RESONANCE IN THE RESTRICTED PROBLEM CAUSED BY SOLAR RADIATION PRESSURE

by K. B. BHATNAGAR *and* BEENA GUPTA, *Department of Mathematics, University of Delhi, Delhi-110007*

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Resonance is discussed in the motion of an artificial Earth satellite caused by Solar-Radiation pressure. The Hamiltonian and the generating functions occurring in the problem are expanded in the power series of small parameter β, which depends on Solar Radiation pressure. Also the perturbations in the osculating elements are obtained up to $O(\beta^{1/2})$.

INTRODUCTION

Peter Musen (1960) and D. Brouwer (1967) have studied the resonance in an artificial satellite motion caused by Solar-radiation pressure. Musen *et al.* (1960) have found out the perturbations in the osculating elements by using Lagrange's planetary equations. He has found out the resonance in the case in which the perigee of the Sun closely follows the perigee of the satellite. Brouwer (1967) has given a qualitative analytical treatment of this problem and also has shown that a development in powers of the time for polar orbits in the resonance region confirms the results obtained by numerical integration. Also P. Musen *et al.* (1960) have observed that the major part of the solar gravitational effect disappears because the Earth undergoes approximately the same acceleration as the satellite, i.e., the net effect of the geocentric solar gravitational force around a complete orbit is nearly zero. In the case of the radiation pressure the resultant acceleration of the Earth is many orders of magnitude less than that of the satellite.

Here we have studied the problem of resonance under different conditions and with different approach. We have studied it by applying Von-Zeipel method and expanding the series of the Hamiltonian and the generating function in the power series of small parameter β which is proportional to the intensity of the solar-radiation and the cross section of the satellite and is treated as a constant. First we have eliminated short-period perturbations and then we have studied the resonance in the case where the mean motion of the Sun and that of the satellite are approximately in the ratio of 2 : 1. In the present study we have ignored the solar gravitational effect.

EQUATIONS OF MOTION

Suppose that the Sun moves in an unperturbed circular orbit around the Earth with mean motion n', mean anomaly l', semi-major axis a', eccentricity e' and its orbital plane coinciding with xy-plane.

The equations of motion of an artificial satellite are given by

$$\left.\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial \dot{x}}, & \frac{d\dot{x}}{dt} &= -\frac{\partial H}{\partial x}, \\ \frac{dy}{dt} &= \frac{\partial H}{\partial \dot{y}}, & \frac{d\dot{y}}{dt} &= -\frac{\partial H}{\partial y} \\ \text{and} \quad \frac{dz}{dt} &= \frac{\partial H}{\partial \dot{z}}, & \frac{d\dot{z}}{dt} &= -\frac{\partial H}{\partial z}, \end{aligned}\right\} \qquad \ldots(1)$$

where $H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \mu/r - \beta \dfrac{xx' + yy'}{r'^3}$.

(x, y, z) are the generalized co-ordinates.

$(\dot{x}, \dot{y}, \dot{z})$ are the generalized momenta variables.

(x', y', z') are the co-ordinates of the Sun with origin at the centre of the Earth.

'r' is the distance between satellite and the Earth, and r' is the distance between the Earth and the Sun. As in Bhatnagar and Beena (1974) the equations of motion (1) can be written as

$$\frac{dx_i}{dt} = \frac{\partial K}{\partial y_i}; \quad \frac{dy_i}{dt} = -\frac{\partial K}{\partial x_i}, \qquad \ldots(2)$$

$$(i = 0, 1, 2)$$

where

$$K = K_0 + K_1, \qquad \ldots(3)$$

$$K_0 = \frac{\mu^2}{2(x_0 + 2x_1)^2} + n'x_1, \qquad \ldots(4)$$

$$K_1 = \beta R, \qquad \ldots(5)$$

$$\begin{aligned} R = \frac{a}{a'^2} \Bigg[& \left(-\frac{e}{2}\right) \sin^2 \frac{i}{2} \{\cos(4y_0 - y_1 + 2y_2) \\ & 3\cos(2y_0 - y_1 + 2y_2)\} \\ & - \sin^2 \frac{i}{2} \cos(3y_0 - y_1 + 2y_2) - \frac{e^2}{2} \sin^2 \frac{i}{2} \{\cos(y_0 + 2y_2 - y_1) \\ & - \cos(5y_0 + 2y_2 - y_1)\} + \left(-\frac{e}{2}\right) \cos^2 \frac{i}{2} \{\cos y_1 \\ & 3\cos(y_1 - 2y_0)\} - \cos^2 \frac{i}{2} \cos(y_1 - y_0) \\ & - \frac{e^2}{2} \cos^2 \frac{i}{2} \{\cos(y_1 - 3y_0) - \cos(y_0 + y_1)\} \Bigg], \end{aligned} \qquad \ldots(6)$$

where x_i, y_i, the canonical variables are defined as

$$\left.\begin{aligned} x_0 &= L - 2H, & y_0 &= l, \\ x_1 &= H, & y_1 &= 2l + \omega + (\Omega - \lambda'), \\ x_2 &= G - H, & y_2 &= \omega \end{aligned}\right\} \qquad \ldots(7)$$

and a, e, i, ω, Ω, l, n are the usual Keplerian elements and L, G, H are the Delauny variables of the satellite. We have expanded the series for eccentric anomaly and true anomaly and have taken the series up to first approximation of e.

We assume that

$$| 2n - n' | \leqslant n\beta^{1/2}. \qquad \ldots(8)$$

Now first of all we will eliminate the short-periodic terms. For this we will change the Hamiltonian $K(x, y, \beta)$ to the new Hamiltonian $\varphi(\xi, \eta, \beta)$ such that it is independent of the angular variable η_0.

Let us introduce the new variables

$$(\xi_0, \xi_1, \xi_2;\ \eta_0, \eta_1, \eta_2)$$

defined by the transformations

$$x_i = \frac{\partial W}{\partial y_i};\quad \eta_i = -\frac{\partial W}{\partial \xi_i} \qquad (i = 0, 1, 2),$$

where $W(\xi, y, \beta)$ is a generating function and it is assumed to have a property that the Hamiltonian φ is independent of the angular variable η_0.

Then the corresponding Hamilton's canoncial equations in the transformed variables are

$$\frac{d\xi_i}{dt} = \frac{\partial \varphi}{\partial \eta_i};\quad \frac{d\eta_i}{dt} = -\frac{\partial \varphi}{\partial \xi_i}. \qquad \ldots(9)$$

$$(i = 0, 1, 2).$$

Also we assume that the generating function W and the Hamiltonian φ are developable in power series of $\beta^{1/2}$.

Suppose $W_0 = \xi \cdot y$. $\qquad \ldots(10)$

Now we have to solve the Hamilton-Jacobi equation

$$K\left(\frac{\partial W}{\partial y}, y, \beta\right) = \varphi\left(\xi, \frac{\partial W}{\partial \xi}, \beta\right), \qquad \ldots(11)$$

where x and y mean the vectors x_0, x_1, x_2 and y_0, y_1, y_2 respectively. Similarly ξ and η are defined. By applying Von-Zeipel method and considering that

$$\left|\frac{\partial K_0}{\partial x_1}\right| = | 2n - n' | \leqslant n\beta^{1/2}, \qquad \ldots(12)$$

we get the solution of the Hamilton-Jacobi equation as

$$\varphi_0 = \frac{\mu^2}{2(\xi_0 + 2\xi_1)^2} + n'\xi_1,$$

$$\varphi_{1/2} = 0,$$

$$W_{1/2} = 0,$$

$$\varphi_1 = \beta \frac{a^*}{a'^2}\left(-\frac{e^*}{2}\right)\cos^2\frac{i^*}{2}\cos\eta_1,$$

$$W_1 = \beta\frac{a}{na'^2}\Bigg[\left(-\frac{e}{2}\right)\sin^2\frac{i}{2}\left\{\frac{1}{4}\sin(4y_0 - y_1 + 2y_2)\right.$$
$$\left. - \tfrac{3}{2}\sin(2y_0 - y_1 + 2y_2)\right\} - \frac{1}{3}\sin^2\frac{i}{2}\sin(3y_0 - y_1 + 2y_2)$$
$$- \frac{e^2}{2}\sin^2\frac{i}{2}\left\{\sin(y_0 + 2y_2 - y_1) - \frac{1}{5}\sin(5y_0 + 2y_2 - y_1)\right\}$$
$$+ \tfrac{3}{2}\left(-\frac{e}{2}\right)\cos^2\frac{i}{2}\sin(y_1 - 2y_0) - \cos^2\frac{i}{2}\sin(y_1 - y_0)$$
$$+ \frac{e^2}{2}\cos^2\frac{i}{2}\left\{\frac{1}{3}\sin(y_1 - 3y_0) + \sin(y_0 + y_1)\right\}\Bigg],$$

$$\varphi_{3/2} = 0,$$

$$W_{3/2} = -\frac{a\beta^{3/2}}{na'^2}\Bigg[-\left(-\frac{e}{2}\right)\sin^2\frac{i}{2}\left\{\frac{1}{16}\sin(4y_0 - y_1 + 2y_2)\right.$$
$$\left. - \frac{3}{4}\sin(2y_0 - y_1 + 2y_2)\right\} + \frac{1}{9}\cos^2\frac{i}{2}\sin(3y_0 - y_1 + 2y_2)$$
$$+ \frac{e^2}{2}\sin^2\frac{i}{2}\left\{\sin(y_0 + 2y_2 - y_1) - \frac{1}{25}\sin(5y_0 + 2y_2 - y_1)\right\}$$
$$- \frac{3e}{8}\cos^2\frac{i}{2}\sin(y_1 - 2y_0) - \cos^2\frac{i}{2}\sin(y_1 - y_0)$$
$$+ \frac{e}{2}\sin^2\frac{i}{2}\left\{\frac{1}{9}\sin(y_1 - 3y_0) + \sin(y_0 + y_1)\right\}\Bigg], \qquad \ldots(13)$$

where the asterisk * indicates the average value of the corresponding element.

We have eliminated the short-periodic perturbations with the help of the generating function W and have found out the new Hamiltonian φ. Both φ and W are calculated up to $O(\beta^{3/2})$.

i.e., $\varphi = \varphi_0 + \varphi_{1/2} + \varphi_1 + \varphi_{3/2} + \cdots$

and $W = W_0 + W_{1/2} + W_1 + W_{3/2} + \cdots$

Now since φ is independent of η_0,

$$\xi_0 = \text{const.}$$

or $\quad L^* - 2H^* = \text{const.}$

The short-periodic perturbations are given by the equations

$$x_j = \xi_j + \frac{\partial W_1}{\partial y_j} + \frac{\partial W_{3/2}}{\partial y_j} + \ldots,$$

and $\quad \eta_j = y_j + \dfrac{\partial W_1}{\partial \xi_j} + \dfrac{\partial W_{3/2}}{\partial \xi_j} + \ldots \quad (j = 0, 1, 2).$

They can be found out with the help of equations (13).

Motion Near the Critical Argument

In this case the critical point occurs when $2n - n' = 0$. Also we have assumed that $| 2n - n' |$ is a slowly varying quantity. When this quantity varies, the argument η_1 will be effected and this will be the critical argument variable.

Again we will remove this term with the help of the generating function $S(X, \eta, \beta)$. Let us change the Hamiltonian $\varphi(\xi, \eta_1, \eta_2, \beta)$ to $F(X, Y, \beta)$ such that it is free from the angular variable Y_1.

Let us introduce the new variables

$$X_0, X_1, X_2, \; Y_0, Y_1, Y_2$$

with the help of the transformations defined as

$$\xi_j = \frac{\partial S}{\partial \eta_j}; \; Y_j = \frac{\partial S}{\partial X_j} \quad (j = 0, 1, 2).$$

Here also we assume that the generating function S and the new Hamiltonian F are developable in the power series of $O(\beta^{1/2})$.

Suppose $S_0 = X.\eta$.

and $\quad S = S_0 + \triangle S,$

where $\quad \triangle S = S_{1/2} + S_1 + S_{3/2} + \cdots$

and $\quad F = F_0 + F_{1/2} + F_1 + F_{3/2} + \cdots$

Here $\quad F_j = O(\beta^j)$ and $S_j = O(\beta^j)$.

Obviously S will not contain time explicitly. Hence the Hamilton-Jacobi equation is

$$\varphi\left(X + \frac{\partial \triangle S}{\partial \eta}, \eta_1, \eta_2, \beta\right) = F\left(X, \eta_2 - \frac{\partial \triangle S}{\partial X_2}, \beta\right). \qquad \ldots(14)$$

For solving this equation, we will apply Bohlin's method of successive approximations.

Now the stationary solution exists for the mean motion of the orbit and it will correspond to exact mean resonance. At that point, we have

$$\left.\begin{aligned}\dot{\xi}_1 &= \frac{\partial \varphi}{\partial \eta_1} = 0\\ \dot{\eta}_1 &= -\frac{\partial \varphi}{\partial \xi_1} = 0\end{aligned}\right\} \quad \ldots(15)$$

and $\quad 2n^* - n' = 0. \quad \ldots(16)$

Hence at the stationary point these equations will give the value of η_1, ξ_0 and ξ_1.

(i) *Solution of Hamilton-Jacobi equation by taking the approximation up to* $O(\beta^0)$ — In this case

$$F = F_0,\ \varphi = \varphi_0 \text{ and } S = S_0,$$

and the transformations are identity transformations. So we can easily find out its solution as

$$F_0 = \frac{\mu^2}{2}(X_0 + 2X_1)^{-2} + n'X_1. \quad \ldots(17)$$

(ii) *Solution of Hamilton-Jacobi equation by taking the approximations up to* $O(\beta^{1/2})$ — Now expanding the equation (14) by Taylor's series and taking the approximations up to $O(\beta^{1/2})$, also by considering $|\,2n^* - n'\,| \leqslant n\beta^{1/2}$, we get the solution as

$$F_{1/2} = 0. \quad \ldots(18)$$

Here $S_{1/2}$ is not determined. For finding out $S_{1/2}$ we have to consider next higher approximation.

(iii) *Solution of Hamilton-Jacobi equation by taking the approximations up to* $O(\beta)$ — As in Bhatnagar and Beena (1974) by taking the approximations up to $O(\beta)$ and considering that $|\,n^* - 2n'\,| \leqslant n\beta^{1/2}$, we get

$$\tfrac{1}{2}\frac{\partial^2\varphi_0}{\partial X_1^2}\left(\frac{\partial S_{1/2}}{\partial \eta_1}\right)^2 + \frac{\partial \varphi_0}{\partial X_1}\frac{\partial S_{1/2}}{\partial \eta_1} + \varphi_1(X, \eta_1, \beta) = F_1(x, \beta). \quad \ldots(19)$$

In this equation both F_1 and $S_{1/2}$ are unknown functions. For determining these two we consider the approximate transformations as

$$\xi_1 = X_1 + \frac{\partial S_{1/2}}{\partial \eta_1},$$

$$Y_1 = \eta_1 + \frac{\partial S_{1/2}}{\partial X_1}.$$

At the stationary point ($\dot{\xi}_1 = 0$, $\dot{\eta}_1 = 0$ and $2n^* - n' = 0$) we find $\xi_1 =$ const. and X_1 is constant at any moment, so $\frac{\partial S_{1/2}}{\partial \eta_1}$ must be zero at the stationary point. Also from equations (15) and (16) up to first approximations we have $\frac{\partial \varphi_1}{\partial \eta_1} = 0$, which is the necessary conditions for the solution.

Let $\eta_1 = \eta_1^0$ where $\dfrac{\partial \varphi_1}{\partial \eta_1} = 0$, then this point will be the minimum for φ and it will correspond to the Libration Centre. So from the above condition we get

$$\eta_1^0 = 0.$$

i.e., the oscillation in η_1 will be about $\eta_1 = 0$.
And for solving the equation (19) we define

$$F_1(X, \beta) = \varphi(X, \eta_1^0, \beta).$$

Therefore $F_1 = \beta \dfrac{a^{**}}{a'^2}\left(-\dfrac{e^{**}}{2}\right)\cos^2\dfrac{i^{**}}{2}$. ...(20)

For stable stationary solution the equation (19) reduces to

$$\tfrac{1}{2}\frac{\partial^2 \varphi_0}{\partial X_1^2}\left(\frac{\partial S_{1/2}}{\partial \eta_1}\right)^2 = 0.$$

Now $\dfrac{\partial^2 \varphi_0}{\partial X_1^2} \neq 0$ which is a finite quantity.

Therefore, $\dfrac{\partial S_{1/2}}{\partial \eta_1} = 0$ is the required necessary condition at the stationary point. In general $S_{1/2}$ is real and the equation for $S_{1/2}$ is given by

$$\frac{6n^{**}}{L^{**}}\left(\frac{\partial S_{1/2}}{\partial \eta_1}\right)^2 + (n' - 2n^{**})\frac{\partial S_{1/2}}{\partial \eta_1} + \beta\frac{a^{**}}{a'^2}\left(-\frac{e^{**}}{2}\right)$$
$$\cos^2\frac{i^{**}}{2}(\cos\eta_1 - 1) = 0.$$

Solving this equation we get

$$\frac{\partial S_{1/2}}{\partial \eta_1} = \frac{L^{**}}{12n^{**}}\left[-(n' - 2n^{**}) \pm \left\{(n' - 2n^{**})^2\right.\right.$$
$$\left.\left. - \frac{24\beta}{a'^2 a^{**}}\left(-\frac{e^{**}}{2}\right)\cos^2\frac{i^{**}}{2}(\cos\eta_1 - 1)\right\}^{1/2}\right]. \qquad ...(21)$$

We observe here that the motion will be circular, asymptotic or libration in η_1 if

$$\max_{\{\eta_1\}} \frac{24\beta}{a'^2 a^{**}}\left(-\frac{e^{**}}{2}\right)\cos^2\frac{i^{**}}{2}(\cos\eta_1 - 1) \lesseqgtr (n' - 2n^{**})^2 \qquad ...(22)$$

respectively. In the first two cases we will choose the plus sign (or minus) defining retrograde (or direct) orbit while in the Libration case the sign changes at the end point of the oscillation in η_1 around the Libration centre (η_1^0). There exists a set of variables (ξ_2, η_2) for which the trajectory of the orbit exists.

The amplitude of vibrations is given by the equation

$$\frac{24\beta}{a'^2 a^{**}}\left(-\frac{e^{**}}{2}\right)\cos^2\frac{i^{**}}{2}(\cos\eta_1 - 1) = (n' - 2n^{**})^2.$$

i.e., $$\eta_1 = \cos^{-1}\left[1 + \frac{(n' - 2n^{**})^2}{24\beta\left(-\dfrac{e^{**}}{2}\right)}a'^2 a^{**}\cos^2\frac{i^{**}}{2}\right], \qquad ...(23)$$

which is a constant quantity. Up to $O(\beta)$ we have

$$\dot{X}_2 = \frac{\partial F_1}{\partial Y_2} = 0 \quad \text{or } X_2 = \text{const.}$$

i.e., $$G^* + H^* = G^{**} + H^{**} = \text{const.}$$

This exists only at the Libration centre;

and $$\dot{Y}_2 = -\frac{\partial F_1}{\partial X_2}.$$

Now since $F_1 = \text{const.}$, Y_2 has a secular motion. Thus we have found out the three integrals as

$$\left.\begin{aligned} & a^{**} = \text{const.} \\ & (1 - e^{**2})^{1/2} = \text{const.} \\ \text{and} \quad & (1 - e^{**2})^{1/2} \cos i^{**} = \text{const.} \end{aligned}\right\} \qquad \ldots(24)$$

The other parameters of the trajectory are given by

$$\left.\begin{aligned} \dot{Y}_0 &= n^{**} - \frac{\partial F_1}{\partial X_0}, \\ \dot{Y}_1 &= 2n^{**} - n' - \frac{\partial F_1}{\partial X_1}, \\ \dot{Y}_2 &= \frac{\partial F_1}{\partial X_2}. \end{aligned}\right\} \qquad \ldots(25)$$

Here the period of Y_0 is short $(2\pi/n^*)$ and that of Y_1 is long and is given by $2\pi/n^{**}\,\beta^{1/2}$. Also here we observe that Y_2 has a secular motion.

Perturbations in the Osculating Elements up to $O(\beta^{1/2})$

From equations (7) we have

$$\begin{aligned} L &= x_0 + 2x_1, & l &= y_0 \\ G &= x_1 + x_2, & (\Omega - \lambda') &= y_1 - 2y_0 + y_2, \\ H &= x_1, & \omega &= y_2. \end{aligned}$$

From these equations we can find out the perturbations in the osculating elements.

We know that

$$L^* = \xi_0 + 2\xi_1 = X_0 + \frac{\partial S_{1/2}}{\partial \eta_0} + 2\left(X_1 + \frac{\partial S_{1/2}}{\partial \eta_1}\right).$$

Now up to $O(\beta^{1/2})$, $\dfrac{\partial S_{1/2}}{\partial \eta_0} = 0$.

Therefore $L^* = X_0 + 2X_1 + 2\,\frac{\partial S_{1/2}}{\partial \eta_1}$

or

$$\left.\begin{aligned} L^* &= L^{**} + 2\,\frac{\partial S_{1/2}}{\partial \eta_1}, \\ \text{Similarly}\quad G^* &= G^{**} + \frac{\partial S_{1/2}}{\partial \eta_1} \\ \text{and}\quad H^* &= H^{**} + \frac{\partial S_{1/2}}{\partial \eta_1}. \end{aligned}\right\} \qquad \ldots(26)$$

The variation in the semi-major axis can be found out as follows :

We know that

$$a^* = \frac{L^{*2}}{\mu} = \frac{1}{\mu}\left(L^{**} + 2\,\frac{\partial S_{1/2}}{\partial \eta_1}\right)^2$$

Putting the values of $\frac{\partial S_{1/2}}{\partial \eta_1}$ from equation (21) and neglecting the higher order terms i.e., $O(\beta)$ we get

$$\left.\begin{aligned} a^* &= a_0^* \pm \triangle a^*, \\ \text{where}\quad a_0^* &= a^{**}\left(\frac{5}{3} - \frac{n'}{3n^{**}}\right) \\ \text{and}\quad \triangle a^* &= \frac{2a^{**}}{3}\left[\left(1 - \frac{n'}{2n^{**}}\right)^2 - \frac{6\beta}{a'^2 n^{**} a^{**}}\left(-\frac{e^{**}}{2}\right) \cos^2\frac{i^{**}}{2}\,(\cos\eta_1 - 1)\right]^{1/2} \end{aligned}\right\} \qquad \ldots(27)$$

Here for exact resonance we have

$$a^* = a_0^* = a^{**};$$

and the maximum variation of the mean semi-major from its mean value a_0^* is given by

$$(\triangle a^*)_{max.} = \frac{2}{3}\,a^{**}\left|\,1 - \frac{n'}{2n^{**}}\,\right|.$$

Similarly we can find out the variation in eccentricity and inclination.

From equations (26) we have

$$G^* = G^{**} + \frac{\partial S_{1/2}}{\partial \eta_1},$$

or $$L^*(1 - e^{*2})^{1/2} = L^{**}(1 - e^{**2})^{1/2} + \frac{\partial S_{1/2}}{\partial \eta_1}.$$

Neglecting the terms of order higher than $\beta^{1/2}$, we get

$$(1 - e^{*2})^{1/2} = (1 - e^{**2})^{1/2} + \frac{2}{L^{**}} \frac{\partial S_{1/2}}{\partial \eta_1} [\tfrac{1}{2} - (1 - e^{**2})^{1/2}].$$

or

$$(1 - e^{*2})^{1/2} = (1 - e^{**2})^{1/2} + (\tfrac{1}{2} - (1 - e^{**2})^{1/2}) \Bigg[\left(\frac{1}{3} - \frac{n'}{6n^{**}}\right) \pm \frac{1}{3}\left\{\left(1 - \frac{n'}{2n^{**}}\right)^2 - \frac{6\beta}{a'^2 a^{**} n^{**}} \left(1 - \frac{e^{**}}{2}\right) \cos^2 \frac{i^{**}}{2} (\cos \eta_1 - 1)\right\}^{1/2}\Bigg]$$

or

$$(1 - e^{*2}) = (1 - e^{**2})^{1/2} + (\tfrac{1}{2} - (1 - e^{**2})^{1/2}) \left(\frac{1}{3} - \frac{n'}{6n^{**}}\right) \pm \frac{1}{3} \{\tfrac{1}{2} - (1 - e^{**2})^{1/2}\} \times \left[\left(1 - \frac{n'}{2n^{**}}\right)^2 - \frac{6\beta}{a'^2 a^{**} n^{**}} \left(- \frac{e^{**}}{2}\right) \cos^2 \frac{i^{**}}{2} (\cos \eta_1 - 1)\right]^{1/2}. \quad \ldots(28)$$

Therefore, in the case of exact resonance we have

$$e^* = e^{**} = \text{const.}$$

Similarly the variation in inclination is given by

$$\cos i^* = \cos i^{**} + \frac{L^{**}}{2G^{**}} \left\{1 - \cos i^{**}\right\} \Bigg[\left(\frac{1}{3} - \frac{n'}{6n^{**}}\right) \pm \frac{1}{3} \left\{\left(1 - \frac{n'}{2n^{**}}\right)^2 - \frac{6\beta}{a'^2 a^{**} n^{**}} \left(- \frac{e^{**}}{2}\right) \cos^2 \frac{i^{**}}{2} (\cos \eta_1 - 1)\right\}^{1/2}\Bigg]; \quad \ldots(29)$$

and at the exact resonance we have

$$i^* = i^{**}.$$

These constants of integration e^{**} and i^{**} can be found out by a numerical integration over a complete cycle in η_1. Similarly the perturbations in the momentum variables are given by the equations

$$\left.\begin{aligned} l^* &= l^{**} - \frac{\partial S_{1/2}}{\partial L^{**}} + \frac{\partial S_{1/2}}{\partial G^{**}}, \\ \omega^* &= \omega^{**} - \frac{\partial S_{1/2}}{\partial G^{**}} \\ \text{and} \quad \Omega^* &= \Omega^{**} - \frac{\partial S_{1/2}}{\partial G^{**}} - \frac{\partial S_{1/2}}{\partial H^{**}}. \end{aligned}\right\} \quad \ldots(30)$$

By integrating equations (21) and putting the value of $S_{1/2}$ in these equations we can find out the perturbations in the angular variables. Hence we have obtained perturbations in all the osculating elements up to $O(\beta^{1/2})$.

CONCLUSION

Peter Musen *et al.* (1960) have found out that the critical argument oscillates around 180° in the case in which the perigee of the Sun closely follows with that of the satellite. And it has a secular motion. In our case in which the mean motion of Sun and that of the satellite are in the ratio 2:1, we have found out that the critical argument η_1 oscillates about 0°. It is also seen that η_2 has a secular motion. Further we observe that the effect of solar radiation pressure on the orbit of the satellite will be in a long period. The solution obtained here can be found out up to any order of approximation by this method.

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