \quad \frac{dg}{dt} = -\frac{\partial F}{\partial G}, \\ \frac{dH}{dt} &= \frac{\partial F}{\partial h}, \quad \frac{dh}{dt} = -\frac{\partial F}{\partial H} \\ \text{and} \quad \frac{dK}{dt} &= \frac{\partial F}{\partial k}, \quad \frac{dk}{dt} = -\frac{\partial F}{\partial K} \end{aligned}\right\} \quad \ldots(1)$$

with

$$F = F_0 + F_1,$$

and

$$F_0 = \frac{\mu^2}{2L^2} - n'K$$

$$F_1 = \epsilon \sum C^{m_2, m_3, m_4}_{p_1, p_2, p_3, p_4} \left(\text{Sin}\ \frac{i}{2}\right)^{2m_3} e^{m_2} e'^{m_4} \times \cos\,(p_1 l + p_2 g + p_3 h + p_4 k).$$

The coefficient C's are functions of a and a' of degree—1; and the D'Alembert's relationship gives

$$\left.\begin{aligned} m_2 &= |\,j_2\,| + 2k_2 = |\,p_1 - p_2\,| + 2k_2, \\ 2m_3 &= |\,j_3\,| + 2k_3 = |\,p_2 - p_3\,| + 2k_3 \\ \text{and} \quad m_4 &= |\,j_4\,| + 2k_4 = |\,p_3 + p_4\,| + 2k_4, \end{aligned}\right\} \quad ...(2)$$

in which k_2, k_3 and k_4 are positive integers or zero.

Let us change the variables

$$(L, G, H, K;\ l, g, h, k) \rightarrow (x_1, x_2, x_3, x_4;\ y_1, y_2, y_3, y_4)$$

defined by the following canonical transformations :—

$$\left.\begin{aligned} x_1 &= L + \frac{p}{q} K\ , & y_1 &= l, \\ x_2 &= -\frac{1}{q} K\ , & y_2 &= pl + q\omega + q(\Omega - \lambda'), \\ x_3 &= G + K\ , & y_3 &= \omega \\ \text{and} \quad x_4 &= -H - K\ , & y_4 &= \omega'. \end{aligned}\right\} \quad ...(3)$$

The system of equations (1) reduces to

$$\frac{\partial x_i}{dt} = \frac{\partial K'}{\partial y_i}\ ;\ \frac{dy_i}{dt} = -\frac{\partial K'}{\partial x_i} \quad ...(4)$$

$$(i = 1, 2, 3, 4)$$

with

$$K' = K_0 + K_1,$$

$$K_0 = \frac{\mu^2}{2(x_1 + px_2)^2} + qn'x_2 \quad ...(5)$$

and

$$K_1 = \epsilon R,$$

where

$$R = \Sigma f(a, e, i, a', e') \cos\,(p_1 l + p_2\omega - p_3\omega' + p_4 l'). \quad ...(6)$$

and restrictions on p_1, p_2, p_3 and p_4 are given by the equations (2).

Short Period Perturbations

Let us eliminate the short periodic terms, i.e., the terms which contain mean anomaly in their argument. The elimination is achieved through well known Von-Zeipel method. Here we assume canonical transformations (x, y) to (ξ, η) defined by the generating function $W(\xi, y, \epsilon)$ such that the new Hamiltonian $\varphi(\xi, \eta, \epsilon)$ is free from the angular variable η_1. Also we assume the two series

$$W = W_0 + W_{1/2} + W_1 + W_{3/2} + \ldots,$$

and

$$\varphi = \varphi_0 + \varphi_{1/2} + \varphi_1 + \varphi_{3/2} + \ldots;$$

where

$$W_j = O(\epsilon^j) \text{ and } \varphi_j = O(\epsilon^j).$$

We consider the problem by assuming that

$$| pn - qn' | \leqslant n\,\epsilon^{1/2}, \qquad \ldots(7)$$

where p and q are mutually prime integers. Since W does not contain time explicitly, the Hamilton-Jacobi equation will be

$$\varphi\left(\xi; \frac{\partial W}{\partial \xi_2}, \frac{\partial W}{\partial \xi_3}, \frac{\partial W}{\partial \xi_4}, \epsilon\right) = K\left(\frac{\partial W}{\partial y}; y; \epsilon\right). \qquad \ldots(8)$$

Here ξ means ξ_1, ξ_2, ξ_3 and ξ_4 and y means y_1, y_2, y_3 and y_4.

Following the procedure of Giacaglia (1969) we shall have

$$\left.\begin{aligned}
&\varphi_0 = K_0, \\
&\varphi_{1/2} = W_{1/2} = 0, \\
&\varphi_1 = \frac{1}{2\pi q}\int_0^{2\pi q} R\,dy_1, \\
&W_1(\xi, y, \epsilon) = -\left(\frac{\partial K_0}{\partial \xi_1}\right)^{-1}\int (K_1 - \varphi_1)\,dy_1, \\
&\varphi_{3/2} = 0 \\
\text{and}\quad &W_{3/2} = -\left(\frac{\partial K_0}{\partial \xi_1}\right)^{-1}\int \left(\frac{\partial K_0}{\partial \xi_2}\right)\left(\frac{\partial W_1}{\partial y_2}\right) dy_1.
\end{aligned}\right\} \qquad \ldots(9)$$

Thus we have established the two series of W and φ upto $O(\epsilon^{3/2})$. Since we are limited upto $O(\epsilon^{3/2})$, we neglect the terms of $O(\epsilon^2)$. It may be noted that the series are of the same form as in the circular case except that the value of K differs in both the cases.

Thus in this case, i.e., up to $O(\epsilon^{3/2})$, the Hamiltonian becomes

$$\varphi^{(3/2)} = \varphi_0 + \varphi_1, \qquad \ldots(10)$$

where

$$\varphi_0 = \frac{\mu^2}{2(\xi_1 + p\xi_2)^2} + qn'\xi_2 \qquad \ldots(11)$$

and

$$\varphi_1 = \epsilon\, \Sigma\, C(a^*, e^*, i^*, a'^* e'^*) \cos[-p_1\eta_2 + (p_2 + p_1 q) \times \eta_3 - (p_3 + p_1 q)\eta_4]. \qquad \ldots(12)$$

Also up to this order we have

$$\dot{\xi}_1 = \frac{\partial \varphi^{(3/2)}}{\partial \eta_1} = 0.$$

Hence

$$\xi_1 = \text{const.}$$

or

$$L^* + \frac{p}{q} K^* = \text{const.}$$

And the short-period perturbations are given by the generating function W in an implicit form as

$$\left.\begin{aligned} x_j &= \xi_j + \frac{\partial W_1}{\partial y_j} + \frac{\partial W_{3/2}}{\partial y_j} = \xi_j + \epsilon \triangle x_j \\ \text{and} \quad \eta_j &= y_j + \frac{\partial W_1}{\partial \xi_j} + \frac{\partial W_{3/2}}{\partial \xi_j} = y_j + \epsilon \triangle y_j \end{aligned}\right\}, \qquad \ldots(13)$$

where $\triangle x_j$ and $\triangle y_j$ are short periodic terms.

Elimination of the Critical Argument

At the critical point the motion is stationary and it occurs when $pn = qn'$. Now we will further decrease the degrees of freedom by introducing a generating function S.

Here φ is a function of $(\xi; \eta_2, \eta_3, \eta_4, \epsilon)$. Let us change the Hamiltonian φ to $F(X; Y; \epsilon)$ by introducing a generating function S such that the new Hamiltonian F is independent of Y_2.

Let us change the variables

$$(\xi_1, \xi_2, \xi_3, \xi_4; \eta_2, \eta_3, \eta_4, \epsilon) \rightarrow (X_1, X_2, X_3, X_4; -, Y_3\; Y_4, \epsilon)$$

with the transformations defined by the equation

$$\xi_j = \frac{\partial S}{\partial \eta_j};\; Y_j = \frac{\partial S}{\partial X_j},\; (j = 1, 2, 3, 4)$$

We also assume that

$$S = S_0 + S_{1/2} + S_1 + S_{3/2} + \ldots,$$
$$F = F_0 + F_{1/2} + F_1 + F_{3/2} + \ldots$$

and

$$S_0 = X_1\eta_1 + X_2\eta_2 + X_3\eta_3 + X_4\eta_4,$$

where

$$S_j = O(\epsilon^j) \text{ and } F_j = O(\epsilon^j).$$

In general, the stationary solution will exist for the mean motion of the orbit and it will correspond to exact mean reasonance, i.e., at that point,

$$\left.\begin{aligned} \dot{\xi}_2 &= \frac{\partial\varphi}{\partial\eta_2} = 0, \\ \dot{\eta}_2 &= -\frac{\partial\varphi}{\partial\xi_2} = 0 \end{aligned}\right\} \quad \ldots(14)$$

and

$$pn^{**} - qn' = 0. \quad \ldots(15)$$

Here the double astrisks denote the averaged value over η_1 and η_2.

To obtain the series of S and F we will solve the Hamilton-Jacobi equation by successive approximations.

(a) If we take the case of zero order approximation then we will get

$$F_0(X_1, X_2) = \varphi_0(X_1, X_2) = \frac{\mu^2}{2}(X_1 + pX_2)^{-2} + qn'X_2. \quad \ldots(16)$$

which is constant.

(b) *Approximation of order* $(\epsilon^{1/2})$ — Taking the approximation upto $O(\epsilon^{1/2})$ we have

$$F_{1/2} = 0. \quad \ldots(17)$$

(c) *Approximation of order* (ϵ) — In this case

$$\varphi = \varphi_0 + \varphi_1,$$
$$F = F_0 + F_1$$

and

$$S = S_0 + S_{1/2} + S_1$$

Also from Eqn. (8) and taking transformations up to this order we have Hamilton-Jacobi equation as

$$\varphi\left(X + \frac{\partial S_{1/2}}{\partial\eta} + \frac{\partial S_1}{\partial\eta}; \eta_2, \eta_3, \eta_4, \epsilon\right) = F\left(X; \eta_3 + \frac{\partial S_{1/2}}{\partial X_3} + \frac{\partial S_1}{\partial X_3}, \eta_4 + \frac{\partial S_{1/2}}{\partial X_4} + \frac{\partial S_1}{\partial X_4}, \epsilon\right).$$

Expanding this equation by Taylors's expansion and considering the terms up to $O(\epsilon)$ we have

$$F_1(X;\eta_3,\eta_4,\epsilon)=\varphi_1(X;\eta_2,\eta_3,\eta_4,\epsilon)+\tfrac{1}{2}\left(\frac{\partial S_{1/2}}{\partial\eta_2}\right)^2\frac{\partial^2\varphi_0}{\partial X_2{}^2}+\frac{\partial S_{1/2}}{\partial\eta_2}\frac{\partial\varphi_0}{\partial X_2}$$

In this equation both F_1 and $S_{1/2}$ are unknown quantities. For determining these two we consider the approximate relations :

$$\left.\begin{aligned}\xi_2&=X_2+\frac{\partial S_{1/2}}{\partial\eta_2}\\ Y_2&=\eta_2+\frac{\partial S_{1/2}}{\partial X_2}\end{aligned}\right\}\qquad\ldots(18)$$

We know that X_2 is constant at any event. And by considering the Eqn. (14) we see that ξ_2 is constant. From Eqn. (14) we see, that upto $O(\epsilon)\,\dfrac{\partial^{\varphi(1)}}{\partial\eta_2}=0$ is the necessary condition for the solution. Similarly, for satisfying Eqn. (18) we find that $S_{1/2}$ should also be zero for the stable stationary solution.

Let $\eta_2=\eta_2^0(\xi;\eta_3,\eta_4,\epsilon)$ be the point of minimum of $\varphi(\xi;\eta;\epsilon)$ such that

$$\left|\frac{\partial\varphi_1}{\partial\eta_2}\right|_{\eta_2=\eta_2^0}=0.\qquad\ldots(19)$$

This point will exist because φ_1 is periodic in η_2 with period π. Now to make the condition ($S_{1/2}=0$) sufficient for the stable stationary solution we take

$$F_1(X;\eta_3,\eta_4,\epsilon)=\varphi_1(X;\eta_2^0(X;\eta_3,\eta_4,\epsilon)\,\eta_3,\eta_4,\epsilon)\qquad\ldots(20)$$

where φ_1 is given by Eqn. (12).

And the general equation defining $S_{1/2}$ is given by

$$\frac{\partial S_{1/2}}{\partial\eta_2}=\frac{L^{**}}{3p^2n^{**}}\left[-(qn'-pn^{**})\pm\left\{(qn'-pn^{**})^2-6\,\frac{p^2n^{**}}{L^{**}}\,U_1\right\}^{1/2}\right],\qquad\ldots(21)$$

where

$$U_1(X;\eta_2,\eta_3,\eta_4,\epsilon)=\varphi_1(X;\eta_2,\eta_3,\eta_4,\epsilon)-\varphi_1(X;\eta_2^0(X,\eta_3,\eta_4,\epsilon),\eta_3,\eta_4,\epsilon).$$

At the stationary solution the condition $S_{1/2}=0$ is satisfied by Eqn. (21). Also from this equation we see that in general the motion will be circular, asymptotic or libration in η_2 if

$$6\,\frac{p^2n^{**}}{L^{**}}\,U_1\lesseqqgtr(qn'-pn^{**})^2,$$

provided η_2 is taken to be maximum. U_1 is minimum at the libration centre ($\eta_2=\eta_2^0$) where it is zero and it is maximum at the end point of the oscillation.

The amplitude of vibration is given by the equation

$$U_1(X;\eta_2,\eta_3,\eta_4,\epsilon)=\frac{L^{**}}{6p^2n^{**}}(qn'-pn^{**})^2$$

and is obtained as

$$\eta_2=\bar{\eta}_2(X;\eta_3,\eta_4,\epsilon),$$

which is of order (ϵ) in this case.

Finally upto $O(\epsilon^{3/2})$ the Hamiltonian is given by

$$F=\frac{\mu^2}{2}(X_1+pX_2)^{-2}+qn'X_2+F_1(X,Y_3,Y_4,\epsilon),$$

which is a system with two degrees of freedom.

Also the parameters of the trajectory are given by the following equations :

$$\left.\begin{aligned}
a^* &= a^{**}=\text{const.}=\left(\frac{p^2\mu}{q^2n'}\right)^{1/3},\\
K^{**} &= \text{const.},\\
\dot{Y}_1 &= n^{**}-\frac{\partial F_1}{\partial X_1}=n^{**}-\epsilon R'(X;Y_3,Y_4),\\
\dot{Y}_2 &= pn^{**}-qn'-\frac{\partial F_1}{\partial X_2}=pn^{**}-qn'-\epsilon R''(X;Y_3,Y_4),\\
\dot{X}_3 &= \frac{\partial F_1}{\partial Y_3}=\epsilon h'(X;Y_3,Y_4),\\
\dot{X}_4 &= \frac{\partial F_1}{\partial Y_4}=\epsilon h''(X;Y_3,Y_4),\\
\dot{Y}_3 &= -\frac{\partial F_1}{\partial X_3}=\epsilon F'(X;Y_3,Y_4,t)\\
\text{and}\quad \dot{Y}_4 &= -\frac{\partial F_1}{\partial X_4}=\epsilon G'(X;Y_3,Y_4,t).
\end{aligned}\right\}\quad \ldots(22)$$

The period of Y_1 is $2\pi/n^{**}$ which is short, and of Y_2 is given by $2\pi/(pn^{**}-qn')$ which is long and of Y_3, X_3, Y_4 and X_4 are very long which are given by $2\pi/n^{**}\epsilon$.

(d) *Approximation of* $O(\epsilon^{3/2})$ — Taking the approximations upto $O(\epsilon^{3/2})$, we will get

$$F_{3/2}=P_{3/2}(X;\eta_2^0(X;\eta_3,\eta_4,\epsilon),\eta_3,\eta_4,\epsilon),$$

where

$$\begin{aligned}P_{3/2}(X;\eta_3,\eta_4,\epsilon)=&\frac{\partial S_{1/2}}{\partial\eta_2}\frac{\partial\varphi_1}{\partial X_2}+\frac{\partial S_1}{\partial\eta_3}\frac{\partial\varphi_1}{\partial X_3}+\frac{\partial S_{1/2}}{\partial\eta_4}\frac{\partial\varphi_1}{\partial X_4}\\
&+\tfrac{1}{6}\left(\frac{\partial S_{1/2}}{\partial\eta_2}\right)^3\frac{\partial^3\varphi_0}{\partial X_2^3}-\frac{\partial S_{1/2}}{\partial X_3}\frac{\partial F_1}{\partial\eta_3}-\frac{\partial S_{1/2}}{\partial X_4}\frac{\partial F_1}{\partial\eta_4}.\end{aligned}$$

Therefore

$$F_{3/2} = \left| \frac{\partial S_{1/2}}{\partial \eta_3} \frac{\partial \varphi_1}{\partial X_3} - \frac{\partial S_{1/2}}{\partial X_3} \frac{\partial F_1}{\partial \eta_3} + \frac{\partial S_{1/2}}{\partial \eta_4} \frac{\partial \varphi_1}{\partial X_4} - \frac{\partial S_{1/2}}{\partial X_4} \frac{\partial F_1}{\partial \eta_4} \right|_{\eta_2 = \eta_2^0} \quad \ldots(23)$$

and η_2^0 in this case is given by the equation

$$\left| \frac{\partial \varphi^{(3/2)}}{\partial \eta_2} \right|_{\eta_2 = \eta_2^0} = 0 \quad \text{or} \quad \left| \frac{\partial \varphi_1}{\partial \eta_2} \right|_{\eta_2 = \eta_2^0} = 0$$

for $\varphi_{1/2}$ and $\varphi_{3/2}$ are zero. Hence in this case location of the libration centre remains unchanged and is of the same order as in the previous case.

Also S_1 is given by the equation

$$\frac{\partial S^{(1)}}{\partial \eta_2} = \frac{L^{**}}{3p^2 n^{**}} \left[-(qn' - pn^{**}) \pm \left\{ (qn' - pn^{**})^2 - 6 \frac{p^2 n^{**}}{L^{**}} (U_1 + U_{3/2}) \right\}^{1/2} \right], \quad \ldots(24)$$

where

$$S^{(1)} = S_{1/2} + S_1$$

and

$$U_{3/2}(X;\, \eta_2, \eta_3, \eta_4, \epsilon) = P_{3/2}(X;\, \eta_2, \eta_3, \eta_4, \epsilon) - F_{3/2}(X;\, \eta_2, \eta_3, \eta_4, \epsilon). \quad \ldots(25)$$

Since in general S_1 is real, there are three possible motions in the variable η_2. The case of circular, asymptotic and libration in η_2 occurs when

$$6 \frac{p^2 n^{**}}{L^{**}} (U_1 + U_{3/2}) \lesseqgtr (qn' - pn^{**})^2$$

provided η_2 is taken to be maximum.

In the circular and asymptotic motion, S_1 is defined by choosing plus or minus sign. But in libration case the sign changes at the end point of oscillation where

$$6 \frac{p^2 n^{**}}{L^{**}} (U_1 + U_{3/2}) = (qn' - pn^{**})^2,$$

which also gives the amplitude of vibration and can be found as

$$\eta_2 = \bar{\bar{\eta}}_2(X;\, \eta_3, \eta_4, \epsilon).$$

Now upto $O(\epsilon^{3/2})$ the system is reduced to two degrees of freedom with the Hamiltonian given by

$$F = F_0 + F_1 + F_{3/2},$$

where F_0, F_1 and $F_{3/2}$ are given by Eqns. (16), (20) and (23).

Also the two integrals of motion can be found from the equations

and
$$\left.\begin{aligned} a^{**} &= \text{const.} \\ K^{**} &= \text{const.} \end{aligned}\right\} \quad \ldots(26)$$

and the other parameters of the trajectory can be found from the following six equations

and
$$\left.\begin{aligned}
\dot{Y}_1 &= n^{**} - \frac{\partial F_1}{\partial X_1} - \frac{\partial F_{3/2}}{\partial X_1} = n^{**} - \epsilon U(X;\, Y_3, Y_4, \epsilon), \\
\dot{Y}_2 &= pn^{**} - qn' - \frac{\partial F_1}{\partial X_2} - \frac{\partial F_{3/2}}{\partial X_2} = pn^{**} - qn' - \epsilon U'(X; Y_3, Y_4, \epsilon), \\
\dot{X}_3 &= \frac{\partial F_1}{\partial Y_3} + \frac{\partial F_{3/2}}{\partial Y_3} = \epsilon V(X;\, Y_3,\, Y_4, \epsilon), \\
\dot{X}_4 &= \frac{\partial F_1}{\partial Y_4} + \frac{\partial F_{3/2}}{\partial Y_4} = \epsilon V'(X;\, Y_3,\, Y_4, \epsilon), \\
\dot{Y}_3 &= -\frac{\partial F_1}{\partial X_3} - \frac{\partial F_{3/2}}{\partial X_3} = \epsilon \bar{W}(X;\, Y_3,\, Y_4, \epsilon, t) \\
\dot{Y}_4 &= -\frac{\partial F_1}{\partial X_4} - \frac{\partial F_{3/2}}{\partial X_4} = \epsilon \bar{\bar{W}}(X;\, Y_3,\, Y_4, \epsilon, t).
\end{aligned}\right\} \quad \ldots(27)$$

The period of Y_1 is $2\pi/n^{**}$ which is short. The period of Y_2 is given by $2\pi/(pn^{**} - qn')$ which is long but of X_3, X_4, Y_3 and Y_4 are very long in this case and are given by $2\pi/n^{**}\epsilon^{3/2}$.

PERTURBATIONS IN THE OSCULATING ELEMENTS UPTO $O\,(\epsilon^{1/2})$.

We see that upto $O\,(\epsilon^0)$ there are no perturbations in the osculating elements. Up to $O\,(\epsilon^{1/2})$ the variations in the osculating elements can be found out by considering the transformations:

$$\xi_j = X_j + \frac{\partial S_{1/2}}{\partial \eta_j};$$

and

$$\eta_j = Y_j - \frac{\partial S_{1/2}}{\partial X_j},\ (j = 1, 2, 3, 4).$$

We shall find and first the perturbations in Delaunay's variables and then we shall deduce the variations in the osculating elements taking terms up to $O\,(\epsilon^{1/2})$.

From Eqns. (3) we have

and
$$\begin{aligned}
L &= x_1 + px_2, & l &= y_1, \\
G &= x_3 + qx_2, & \Omega - \lambda' &= \frac{1}{q}\, y_2 - \frac{p}{q}\, y_1 - y_3, \\
H &= qx_2 - x_4, & \omega &= y_3 \\
K &= -\, qx_2, & \omega' &= y_4.
\end{aligned}$$

Also we know that

$$L^* = \xi_1 + p\xi_2 = X_1 + \frac{\partial S_{1/2}}{\partial \eta_1} + pX_2 + p\,\frac{\partial S_{1/2}}{\partial \eta_2}$$

and

$$\frac{\partial S_{1/2}}{\partial \eta_1} = 0.$$

Therefore,

$$L^* = L^{**} + p\,\frac{\partial S_{1/2}}{\partial \eta_2}.$$

Similarly,

$$G^* = G^{**} + q\,\frac{\partial S_{1/2}}{\partial \eta_2} + \frac{\partial S_{1/2}}{\partial \eta_3},$$

$$H^* = H^{**} + q\,\frac{\partial S_{1/2}}{\partial \eta_2} - \frac{\partial S_{1/2}}{\partial \eta_4} \qquad \ldots(28)$$

and

$$K^* = K^{**} - q\,\frac{\partial S_{1/2}}{\partial \eta_2}$$

The variation of the mean semi-major axis is given by

$$a^* = \frac{L^{*2}}{\mu} = \frac{1}{\mu}\left[L^{**} + p\,\frac{\partial S_{1/2}}{\partial \eta_2}\right]^2.$$

Putting the value of $\frac{\partial S_{1/2}}{\partial \eta_2}$ from Eqn. (21) in this equation we have

$$a^* = a_0^* \pm \triangle\, a^*, \qquad \ldots(29)$$

where

$$a_0^* = a^{**}\left(\tfrac{5}{3} - \tfrac{2}{3}\,\frac{qn'}{pn^{**}}\right) \qquad \ldots(30)$$

and

$$\triangle\, a^* = \tfrac{2}{3}\, a^{**}\left[\left(1 - \frac{qn'}{pn^{**}}\right) - \frac{6}{n^{**}L^{**}}\, U_1\right]^{1/2}. \qquad \ldots(31)$$

For stationary solution $a^* = a_0^* = a^{**}$.

But in general the maximum variation from the mean value a_0^* is given by putting $\eta_2 = \eta_2^0$ in Eqn. (31)

i.e., $$(\triangle\, a^*)_{\max} = \tfrac{2}{3}\, a^{**}\left(1 - \frac{qn'}{pn^{**}}\right).$$

Also from the second and third relation of Eqn. (28) we see that for the exact resonance $G^* = G^{**}$ and $H^* = H^{**}$.

The variation in eccentricity and inclination can be found if the system of equations $a^* =$ const. and $K^* =$ const. are completely integrated.
Similarly we can find the variations in the angular variables as follows :

$$\left.\begin{aligned} l^* &= l^{**} - \frac{\partial S_{1/2}}{\partial L^{**}} - \frac{\partial S_{1/2}}{\partial G^{**}}, \\ \omega^* &= \omega^{**} - \frac{\partial S_{1/2}}{\partial G^{**}}, \\ \Omega^* &= \Omega^{**} - \frac{\partial S_{1/2}}{\partial K^{**}} - \frac{\partial S_{1/2}}{\partial G^{**}} \\ \text{and} \quad \omega'^* &= \omega'^{**} - \frac{\partial S_{1/2}}{\partial H^{**}}. \end{aligned}\right\} \quad \ldots(32)$$

PERTURBATIONS IN THE OSCULATING ELEMENTS

Up to $O(\epsilon^{3/2})$

In this case the transformations are

$$\xi_j = X_j + \frac{\partial S_{1/2}}{\partial \eta_j} + \frac{\partial S_1}{\partial \eta_j},$$

$$(j = 1, 2, 3, 4)$$

$$\eta_j = Y_j + \frac{\partial S_{1/2}}{\partial X_j} + \frac{\partial S_1}{\partial X_j}$$

Again in this case also we shall first find the perturbations in the Delauny variables and from that we shall deduce the variation in the osculating elements taking terms upto $O(\epsilon)$.

Following the same procedure as in the section (IV) we have the variations in the Delauny variables as

$$\left.\begin{aligned} L^* &= L^{**} + p\,\frac{\partial S^{(1)}}{\partial \eta_2}, \\ G^* &= G^{**} + q\,\frac{\partial S^{(1)}}{\partial \eta_2} + \frac{\partial S^{(1)}}{\partial \eta_3}, \\ H^* &= H^{**} + q\,\frac{\partial S^{(1)}}{\partial \eta_2} + \frac{\partial S^{(1)}}{\partial \eta_4} \\ \text{and} \quad K^* &= K^{**} - q\,\frac{\partial S^{(1)}}{\partial \eta_2}, \end{aligned}\right\} \quad \ldots(33)$$

where

$$S^{(1)} = S_{1/2} + S_1.$$

Also the variation in the mean semi-major axis is given by

$$a^* = \frac{L^{*2}}{\mu} = \frac{1}{\mu}\left[L^{**} + p\,\frac{\partial S^{(1)}}{\partial \eta_2}\right]^2.$$

Simplifying this result, we get

$$a^* = a_0^* \pm \triangle\, a^*, \qquad \ldots(34)$$

where

$$a_0^* = a^{**}\left(\tfrac{5}{3} - \tfrac{2}{3}\,\frac{qn'}{pn^{**}}\right)$$

and

$$\triangle\, a^* = \tfrac{2}{3}\, a^{**}\left[\left(1 - \frac{qn'}{pn^{**}}\right) - \frac{6}{n^{**}L^{**}}(U_1 + U_{3/2})\right]^{1/2}. \qquad \ldots(35)$$

For exact resonance

$$a^* = a_0^* = a^{**}.$$

In general the maximum variation from the mean semi-major axis a_0^* is obtained by putting $\eta_2 = \eta_2^0$ in Eqn. (35),

$$\text{i.e.,}\quad (\triangle\, a^*)_{\max} = \tfrac{2}{3}\, a^*\left|\,1 - \frac{qn'}{pn^{**}}\right|;$$

And the variations in the eccentricity and inclination can be found if integrals of Eqn. (26) are completely solved.

Similarly we can show the variations in the angular variables as

$$\left.\begin{aligned} l^* &= l^{**} - \frac{\partial S^{(1)}}{\partial L^{**}} - \frac{\partial S^{(1)}}{\partial G^{**}},\\ \omega^* &= \omega^{**} - \frac{\partial S^{(1)}}{\partial G^{**}},\\ \Omega^* &= \Omega^{**} + \frac{\partial S^{(1)}}{\partial K^{**}} + \frac{\partial S^{(1)}}{\partial G^{**}}\\ \text{and}\qquad \omega'^* &= \omega'^{**} + \frac{\partial S^{(1)}}{\partial H^{**}}. \end{aligned}\right\} \qquad \ldots(36)$$

Hence we can find perturbations in all the osculating elements.

References

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