APPLICATION OF FATOU AND DELAUNAY-HILL MEAN SCHEMES IN THE FORMATION OF PERIODIC SOLUTIONS IN THE RESTRICTED PROBLEM OF THREE BODIES

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The existence of periodic solutions in the restricted circular problem of three bodies in the case of small perturbing masses has been shown with the help of Fatou and Delaunay-Hill mean schemes.

INTRODUCTION

As a brief introduction, the problem about the motion of the system consisting of the three material points with finite masses under mutual attractions according to the Newton's Law of Gravitation is known as the Problem of Three Bodies. The problem in which one of the material points has negligible mass as compared with the finite masses of remaining two material points, has been defined by Poincaré as restricted problem of three bodies. Finally, if the two finite bodies move in circular orbits, the problem is named as Restricted Circular Problem of Three Bodies. This problem is the most important one in celestial mechanics and has wide applications in classical celestial mechanics (lunar theory) as well as in the dynamics of cosmic flights.

PLANAR RESTRICTED PROBLEM OF THREE BODIES

Let us now discuss the planar restricted problem of three bodies using first the scheme of Fatou (Fatou 1931, Grebenikov & Ryabov 1971, and Moiseev 1945) as the corresponding differential equations can be integrated in quadrature and therefore it can be used for the formation of generating solutions in the investigation of three body problem.

The differential equations of motion in the planar circular problem of three bodies may be written as

$$\left.\begin{aligned} \frac{da}{dt} &= 2\sqrt{\frac{a}{\mu}}\,\frac{\partial R}{\partial M}, \\ \frac{dp}{dt} &= 2\sqrt{\frac{p}{\mu}}\,\frac{\partial R}{\partial \omega}, \\ \frac{dM}{dt} &= \sqrt{\frac{\mu}{a^3}} - 2\sqrt{\frac{a}{\mu}}\,\frac{\partial R}{\partial a} \\ \text{and}\quad \frac{d\tilde{\omega}}{dt} &= -N - 2\sqrt{\frac{p}{\mu}}\,\frac{\partial R}{\partial p}, \end{aligned}\right\} \quad \ldots(1)$$

where a is the semi-major axis of the osculating elliptic orbit of passively gravitating body, $p = a(1 - e^2)$ the parameter of the ellipse, e being its eccentricity, $\tilde{\omega} = (\omega - Nt)$ the argument of pericentre measured from the line joining the two finite masses, M the mean anamoly and N the relativistic mean motion of the finite masses having the form (Brumberg 1972):

$$N = 1 - \frac{3}{2} \cdot \frac{1}{c^2} \left[1 - \frac{1}{3} (1 - \mu) \mu \right],$$

C being the velocity of light. Other notations are as usual. The above equations have been written considering μ as the main attracting body and $1 - \mu$ as the perturbing body at unit distance from μ. The unit of time is so chosen that the gravitational constant may be equal to unity.

The perturbing function, in equations (1), is given by

$$R = (1 - \mu) \left(\frac{1}{\triangle} - r \cos \theta \right), \qquad \ldots(2)$$

where

$$\triangle = \sqrt{1 - 2r \cos \theta + r^2}.$$

While using Fatou scheme, the function R may be divided into two parts :

$$R = R_1 + R_2 \qquad \ldots(3)$$

where

$$R_1 = \frac{1}{2\pi} \int_0^{2\pi} R(a, p, M, \omega, t) \, dl'$$

$$= \sum_{k=0}^{\infty} C_k(a, p) \cos kM, \qquad \ldots(4)$$

which is the perturbing function in the problem of Fatou, obtained by averaging R with respect to the mean longitude l' of the perturbing mass.

When $R_2 = 0$, the eqns. (1) are integrable in quadrature

$$\int_{t_0}^{t} \sqrt{\frac{p}{\mu}} \frac{\partial R_1}{\partial p} \, dt = - \tfrac{1}{2}(\omega - \omega_0), \qquad \ldots(5)$$

$$\int_0^M \frac{dM}{\sqrt{\frac{\mu}{a^3}} - 2\sqrt{\frac{a}{\mu}} \frac{\partial R_1}{\partial a}} = t - t_0, \qquad \ldots(6)$$

$$p = p_0 = \text{Const.} \qquad \ldots(7)$$

The system of eqns. (1) has Jacobian integral

$$\frac{\mu}{2a} + \sqrt{\mu p}.\, N + R_1 = C. \qquad \ldots(8)$$

If the perturbing body is sufficiently small as compared to the main mass or if the passively gravitating mass is situated in the sufficiently close neighbourhood of the main body, then

$$| R_2 | \ll | R_1 |$$

and in that case the small parameter method can be used.

The ordinary periodicity conditions may be simplified with the help of symmetry theorem. Equations (1), as is clear from (4), are invariant with respect to the substitution

$$t \to -t,\ a(t) \to a(-t),\ p(t) \to p(-t),$$
$$\omega(t) \to -\omega(-t),\ M(t) \to -M(-t),$$

which will allow us to bring the periodicity conditions to the form mentioned by Kurcheeva (1972).

Here, we will assume that the perturbing body is small in mass i.e., in (2) $1 - \mu \approx 0$. In that case, we can use the transformations in the axpression on the left of (6):

$$\frac{1}{\sqrt{\frac{\mu}{a^3}} - 2\sqrt{\frac{a}{\mu}}\left(\frac{\partial C_0}{\partial a} + \sum_{k=1}^{\infty} \frac{\partial C_k}{\partial a} \cos kM\right)}$$

$$= \frac{1}{\sqrt{\frac{\mu}{a^3}} - 2\sqrt{\frac{a}{\mu}}\frac{\partial C_0}{\partial a}} \cdot \left[1 + \frac{2\sqrt{\frac{a}{\mu}}}{\sqrt{\frac{\mu}{a^3}} - 2\sqrt{\frac{a}{\mu}}\frac{\partial C_0}{\partial a}} \times \right.$$

$$\left. \sum_{k=1}^{\infty} \frac{\partial C_k}{\partial a} \cos kM + \ldots \right] = \frac{1}{n_1}\left(1 + \sum_{k=1}^{\infty} L_k \cos kM\right), \qquad \ldots(9)$$

where

$$L_k = \frac{2\sqrt{\frac{a}{\mu}}}{\sqrt{\frac{\mu}{a^3}} - 2\sqrt{\frac{a}{\mu}}\frac{\partial C_0}{\partial a}} \cdot \frac{\partial C_k}{\partial a} \qquad \ldots(10)$$

and

$$n_1 = \sqrt{\frac{\mu}{a^3}} - 2\sqrt{\frac{a}{\mu}}\frac{\partial C_0}{\partial a}. \qquad \ldots(11)$$

The coefficient L_k has the order of perturbing mass.

Integrating (6) we get

$$n_1(t - t_0) = M + \sum_{k=1}^{\infty} \frac{1}{k} L_k \sin kM \qquad ...(12)$$

This equation may be solved with respect to M, using lagrange's series (we mean the equation $z - a - \mu f(z, \mu) = 0$, where μ is small) :

$$M = n_1(t - t_0) + \sum_{k=1}^{\infty} P_k \sin kn_1(t - t_0). \qquad ...(13)$$

The anologous procedure may be used in connection with the equation (5) :

$$\sqrt{\frac{p}{\mu}} \int_{t_0}^{t} \left[\frac{\partial C_0}{\partial p} + \sum_{k=1}^{\infty} \frac{\partial C_k}{\partial p} \cos kM\right] dt$$

$$= \sqrt{\frac{p}{\mu}} \left\{\frac{\partial C_0}{\partial p}(t - t_0) + \sum_{k=1}^{\infty} \frac{\partial C_k}{\partial p} \int_{t_0}^{t} \cos kM \, dt\right\} = \omega - \omega_0.$$

Here, taking the terms $\frac{\partial C_k}{\partial p}$ out from under the integral sign is justified, as C_k do'nt depend on time.

Taking the smallness of coefficients P_k, $\cos kM$ can be expressed in the form of Fourier series in multiple of $n_1(t - t_0)$ and the above can be integrated.

Then we get

$$\omega - \omega_0 = n_2(t - t_0) + \sum_{k=1}^{\infty} S_k \sin kn_1(t - t_0) \qquad ...(14)$$

In the formula (14)

$$n_2 = \sqrt{\frac{p}{\mu}} \frac{\partial C_0}{\partial p}, \qquad ...(15)$$

and the coefficients S_k are proportional to small perturbing mass.

The above discussions are true, provided

$$a\left(1 - \sqrt{1 - \frac{p}{a}}\right) > 1.$$

The generating periodic solution in the Fatou scheme will take place when frequencies n_1 and n_2 are commensurable and has complicated form due to the presence of the trigonometrical series. Moreover, for sufficiently small R_2 the question about the existence of periodic solutions will be discussed with the help of determinant of the type

$$\frac{D(n_1, n_2)}{D(a, p)} \neq 0,$$

in which the values of n_1 and n_2 are taken from the formulae (11) and (15).

The value of the coefficient C_0, obtained from (4), may be written as :

$$C_0 = (1 - \mu)\left[1 + \frac{3a}{2} - \frac{p}{2} + \frac{9}{16} a^2 + \frac{p^2}{16} - \frac{3ap}{8}\right].$$

Here the terms of second order in eccentricity are included.

The determinant defining the existence of periodic solutions is

$$\frac{D(n_1, n_2)}{D(a, p)} = \frac{3(1 - \mu)}{16}\left[\frac{(1 - \mu)(3a - p + 4)}{2\mu\sqrt{ap}}\left(\frac{\mu}{a^2(1 - \mu)} + \frac{9}{4} a - \frac{3}{4} p + 1\right) - \frac{\sqrt{p}}{a^{5/2}}\right] \neq 0.$$

In this way, the periodic solutions obtained from Fatou problem in the circular problem of three bodies, in the case of small perturbing masses exist in both classical as well as relativistic consideration.

We will now discuss in brief the planar resistricted circular problem of three bodies using first scheme of Delaunay-Hill (Grebenikov & Ryabov 1971). In this case the differential equations of motion may be written as in (1), however the perturbing function must be written in the way (Grebenikov 1970) :

$$R = R_b + R_g + R_H, \qquad \text{...(16)}$$

where R_b is the secular part of perturbing function, R_g the long-periodic part of perturbing function and R_H the short-periodic. This representation has a meaning in the case of exact or acute commensurability of mean motions of perturbing or perturbed bodies, i.e., in the neighbourhood of resonance motions. In this case it is worth taking the simplified system with pertubing function

$$R_0 = R_b + R_g \qquad \text{...(17)}$$

the expansion of which has the form

$$R_0 = \mu \sum_{s=0}^{\infty} C_{s\bar{k}_1, s\bar{k}_2}(a, p) \cos sD, \qquad \text{...(18)}$$

where $C_{s\bar{k}_1, s\bar{k}_2}$ are coefficients of Laplace and D the anomaly of Delaunay :

$$D = \bar{k}_1 M + \bar{k}_2(\omega - l'). \qquad \text{...(19)}$$

Here $\bar{k}_1$, $\bar{k}_2$ are the fixed integral numbers and l^1 is the longitude of perturbing mass. This function, as is clear (Grebenikov 1970) is obtained by averaging the perturbing function R with respect to M remained in R after expressing $\omega - l'$ through D and M.

As $\frac{\partial}{\partial M} = \bar{k}_1 \frac{\partial}{\partial D}, \frac{\partial}{\partial \omega} = \bar{k}_2 \frac{\partial}{\partial D}$

we will have, in place of (1)

$$\left.\begin{aligned} \frac{da}{dt} &= 2\bar{k}_1 \sqrt{\frac{a}{\mu}} \frac{\partial R}{\partial D}, \\ \frac{dp}{dt} &= 2\bar{k}_2 \sqrt{\frac{p}{\mu}} \frac{\partial R}{\partial D}, \\ \frac{dM}{dt} &= \sqrt{\frac{\mu}{a^3}} - 2\sqrt{\frac{a}{\mu}} \frac{\partial R}{\partial a} \\ \text{and} \quad \frac{d\bar{\omega}}{dt} &= -N - 2\sqrt{\frac{p}{\mu}} \frac{\partial R}{\partial p}. \end{aligned}\right\} \qquad \ldots(20)$$

When $R_H = 0$, we get simplified system which gives first integrals :

$$\sqrt{a} - \frac{\bar{k}_1}{\bar{k}_2} \sqrt{p} = C_1, \qquad \ldots(21)$$

$$\frac{\mu}{2a} + \frac{\bar{k}_2}{\bar{k}_1} N\sqrt{\mu a} + R_0 = C_2. \qquad \ldots(22)$$

The integral (22) is obtained by combining the Jacobian integral of the system with the integral (21).

We have from (21) and (22)

$$a = a(D, C_1, C_2), \; p = p(D, C_1, C_2)$$

and we substitute for M and $\bar{\omega}$ in the differential equations.

The last but one equation in (20) may be replaced by the differential equation in D, the anomaly of Delaunay :

$$\frac{dD}{dt} = \bar{k}_1 \sqrt{\frac{\mu}{a^3}} - \bar{k}_2 N - 2\bar{k}_1 \sqrt{\frac{a}{\mu}} \frac{\partial R_0}{\partial a} - 2\bar{k}_2 \sqrt{\frac{p}{\mu}} \frac{\partial R_0}{\partial p}. \qquad \ldots(23)$$

Then, the problem is integrable in quadrature of the type

$$\phi(D, C_1, C_2) = t - t_0,$$

$$\bar{\bar{\omega}} - \bar{\omega}_0 - Nt = -\frac{2}{\sqrt{\mu}} \int_{t_0}^{t} \sqrt{p} \frac{\partial R_0}{\partial p} dt \qquad \ldots(24)$$

The perturbing function of circular problem of three bodies may, with the help of (19), be given the form :

$$R = \mu \sum_{k_1, k_2} C_{k_1, k_2}(a, p) \cos\left[\left(k_1 - k_2 \frac{\bar{k}_1}{\bar{k}_2}\right) M + \frac{k_2}{\bar{k}_2} D\right] \qquad \ldots(25)$$

It is clear from the expansion (25) that the equations of perturbed motion are invariants with respect to the substitution

$$
\begin{aligned}
& t \to -t,\ a(t) \to a(-t),\ p(t) \to p(-t), \\
& \omega(t) \to -\omega(-t),\ M(t) \to -M(-t), \\
& [\text{or } D(t) \to -D(-t)].
\end{aligned}
$$

Therefore, as in the problem of Fatou, here also it is possible to show the existence of a family of symmetric periodic solutions in the circular problem of three bodies. The further analysis is parallel to that in the case of Fatou problem and therefore we will omit them here. It is only necessary that the perturbed mean motions for the elements M and $\bar{\omega}$ (or D and $\bar{\omega}$) be commensurable. The commensurability condition is obtained from eqns. (20) with the help of (24)

$$s_1 n_1 + s_2 n_2 = 0,$$

where s_1, s_2 — any integral numbers and

$$n_1 = \sqrt{\frac{\mu}{a^3}} - \sqrt{\frac{a}{\mu}} \cdot \frac{1}{\pi} \int_0^{2\pi} \frac{\partial \bar{R}_0}{\partial a}\, dD,$$

$$n_2 = -N - \sqrt{\frac{p}{\mu}} \cdot \frac{1}{\pi} \int_0^{2\pi} \frac{\partial \bar{R}_0}{\partial p}\, dD,$$

in which $\bar{R}_0$ represents the result of substitution in R_0 the expressions, in terms of D, of a and p.

Hence we conclude by proving the existence of symmetric periodic solutions in the circular problem of three bodies using first scheme of Delaunay-Hill. This result holds good equally in the classical and relativistic treatments.

The above results, proving the existence of symmetric periodic solutions in the restricted circular problem of three bodies with small perturbing masses using Fatou and Delaunay-Hill Mean Schemes, are an improvement upon the various methods in the literature and in many aspects they appear to be the new ones. These considerations give a picture more close to the real existing problems.

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