A TWO PARAMETER SINGULAR PERTURBATION SOLUTION OF ONE DIMENSIONAL FLOW THROUGH UNSATURATED POROUS MEDIA

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A two parameter singular perturbation solution of one dimensional flow through unsaturated porous media has been discussed here.

Introduction

The present paper analytically discusses an one dimensional flow through unsaturated porous media by using two parameter singular perturbation technique. The mathematical formulation yields a nonlinear diffusion type equation which is transformed by similarity method into an ordinary differential equation containing two small parameters.

The mathematical model conforms to the hydrological situation of an one dimensional vertical ground water recharge by Spreading (Verma 1969). Such flow of great importance in water resources science and many authors have discussed it from a different view point, e.g., Klute (1952) employs a finite difference approach, Phillips (1970) uses a transformation of variables technique, Sharma (1965) discusses a variational approach, Mehta (1975) discusses multiple scale method and Verma (1969) and Verma & Mishra (1973) obtain a Laplace transformation and similarity solution. For definiteness, we consider the physical problem as given by Verma (1969).

Formulation of Problem

The equation of continuity for an unsaturated porous media is given by

$$\frac{\partial}{\partial t}(\rho_r \theta) = \nabla \cdot \vec{M}, \qquad ...(1)$$

where ρ_r is the bulk density of the medium, θ is moisture content on a dry weight basis, $\vec{M}$ is the mass flux of moisture and ∇ is the vector differential operator. From Darcey's law, we get

$$\vec{V} = -K \nabla \phi, \qquad ...(2)$$

where ϕ represents the moisture potentials, V the volume flux of moisture and K the coefficient of aqueous conductivity. Combining (1) and (2) and recalling that the flow takes place only in the vertical direction, we have

$$\rho_r \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(\rho K \frac{\partial \psi}{\partial z}\right) - \frac{\partial}{\partial z}(\rho K g), \qquad \ldots(3)$$

where ψ is the capillary pressure potential, g the gravitational constant, ρ the fluid density and $\phi = \psi - gz$. The positive direction of Z-axis is the same as that of the gravity.

Considering θ and ψ to be connected by a single valued function, we may write (3) as

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(D \frac{\partial \theta}{\partial z}\right) + \frac{\rho}{\rho_r} g \frac{\partial K}{\partial z}, \qquad \ldots(4)$$

where $D = \frac{\rho}{\rho_r} K \frac{\partial \psi}{\partial \theta}$, and is called small diffusivity coefficient.

Replacing D by its average value D_1 and assuming $K = \frac{K_0 \theta}{\sqrt{t}}$ (Mehta 1975), we get

$$\frac{\partial \theta}{\partial t} = D_1 \frac{\partial^2 \theta}{\partial z^2} + \frac{\rho}{\rho_r} g K_0 \frac{\partial \theta}{\partial z} \qquad \ldots(5)$$

Considering that the watertable is situated at a depth L and setting

$$\frac{z}{L} = \xi, \frac{t}{L^2} = T$$

we may write (5) as

$$\frac{\partial \theta}{\partial T} = D_1 \frac{\partial^2 \theta}{\partial \xi^2} + \frac{M_1}{\sqrt{T}} \frac{\partial \theta}{\partial \xi}, \; M_1 = \frac{K_0 g \rho}{\rho_r} \qquad \ldots(6)$$

A set of suitable boundary condition may be taken as

$$\theta(0, T) = \theta_0; \;\; \theta(1, T) = 1 \qquad \ldots(7)$$

Assuming that the moisture content is expressible in the separation of variable form viz., $\theta(\xi, T) = H(T)\, F(\eta)$; $H(T) = C_2 T^{\eta}$, where η is any integer, C_2 is an arbitrary constant and using Boltazmanu transformation $\eta = \xi/2\sqrt{T}$ (see Hansen 1964) equations (6) and (7) become

$$\left.\begin{array}{ll} & \epsilon F''(\eta) + 2\mu F'(\eta) - F(\eta) = 0 \\ \text{where} & \epsilon = \dfrac{D_1}{4n} \cdot \mu = \dfrac{M_1 - \eta}{4n} \\ \text{and} & F(0) = \theta_0, \; F\left(\dfrac{1}{2\sqrt{T}}\right) = 1 \end{array}\right\} \qquad \ldots(8)$$

This is the desired differential equation whose solution is discussed in the following section.

SOLUTION

If ϵ and μ are small, we obtain the solution of (8) by the two parameter singular perturbation method (O'Malley 1966).

Two cases depending on the order of parameter arises.

Case I — Suppose $\frac{\epsilon}{\mu^2} \to 0$ as $\mu \to 0$.

In this case the auxiliary polynomial of (8) is

$$\epsilon m^2 + 2\mu m - 1 = 0 \qquad \text{...(9)}$$

whose roots are

$$m_1 = \frac{\mu}{\epsilon}\left[-1 + \sqrt{1 + \frac{\epsilon}{\mu^2}}\right] \text{ and } m_2 = \frac{\mu}{\epsilon}\left[-1 - \sqrt{1 + \frac{\epsilon}{\mu^2}}\right].$$

Therefore, we may write the explicit solution of the boundary value problem (8) as

$$\begin{aligned} F(\eta) = & \left[1 - \exp\left\{\frac{\mu}{\epsilon\sqrt{T}}\sqrt{1 + \frac{\epsilon}{\mu^2}}\right\}\right]^{-1} \\ & \times\left[\left(1 - \theta_0 \exp\left\{\frac{\mu}{2\epsilon\sqrt{T}}\left(-1 + \sqrt{1 + \frac{\epsilon}{\mu^2}}\right)\right\}\right)\right. \\ & \times \exp\left\{\frac{\mu}{\epsilon}\left(-1 - \sqrt{1 + \frac{\epsilon}{\mu^2}}\right)\left(\eta - \frac{1}{2\sqrt{T}}\right)\right\} \\ & + \left(\theta_0 - \exp\left\{\frac{-\mu}{2\epsilon\sqrt{T}}\left(-1 - \sqrt{1 + \frac{\epsilon}{\mu^2}}\right\}\right)\right) \\ & \left.\times \exp\left\{\frac{\mu}{\epsilon}\left(-1 + \sqrt{1 + \frac{\epsilon}{\mu^2}}\right)\eta\right\}\right]. \end{aligned} \qquad \text{...(10)}$$

To any number of terms the asymptotic expansion of (10) is given by

$$\begin{aligned} F(\eta) \sim & \exp\left\{\frac{-\mu}{\epsilon}\left(1 + \sqrt{1 + \frac{\epsilon}{\mu^2}}\right)\left(\eta - \frac{1}{2\sqrt{T}}\right)\right\} \\ & + \theta_0 \exp\left\{\frac{-\mu}{\epsilon}\left(1 - \sqrt{1 + \frac{\epsilon}{\mu^2}}\right)\eta\right\} \end{aligned} \qquad \text{...(11)}$$

Case II — Suppose ϵ and $\frac{\mu^2}{\epsilon} \to 0$ as $\mu \to 0$.

Again considering the polynomial equation (9) viz.

$$\epsilon m^2 + 2\mu m - 1 = 0,$$

whose roots are

$$m_1 = \frac{1}{\sqrt{\epsilon}}\left[-\sqrt{\frac{\mu^2}{\epsilon}} + \sqrt{1 + \frac{\mu^2}{\epsilon}}\right]$$

and

$$m_2 = \frac{1}{\sqrt{\epsilon}}\left[-\sqrt{\frac{\mu^2}{\epsilon}} - \sqrt{1 + \frac{\mu^2}{\epsilon}}\right]$$

the solution of the boundary value problems (8) is given by

$$F(\eta) = \left[1 - \exp\left\{\frac{1}{\sqrt{\epsilon T}}\sqrt{1+\frac{\mu^2}{\epsilon}}\right\}\right]^{-1}$$

$$\times\left[\left(\theta_0 - \exp\left\{\frac{1}{2\sqrt{\epsilon T}}\left(\sqrt{\frac{\mu^2}{\epsilon}} + \sqrt{1+\frac{\mu^2}{\epsilon}}\right)\right\}\right)\right.$$

$$\times \exp\left\{\frac{-1}{\sqrt{\epsilon}}\left(\sqrt{\frac{\mu^2}{\epsilon}} - \sqrt{1+\frac{\mu^2}{\epsilon}}\right)\eta\right\}$$

$$+\left(1 - \theta_0 \exp\left\{\frac{-1}{2\sqrt{\epsilon T}}\left(\sqrt{\frac{\mu^2}{\epsilon}} - \sqrt{1+\frac{\mu^2}{\epsilon}}\right)\right\}\right)$$

$$\left.\times \exp\left\{\frac{-1}{\sqrt{\epsilon}}\left(\sqrt{\frac{\mu^2}{\epsilon}} + \sqrt{1+\frac{\mu^2}{\epsilon}}\right)\left(\eta - \frac{1}{2\sqrt{T}}\right)\right\}\right] \qquad \text{...(12)}$$

To any number of terms the asymptotic expansion of (12) is given by

$$F(\eta) \sim \exp\left\{-\frac{1}{\sqrt{\epsilon}}\left(\sqrt{\frac{\mu^2}{\epsilon}} + \sqrt{1+\frac{\mu^2}{\epsilon}}\right)\left(\eta - \frac{1}{2\sqrt{T}}\right)\right\}$$

$$+ \theta_0 \exp\left\{-\frac{1}{\sqrt{\epsilon}}\left(\sqrt{\frac{\mu^2}{\epsilon}} - \sqrt{1+\frac{\mu^2}{\epsilon}}\right)\eta\right\} \qquad \text{...(13)}$$

Expanding the exponents in powers of μ^2/ϵ, we can split those terms which are not singular as $\mu \to 0$. In particular if $\mu = O(\epsilon)$

$$F(\eta) \sim \exp\left\{-\frac{1}{\sqrt{\epsilon}}\left(\eta - \frac{1}{2\sqrt{T}}\right)\right\} + \theta_0 \exp\left\{\frac{-\eta}{\sqrt{\epsilon}}\right\} \qquad \text{...(14)}$$

Concluding Remarks

Here we discuss the following cases:

When $\epsilon/\mu^2 \to 0$ as $\mu \to 0$, we get, on expanding (10) in terms of ϵ/μ^2 and retaining the terms of first order,

$$F(\eta) = \left[1 - \exp\left\{\frac{\mu}{\epsilon\sqrt{T}}\right\}\right]^{-1}\left[(1-\theta_0)\exp\left\{\frac{-2\mu}{\epsilon}\left(\eta - \frac{1}{2\sqrt{T}}\right)\right\}\right.$$

$$\left.+\left(\theta_0 - \exp\left\{\frac{\mu}{\epsilon\sqrt{T}}\right\}\right)\right] \qquad \text{...(15)}$$

Expanding the exponent in (15) and neglecting the terms of $O\left(\frac{\mu^2}{\epsilon^2}\right)$, we have

$$F(\eta) = (1-\theta_0)(2\sqrt{T}\eta - 1) - 1. \qquad \text{...(16)}$$

Equation (16) gives the perturbation solution of the equation (8).

In particular if $\theta_0 = 0$, the equation (16) reduces to

$$F(\eta) = 2(\sqrt{T}.\eta - 1) \quad ...(17)$$

and, if we keep T fixed and equal to $\frac{1}{4}$, then

$$F(\eta) = (\eta - 2). \quad ...(18)$$

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