ON HYDROGEN ATOM IN EINSTEIN'S UNIFIED FIELD THEORY

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A static unified field, which exhibits axial symmetry, is considered. The solution of Einstein's field equations (1953) gives the structure of the space-time of an electrically neutral Hydrogen atom.

1. INTRODUCTION

According to electron theory the picture of an atom is like that of the solar system. The simplest atom is that of hydrogen which consists of one proton in its nucleus and a revolving electron. Its orbital plane behaves like a magnet. Following the theory due to which there is magnetism in the hydrogen atom it looks as if the revolving moon plays as well a role in the formation of terrestrial magnetism as the earth with its moon in the solar system is an analogue of hydrogen atom. In the present paper, it has been merely attempted to report what Einstein's unified field theory (1953) gives about hydrogen atom.

In Einstein's unified field theory (1953) the total field is given by the real non-symmetric tensor $g_{\lambda\mu}$ which has sixteen components. It may be split into its symmetric part $g_{\underline{\lambda\mu}}$ and its skew-symmetric part $g_{\underset{\vee}{\lambda\mu}}$:

$$g_{\lambda\mu} = g_{\underline{\lambda\mu}} + g_{\underset{\vee}{\lambda\mu}} \qquad \ldots \quad (1.1)$$

The symmetric $g_{\underline{\lambda\mu}}$ coincides with the metric tensor of Riemannian space-time and the skew-symmetric $g_{\underset{\vee}{\lambda\mu}}$ is used to interpret the electromagnetic phenomena.

In the present investigation, we consider the total field as characterized by

$$\begin{aligned} g_{11} &= g_{33} = -A \\ g_{22} &= -B, \; g_{44} = C \\ g_{\underset{\vee}{14}} &= \eta, \; g_{\underset{\vee}{34}} = \chi \end{aligned} \qquad \ldots \quad (1.2)$$

all being functions of ρ and z alone. In cylindrical polar coordinates, the corresponding space-time metric is

$$ds^2 = -A(d\rho^2 + dz^2) - B\, d\phi^2 + C\, dt^2 \qquad \ldots \quad (1.3)$$

which exhibits axial symmetry with $\rho = 0$ on the axis of symmetry. The field equations used here are :

$$g_{\lambda\mu,\nu} - g_{\alpha\mu}\Gamma^{\alpha}_{\lambda\nu} - g_{\lambda\alpha}\Gamma^{\alpha}_{\nu\mu} = 0, \qquad \ldots \quad (1.4a)$$

$$\Gamma^{\alpha}_{\underset{\vee}{\lambda\alpha}} = 0, \qquad \text{... (1.4}b\text{)}$$

$$R_{\underline{\lambda\mu}} = 0, \qquad \text{... (1.4}c\text{)}$$

$$R_{\underset{\vee}{\lambda\mu},\nu} + R_{\underset{\vee}{\mu\nu},\lambda} + R_{\underset{\vee}{\nu\lambda},\mu} = 0, \qquad \text{... (1.4}d\text{)}$$

with

$$R_{\underline{\lambda\mu}} = \Gamma^{\alpha}_{\underline{\lambda\mu},\alpha} - \tfrac{1}{2}\left(\Gamma^{\alpha}_{\underline{\lambda\alpha},\mu} + \Gamma^{\alpha}_{\underline{\mu\alpha},\lambda}\right) + \Gamma^{\alpha}_{\underline{\lambda\mu}}\Gamma^{\beta}_{\underline{\alpha\beta}} - \Gamma^{\alpha}_{\underline{\lambda\beta}}\Gamma^{\beta}_{\underline{\alpha\mu}} - \Gamma^{\alpha}_{\underset{\vee}{\lambda\beta}}\Gamma^{\beta}_{\underset{\vee}{\alpha\mu}}, \qquad \text{... (1.5}a\text{)}$$

$$R_{\underset{\vee}{\lambda\mu}} = \Gamma^{\alpha}_{\underset{\vee}{\lambda\mu},\alpha} + \Gamma^{\alpha}_{\underset{\vee}{\lambda\mu}}\Gamma^{\beta}_{\underline{\alpha\beta}} - \Gamma^{\alpha}_{\underline{\lambda\beta}}\Gamma^{\beta}_{\underset{\vee}{\alpha\mu}} - \Gamma^{\alpha}_{\underset{\vee}{\lambda\beta}}\Gamma^{\beta}_{\underline{\alpha\mu}}, \qquad \text{... (1.5}b\text{)}$$

in the usual notations. Here and in what follows a comma (,) preceding a suffix denotes partial differentiation. Differentiation with respect to ρ and z is represented throughout by the suffixes 1 and 3 respectively.

2. Field Equations

The field equations (1.4a) determine the affine connections. For the total field (1.2) there are twenty nonvanishing components of the affine connections out of which 14 are symmetric and the remaining 6 are skew-symmetric. In terms of these affine connections we obtain the components of $R_{\underline{\lambda\mu}}$ and $R_{\underset{\vee}{\lambda\mu}}$ from (1.5), Thus, from (1.4b) — (1.4d), we have the field equations as follows :

$$\Gamma^{\alpha}_{\underset{\vee}{4\alpha}} = 0 \qquad \text{... (2.1}a\text{)}$$

$$\begin{aligned}
K_{11} &+ \frac{2\eta}{A}\Gamma^{4}_{\underset{\vee}{13},3} - \frac{\chi}{A}\Gamma^{4}_{\underset{\vee}{13},1} - \frac{\chi}{C}\Gamma^{3}_{\underset{\vee}{14},1} + \frac{\eta}{C}\Gamma^{1}_{\underset{\vee}{41},1} + \frac{\eta}{C}\left(\frac{\eta_{,1}}{\eta} - \frac{A_{,1}}{2A} - \frac{\eta}{C}\Gamma^{1}_{\underset{\vee}{41}}\right)\Gamma^{1}_{\underset{\vee}{41}} \\
&- \frac{\chi}{C}\left(\frac{\chi_{,1}}{\chi} - \frac{A_{,1}}{2A} - \frac{2\eta}{C}\Gamma^{1}_{\underset{\vee}{41}} + \frac{\chi}{C}\Gamma^{3}_{\underset{\vee}{14}}\right)\Gamma^{3}_{\underset{\vee}{14}} - \frac{A_{,3}}{2AC}\left(\eta\Gamma^{1}_{\underset{\vee}{34}} - \chi\Gamma^{3}_{\underset{\vee}{43}}\right) \\
&+ \left[\frac{\eta}{A}\left(\frac{2\eta_{,3}}{\eta} - \frac{5A_{,3}}{2A} + \frac{B_{,3}}{B} + \frac{C_{,3}}{C}\right) - \frac{\chi}{A}\left(\frac{\chi_{,1}}{\chi} - \frac{A_{,1}}{2A}\right)\right. \\
&\left.+ \frac{2\eta}{AC}\left(\eta\Gamma^{1}_{\underset{\vee}{34}} - \chi\Gamma^{3}_{\underset{\vee}{43}}\right) + \frac{2\eta^{2} - \chi^{2}}{A^{2}}\Gamma^{4}_{\underset{\vee}{13}} + 2\Gamma^{3}_{\underset{\vee}{14}}\right]\Gamma^{4}_{\underset{\vee}{13}} = 0,
\end{aligned} \qquad \text{... (2.1}b\text{)}$$

$$
\begin{aligned}
& K_{22} + \frac{\eta}{2A}\Big(\frac{B_{,1}}{C}\underset{v}{\Gamma}{}^{1}_{41} + \frac{B_{,3}}{A}\underset{v}{\Gamma}{}^{4}_{13} - \frac{B_{3,}}{C}\underset{v}{\Gamma}{}^{1}_{34}\Big) + \frac{\chi}{2A}\Big(\frac{B_{3,}}{C}\underset{v}{\Gamma}{}^{3}_{43} - \frac{B_{,1}}{A}\underset{v}{\Gamma}{}^{4}_{13} \\
& \qquad - \frac{B_{,1}}{C}\underset{v}{\Gamma}{}^{3}_{14}\Big) = 0 \qquad \ldots \quad (2.1c)
\end{aligned}
$$

$$
\begin{aligned}
& K_{33} + \frac{\eta}{A}\underset{v}{\Gamma}{}^{4}_{13,3} - \frac{\eta}{C}\underset{v}{\Gamma}{}^{1}_{34,}\ : - \frac{2\chi}{A}\underset{v}{\Gamma}{}^{4}_{13,1} + \frac{\chi}{C}\underset{v}{\Gamma}{}^{3}_{43,3} - \frac{\eta}{C}\Big(\frac{\eta_{,3}}{\eta} - \frac{A_{,3}}{2A} + \frac{\eta}{C}\underset{v}{\Gamma}{}^{1}_{34} \\
& - \frac{2\chi}{C}\underset{v}{\Gamma}{}^{3}_{43}\Big)\underset{v}{\Gamma}{}^{1}_{34} + \frac{\chi}{C}\Big(\frac{\chi_{,3}}{\chi} - \frac{A_{,3}}{2A} - \frac{\chi}{C}\underset{v}{\Gamma}{}^{3}_{43}\Big)\underset{v}{\Gamma}{}^{3}_{43} - \frac{A_{,1}}{2AC}\Big(\chi\underset{v}{\Gamma}{}^{3}_{14} - \eta\underset{v}{\Gamma}{}^{1}_{41}\Big) \\
& + \Big[\frac{\eta}{A}\Big(\frac{\eta_{,3}}{\eta} - \frac{A_{,3}}{2A} - \frac{\eta}{A}\underset{v}{\Gamma}{}^{4}_{13}\Big) - \frac{\chi}{A}\Big(\frac{2\chi_{,1}}{X} - \frac{5A_{,1}}{2A} + \frac{B_{,1}}{B} + \frac{C_{,1}}{C} \\
& - \frac{2\eta}{C}\underset{v}{\Gamma}{}^{1}_{41} - \frac{2\chi}{C}\underset{v}{\Gamma}{}^{4}_{13} + \frac{2\chi}{C}\underset{v}{\Gamma}{}^{3}_{14}\Big) - 2\underset{v}{\Gamma}{}^{1}_{34}\Big]\underset{v}{\Gamma}{}^{4}_{13} = 0, \qquad \ldots \quad (2.1d)
\end{aligned}
$$

$$
\begin{aligned}
& K_{44} - \frac{2}{A}\Big(\eta\underset{v}{\Gamma}{}^{1}_{41,1} - \eta\underset{v}{\Gamma}{}^{1}_{34,3} - \chi\underset{v}{\Gamma}{}^{3}_{14,1} + \chi\underset{v}{\Gamma}{}^{3}_{43,3}\Big) + \frac{\chi C_{,1} - \eta C_{,3}}{2A^{2}}\underset{v}{\Gamma}{}^{4}_{13} + \frac{\eta}{A} \\
& \Big(\frac{2\eta_{,3}}{\eta} + \frac{B_{,3}}{B} - \frac{3C_{,3}}{2C} - \frac{2\eta}{C}\underset{v}{\Gamma}{}^{1}_{34} - \frac{2\eta}{A}\underset{v}{\Gamma}{}^{4}_{13} + \frac{4\chi}{C}\underset{v}{\Gamma}{}^{3}_{43}\Big)\underset{v}{\Gamma}{}^{1}_{34} - \Big[\frac{\eta}{A}\Big(\frac{2\eta_{,1}}{\eta} \\
& + \frac{B_{,1}}{B} - \frac{3C_{,3}}{2C} + \frac{2\chi}{A}\underset{v}{\Gamma}{}^{4}_{13} + \frac{2\eta}{C}\underset{v}{\Gamma}{}^{1}_{41} - \frac{4\chi}{C}\underset{v}{\Gamma}{}^{3}_{14}\Big)\underset{v}{\Gamma}{}^{1}_{41} + \frac{\chi}{A}\Big(\frac{2\chi_{,3}}{\chi} + \frac{B_{,3}}{B} \\
& - \frac{3C_{,3}}{2C} - \frac{2\eta}{A}\underset{v}{\Gamma}{}^{4}_{13} + \frac{2\chi}{C}\underset{v}{\Gamma}{}^{3}_{43}\Big)\underset{v}{\Gamma}{}^{3}_{43}\Big] + \Big[\frac{\chi}{A}\Big(\frac{2\chi_{,1}}{\chi} + \frac{B_{,1}}{B} - \frac{3C_{,1}}{2C} + \frac{2\chi}{A}\underset{v}{\Gamma}{}^{4}_{13} \\
& - \frac{2\chi}{C}\underset{v}{\Gamma}{}^{3}_{14}\Big) + 2\underset{v}{\Gamma}{}^{1}_{34}\Big]\underset{v}{\Gamma}{}^{3}_{14} - 2\Big(\underset{v}{\Gamma}{}^{1}_{41}\underset{v}{\Gamma}{}^{2}_{42} + \underset{v}{\Gamma}{}^{2}_{42}\underset{v}{\Gamma}{}^{3}_{43} + \underset{v}{\Gamma}{}^{3}_{43}\underset{v}{\Gamma}{}^{1}_{41}\Big) = 0 \\
& \qquad \ldots \quad (2.1e)
\end{aligned}
$$

$$
\begin{aligned}
& K_{13} + \frac{\eta}{2AC}\Big(A\underset{v}{\Gamma}{}^{1}_{41,3} - A\underset{v}{\Gamma}{}^{1}_{34,1} - C\underset{v}{\Gamma}{}^{4}_{13,1}\Big) + \frac{\chi}{2AC}\Big(A\underset{v}{\Gamma}{}^{3}_{43,1} - A\underset{v}{\Gamma}{}^{3}_{14,3} + C\underset{v}{\Gamma}{}^{4}_{13,3}\Big) \\
& + \frac{1}{2AC}\Big[C\chi\Big(\frac{\chi_{,3}}{\chi} + \frac{B_{,3}}{B} + \frac{C_{,3}}{C} - \frac{2A_{,3}}{A}\Big) - C\eta\Big(\frac{\eta_{,1}}{\eta} - \frac{2A_{,1}}{A} + \frac{B_{,1}}{B} \\
& + \frac{C_{,1}}{C} - \frac{6\chi}{A}\underset{v}{\Gamma}{}^{4}_{13} + \frac{2\chi}{C}\underset{v}{\Gamma}{}^{3}_{14}\Big) + 2\Big(\eta^{2} + AC\Big)\underset{v}{\Gamma}{}^{1}_{41}\Big]\underset{v}{\Gamma}{}^{4}_{13}
\end{aligned}
$$

$$+\frac{1}{2C}\Big[\chi\Big(\frac{\chi_{,1}}{\chi}-\frac{A_{,1}}{A}-\frac{2\chi}{A}\Gamma^4_{\underset{\vee}{13}}-\frac{2\eta}{C}\Gamma^1_{\underset{\vee}{41}}\Big)\Gamma^3_{\underset{\vee}{43}}-\eta\Big(\frac{\eta_{,1}}{\eta}-\frac{A_{,1}}{A}$$

$$-\frac{2\chi}{A}\Gamma^4_{\underset{\vee}{13}}+\frac{2\chi}{C}\Gamma^3_{\underset{\vee}{14}}-\frac{2\eta}{C}\Gamma^1_{\underset{\vee}{41}}\Big)\Gamma^1_{\underset{\vee}{34}}\Big]+\frac{1}{2C}\Big[\eta\Big(\frac{\eta_{,3}}{\eta}-\frac{A_{,3}}{A}\Big)\Gamma^1_{\underset{\vee}{41}}$$

$$-X\Big(\frac{\chi_{,3}}{\chi}-\frac{A_{,3}}{A}-\frac{2\chi}{C}\Gamma^3_{\underset{\vee}{43}}\Big)\Gamma^3_{\underset{\vee}{14}}\Big]-\Gamma^4_{\underset{\vee}{13}}\Gamma^3_{\underset{\vee}{43}}=0 \qquad \ldots \quad (2.1f)$$

$$R_{\underset{\vee}{14},3}-R_{\underset{\vee}{34},1}=0. \qquad \ldots \quad (2.1g)$$

where

$$K_{11}=-\tfrac{1}{2}\Big(\frac{A_{,11}+A_{,33}}{A}-\frac{A^2_{,1}+A^2_{,3}}{A^2}+\frac{B_{,11}}{B}+\frac{C_{,11}}{C}\Big)+\tfrac{1}{4}\Big[\frac{B^2_{,1}}{B^2}+\frac{C^2_{,1}}{C^2}$$
$$+\frac{A_{,1}}{A}\Big(\frac{B_{,1}}{B}+\frac{C_{,1}}{C}\Big)-\frac{A_{,3}}{A}\Big(\frac{B_{,3}}{B}+\frac{C_{,3}}{C}\Big)\Big]$$

$$K_{22}=-\frac{B_{,11}+B_{,33}}{2A}+\frac{B^2_{,1}+B^2_{,3}}{4AB}-\frac{B_{,1}C_{,1}+B_{,3}C_{,3}}{4AC}$$

$$K_{33}=-\tfrac{1}{2}\Big(\frac{A_{,11}+A_{,33}}{A}-\frac{A^2_{,1}+A^2_{,3}}{A^2}+\frac{B_{,33}}{B}+\frac{C_{,33}}{C}\Big)+\tfrac{1}{4}\Big[\frac{B^2_{,3}}{B^2}+\frac{C^2_{,3}}{C^2}$$
$$-\frac{A_{,1}}{A}\Big(\frac{B_{,1}}{B}+\frac{C_{,1}}{C}\Big)+\frac{A_{,3}}{A}\Big(\frac{B_{,3}}{B}+\frac{C_{,3}}{C}\Big)\Big]$$

$$K_{44}=\frac{C_{,11}+C_{,33}}{2A}-\frac{C^2_{,1}+C^2_{,3}}{4AC}+\frac{B_{,1}C_{,1}+B_{,3}C_{,3}}{4AB}$$

$$K_{13}=-\tfrac{1}{2}\Big(\frac{B_{,13}}{B}+\frac{C_{,13}}{C}\Big)+\tfrac{1}{4}\Big[\frac{B_{,1}B_{,3}}{B^2}+\frac{C_{,1}C_{,3}}{C^2}+\frac{A_{,1}}{A}\Big(\frac{B_{,3}}{B}+\frac{C_{,3}}{C}\Big)$$
$$+\frac{A_{,3}}{A}\Big(\frac{B_{,1}}{B}+\frac{C_{,1}}{C}\Big)\Big]$$

$$R_{\underset{\vee}{14}}=-\Gamma^1_{\underset{\vee}{41},1}+\Gamma^3_{\underset{\vee}{14},3}-\frac{\chi}{A}\Big(\Gamma^1_{\underset{\vee}{41}}-3\Gamma^3_{\underset{\vee}{43}}\Big)\Gamma^4_{\underset{\vee}{13}}+\frac{\eta}{A}\Big(\frac{B^2_{,1}}{4B^2}-4\Gamma^4_{\underset{\vee}{13}}\Gamma^1_{\underset{\vee}{34}}\Big)$$
$$-\tfrac{1}{2}\Big[\Big(\frac{A_{,1}}{A}+\frac{B_{,1}}{B}\Big)\Gamma^1_{\underset{\vee}{41}}-\Big(\frac{A_{,3}}{A}+\frac{B_{,3}}{B}\Big)\Gamma^3_{\underset{\vee}{14}}-\frac{A_{,1}}{A}\Gamma^3_{\underset{\vee}{43}}$$
$$-\frac{A_{,3}}{A}\Gamma^1_{\underset{\vee}{34}}+\frac{C_{,3}}{A}\Gamma^4_{\underset{\vee}{13}}\Big]$$

$$\underset{\vee}{R}_{34} = -\underset{\vee}{\Gamma}^{3}_{43,3} + \underset{\vee}{\Gamma}^{1}_{34,1} + \frac{\eta}{A}\left[\frac{B_{,1}B_{,3}}{4B^2} + \left(\underset{\vee}{\Gamma}^{3}_{43} - 3\underset{\vee}{\Gamma}^{1}_{41}\right)\underset{\vee}{\Gamma}^{4}_{13}\right] + \frac{4\chi}{A}\underset{\vee}{\Gamma}^{3}_{14}\underset{\vee}{\Gamma}^{4}_{13}$$

$$+ \tfrac{1}{2}\left[\left(\frac{A_{,1}}{A} + \frac{B_{,1}}{B}\right)\underset{\vee}{\Gamma}^{1}_{34} - \left(\frac{A_{,3}}{A} + \frac{B_{,3}}{B}\right)\underset{\vee}{\Gamma}^{3}_{43} + \frac{A_{,1}}{A}\underset{\vee}{\Gamma}^{3}_{14}\right.$$

$$\left. + \frac{A_{,3}}{A}\underset{\vee}{\Gamma}^{1}_{41} + \frac{C_{,1}}{A}\underset{\vee}{\Gamma}^{4}_{13}\right]$$

$$\underset{\vee}{\Gamma}^{1}_{41} = \frac{C}{AC-\eta^2}\left[\eta_{,1} - \frac{\eta}{2}\left(\frac{A_{,1}}{A} + \frac{C_{,1}}{C}\right) + \frac{\chi A_{,3}}{2A} - \frac{\eta\chi}{AC}\left(A\underset{\vee}{\Gamma}^{3}_{14} + 2C\underset{\vee}{\Gamma}^{4}_{13}\right)\right]$$

$$\underset{\vee}{\Gamma}^{2}_{42} = \frac{\eta B_{,1}}{2AB}$$

$$\underset{\vee}{\Gamma}^{3}_{43} = \frac{C}{AC-\chi^2}\left[\chi_{,3} - \frac{\chi}{2}\left(\frac{A_{,3}}{A} + \frac{C_{,3}}{C}\right) + \frac{\eta A_{,1}}{2A}\right.$$

$$\left. + \frac{\eta\chi}{AC}\left(2C\underset{\vee}{\Gamma}^{4}_{13} - A\underset{\vee}{\Gamma}^{1}_{34}\right)\right]. \qquad \ldots \quad (2.2a)$$

In (2.1) and (2.2a), the remaining nonvanishing skew-symmetric $\underset{\vee}{\Gamma}^{4}_{13}$, $\underset{\vee}{\Gamma}^{3}_{14}$ and $\underset{\vee}{\Gamma}^{1}_{34}$ are given by

$$\frac{A(AC+\chi^2-\eta^2)}{AC-\eta^2}\underset{\vee}{\Gamma}^{3}_{14} - \frac{A(AC-\chi^2+\eta^2)}{AC-\chi^2}\underset{\vee}{\Gamma}^{1}_{34} + \frac{2\eta^2\chi^2}{A(AC-\chi^2)}\underset{\vee}{\Gamma}^{4}_{13} = L,$$

$$\frac{AC(AC-\eta^2)-\chi^2(\chi^2+\eta^2)}{A(AC-\chi^2)}\underset{\vee}{\Gamma}^{4}_{13} - \frac{A(AC-\chi^2-\eta^2)}{AC-\chi^2}\underset{\vee}{\Gamma}^{1}_{34} = M,$$

$$\frac{AC(AC-\chi^2)-\eta^2(\chi^2+\eta^2)}{A(AC-\eta)}\underset{\vee}{\Gamma}^{4}_{13} + \frac{A(AC-\chi^2-\eta^2)}{AC-\eta^2}\underset{\vee}{\Gamma}^{3}_{14} = -N, \ldots \quad (2.2b)$$

where

$$2L = \frac{\eta C_{,3} - \chi C_{,1}}{C} + \frac{\chi\eta^2}{AC-\eta^2}\left(\frac{2\eta_{,1}}{\eta} - \frac{A_{,1}}{A} - \frac{C_{,1}}{C}\right)$$

$$- \frac{\eta\chi^2}{AC-\chi^2}\left(\frac{2\chi_{,3}}{\chi} + \frac{\eta A_{,1}}{\chi A} - \frac{A_{,3}}{A} - \frac{C_{,3}}{C}\right),$$

$$2M = 2\eta_{,3} - \frac{\eta AC}{AC-\chi^2}\left(\frac{A_{,3}}{A} + \frac{C_{,3}}{C}\right) + \frac{\eta\chi}{AC-\chi^2}\left(2\chi_{,3} + \frac{\eta A_{,1}}{A}\right) - \frac{\chi A_{,1}}{A}$$

$$2N = 2\chi_{,1} - \frac{\chi AC}{AC-\eta^2}\left(\frac{A_{,1}}{A} + \frac{C_{,1}}{C}\right) - \frac{\eta\chi}{AC-\eta^2}\left(2\eta_{,1} - \frac{\chi A_{,3}}{A}\right) - \frac{\eta A_{,3}}{A}.$$

$$\ldots \quad (2.2c)$$

The field equations (2.1) are nonlinear with respect to the field variables $g_{\lambda\mu}$ and are quite complicated and so a rigorous solution is difficult to obtain. Following the technique of the linear theory the effect of current and the electromagnetic induction in Einstein's unified field theory has already been studied (Tiwari & Kashyap 1972; Kashyap & Ram 1975; and Kashyap & Singh 1977). As a linear theory accounts, with a considerable degree of accuracy, for the description of physical phenomena it is not unreasonable to assume that the field variables are small quantities. Let us take

$$A = 1 + \alpha$$
$$B = \rho^2 + \beta$$
$$C = 1 + \gamma. \qquad \dots (2.3)$$

The order of smallness of the field variables is as follows :

$$\eta, \chi = 0(1),$$
$$\alpha, \beta, \gamma = 0(2). \qquad \dots (2.4)$$

Upto the second order of smallness, the field equations (2.1) reduce to

$$\eta_{,1} + \chi_{,3} + \frac{\eta}{\rho} = 0, \qquad \dots (2.5a)$$

$$\alpha_{,11} + \alpha_{,33} - \frac{\alpha_{,1}}{\rho} + \gamma_{,11} + \frac{1}{\rho}\left(\beta_{,11} - \frac{2\beta_{,1}}{\rho} + \frac{2\beta}{\rho_2}\right)$$
$$= 2\left[\eta\left(\eta_{,11} + \eta_{,33} - \chi_{,13}\right) + \chi\chi_{,11} - \eta_{,3}\,\chi_{,1} + \tfrac{1}{2}\left(\eta_{,3}^2 + 3\chi_{,1}^2\right) + \eta_{,1}^2\right] \qquad \dots (2.5b)$$

$$\beta_{,11} + \beta_{,33} - \frac{2}{\rho}\left(\beta_{,1} - \frac{\beta}{\rho}\right) + \rho\gamma_{,1} = 2\rho\,(\eta\eta_{,1} + \chi\chi_{,1}) \qquad \dots (2.5c)$$

$$\alpha_{,11} + \alpha_{,33} + \frac{\alpha_{,1}}{\rho} + \gamma_{,33} + \frac{\beta_{,33}}{\rho^2} = 2\left[\chi\left(\chi_{,11} + \chi_{,33} - \eta_{,13}\right)\right.$$
$$\left. + \eta\eta_{,33} - \frac{\chi}{\rho}\left(\eta_{,3} - \chi_{,1}\right) + \tfrac{1}{2}\left(\chi_{,1}^2 + 3\eta_{,3}^2\right) - \chi_{,1}\,\eta_{,3} + \chi_{,3}^2\right] \dots (2.5d)$$

$$\gamma_{,11} + \gamma_{,33} + \frac{\gamma_{,1}}{\rho} = 2\left[\eta\left(2\eta_{,11} + \eta_{,33} + \chi_{,13}\right) + \chi\left(\chi_{,11} + 2\chi_{,33} + \eta_{,13}\right)\right.$$
$$+ \frac{1}{\rho}\left(2\eta\eta_{,1} + \chi\eta_{,3} + \chi\chi_{,1}\right) - \frac{\eta^2}{\rho^2}$$
$$\left. + \frac{1}{2}\left(\chi_{,1}^2 + \eta_{,3}^2\right) + \chi_{,1}\,\eta_{,3} + \chi_{,3}^2 + \eta_{,1}^2\right] \qquad \dots (2.5e)$$

$$\gamma_{,13} - \frac{\alpha_{,3}}{\rho} + \frac{1}{\rho^2}\left(\beta_{,13} - \frac{\beta_{,3}}{\rho}\right) = \eta(\eta_{,13} + \chi_{,11}) + \chi\,(\chi_{,13} + \eta_{,33})$$
$$+ \frac{\eta}{\rho}\,(\chi_{,1} - \eta_{,3}) + 2\,(\eta_{,1}\,\eta_{,3} + \chi_{,1}\,\chi_{,3}) \qquad \dots (2.5f)$$

$$\left[\eta_{,11} + \eta_{,33}\left(\frac{\eta}{\rho}\right)_{,1}\right]_{,3} - \left[\chi_{,11} + \chi_{,33} + \frac{\chi_{,1}}{\rho}\right]_{,1} = 0. \qquad \dots (2.5g)$$

3. Solution of the Field Equations

The equation (2.5*a*) can be rewritten as

$$(\rho\eta)_{,1} = -(\rho\chi)_{,3} = \omega_{,13}\,(\text{say}), \qquad \dots \quad (3.1)$$

where ω is a function of ρ and z. From (3.1), we obtain

$$\eta = \frac{\omega_{,3}}{\rho}, \qquad \dots \quad (3.2)$$

$$\chi = -\frac{\omega_{,1}}{\rho}. \qquad \dots \quad (3.3)$$

With the help of (3.2) and (3.3), the equation (2.5*g*) can be recast in the form

$$G_{,11} + G_{,33} - \frac{2}{\rho} G_{,1} + \frac{3}{\rho^2} H = 0, \qquad \dots \quad (3.4)$$

where

$$G = \omega_{,11} + \omega_{,33} \qquad \dots \quad (3.5)$$

$$H = \omega_{,11} - \frac{\omega_{,1}}{\rho}. \qquad \dots \quad (3.6)$$

From (3.4) we can take, in particular

$$G = H = 0, \qquad \dots \quad (3.7)$$

which means

$$\omega_{,11} + \omega_{,33} = 0 \qquad \dots \quad (3.8)$$

and $$\omega_{,11} - \frac{\omega_{,1}}{\rho} = 0. \qquad \dots \quad (3.9)$$

The equation (3.9), after integration, gives

$$\omega_{,1} = \delta\rho, \qquad \dots \quad (3.10)$$

where δ is a function of z. Therefore, from (3.8) and (3.10) we have

$$\omega_{,3} = -\int \delta\, dz \qquad \dots \quad (3.11)$$

Substituting the values of η and χ in terms of $\omega_{,1}$ and $\omega_{,3}$ so obtained in (2.5*a*) we find that δ is a constant. Therefore,

$$\eta = -\delta \frac{z}{\rho}, \quad \chi = -\delta. \qquad \dots \quad (3.12)$$

Substituting the values of η and χ in the equation (2.5*e*), we obtain

$$\gamma_{,11} + \gamma_{,33} + \frac{\gamma_{,1}}{\rho} = \frac{\delta^2}{\rho^2}\left(1 + \frac{4z^2}{\rho^2}\right), \qquad \dots \quad (3.13)$$

which provides the solution

$$\gamma = \frac{\delta^2}{2}\left[\frac{2z^2}{\rho^2} - (\log\rho)^2\right]. \qquad \dots \quad (3.14)$$

Making use of the solutions for η, χ and γ in (2.5*c*), we have

$$\left(\frac{\beta}{\rho}\right)_{,11} + \left(\frac{\beta}{\rho}\right)_{,33} = \delta^2 \frac{\log \rho}{\rho}, \qquad \ldots (3.15)$$

which gives the following solution

$$\beta = \tfrac{1}{2}\, \delta^2 \rho^2 (\log \rho - 2) \log \rho. \qquad \ldots (3.16)$$

Putting the values of η, χ, β and γ in the equation (2.5*f*) we obtain

$$\alpha_{,3} = 0, \qquad \ldots (3.17)$$

which suggests that α is a function of ρ only. Let us put

$$\alpha = F(\rho). \qquad \ldots (3.18)$$

Substituting the values of α, β, γ, η and χ so obtained in the equation (2.5*b*) we obtain

$$\left(\frac{F_{,1}}{\rho}\right)_{,1} = 2\, \frac{\delta^2}{\rho^3}(1 - \log\ \rho)\ , \qquad \ldots (3.19)$$

which, after integration, provides

$$F = \tfrac{1}{2}\, \delta^2 (\log \rho - 1) \log \rho. \qquad \ldots (3.20)$$

Therefore, from (3.18) and (3.20), we have

$$\alpha = \tfrac{1}{2}\, \delta^2 (\log \rho - 1) \log \rho. \qquad \ldots (3.21)$$

4. Discussion

In Einstein's unified field theory it has already been attempted to explore the effects of $\underset{\vee}{g}_{14}$ and $\underset{\vee}{g}_{34}$ taken together (Kashyap & Pant 1969). The current in the ϕ-direction was assumed to be absent there and an approximate gravitational field produced by $\underset{\vee}{g}_{14}$ and $\underset{\vee}{g}_{34}$ was found to be given by

$$ds^2 = -\,(d\rho^2 + \rho^2 d\phi^2 + dz^2) + \left(1 + \frac{k^2}{\rho^2}\right) dt^2. \qquad \ldots (4.1)$$

k being a constant of integration, it corresponds to a repulsive gravitational field (Tiwari & Kashyap 1972*a*).

In the present investigation there is a current in the ϕ-direction which varies inversely as the square of the distance from the axis at $\rho = 0$. The surviving components F_{12} and F_{23} of the electromagnetic field tensor $F_{\lambda\mu}$ which is the dual tensor of $\underset{\vee}{g}^{\lambda\mu}$ are given by

$$F_{12} = \delta\rho, \quad F_{23} = \delta z. \qquad \ldots (4.2)$$

In the plane $z = 0$, the component F_{23} vanishes. Classically, this is the case of a magnetic dipole with a magnetic field in the direction perpendicular to the plane $z = 0$. The constant δ, in this case, resembles with the charge of electron which revolves in the circular orbit in the plane $z = 0$. If there exists a massive proton at the origin, this system as a whole gives a picture of Hydrogen atom. We know that a Hydrogen atom with a proton in its nucleus and an election revolving around it constitutes an electrically neutral particle.

In general relativity the external field of a neutral particle is given by the Schwarzschild solution in the approximate form

$$ds^2 = -\left(1 + \frac{2m}{r}\right) d\sigma^2 + \left(1 - \frac{2m}{r}\right)dt^2, \quad \ldots \quad (4.3)$$

where

$$d\sigma^2 = d\rho^2 + \rho^2 \, d\phi^2 + dz^2, \quad r^2 = \rho^2 + z^2. \quad \ldots \quad (4.4)$$

Following the previously mentioned order of smallness we may take

$$m = 0(2). \quad \ldots \quad (4.5)$$

Irrespective of the well-known solution (4.3), the present investigation suggests that the external field of a neutral Hydrogen atom is given by the approximate form

$$ds^2 = -\left(1 + \alpha + \frac{2m}{r}\right) d\sigma^2 + \left(1 + \gamma - \frac{2m}{r}\right) dt^2$$

$$- \frac{1}{2}\delta^2\rho^2 \log \rho d\phi^2. \quad \ldots \quad (4.6)$$

This shows how the external field of an electrically neutral Hydrogen atom differs from Schwarzschild field of a neutral particle.

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