OSCILLATORY FREE CONVECTION FROM A HORIZONTAL CYLINDER

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The free convection flow from a horizontal circular cylinder whose temperature oscillates harmonically in time with frequency ω is studied. Using cylindrical polar coordinates, the boundary layer equations are derived for curvature effects. The method of inner and outer expansions are developed to solve the resulting equations. The flow field is characterised by two parameters, $g\beta\Delta T_0/\omega^2 a$ and $\nu/a^2\omega$. The velocity and temperature fields are affected by curvature to a second-order.

1. INTRODUCTION

The present paper is devoted to the study of free convection flow from a horizontal circular cylinder whose temperature oscillates harmonically in time with frequency ω. The cylinder is immersed in an unbounded mass of fluid maintained at a constant temperature T_0. This problem has been studied by Merkin (1967) using the well known boundary layer equations for a flat plate ignoring altogether the curvature of the cylinder. This is justified if the boundary layer thickness is much smaller than the radius of curvature of the cylinder. However, this condition is not satisfied in the case of a long but thin cylinder. The boundary layer on such a body grows downstream and its thickness becomes comparable with the radius of curvature.

In the present paper, the boundary layer equations are derived using cylindrical polar coordinates and retaining the appropriate curvature terms. The method of inner and outer expansions is used to solve the boundary layer equations. The longitudinal curvature contributes terms to the second order equations in the inner layer solution while in the outer layer the equations are not affected by curvature as expected.

In free convection, temperature plays the same role as velocity in forced convection. The analogous forced convection problem of flow induced by a circular cylinder oscillating perpendicular to its axis in an incompressible fluid which is othewise at rest has been considered by a number of investigators. An essential feature of the flow is the existence of a steady streaming motion which persists outside a "shear layer" of thickness $O(\sqrt{\nu/\omega})$, where ν is the kinematic viscosity and ω the frequency of oscillation. Stuart (1963) has studied the mechanism of this steady outer flow by introducing the concept of double boundary layers — an inner layer of thickness $O(\sqrt{\nu/\omega})$ and an outer layer of thickness

$O(1/\epsilon\sqrt{\nu/\omega})$ where $\epsilon = U_0/\omega d$ is the frequency parameter; in the outer layer the nonlinear inertia terms are of the same order of magnitude as the viscous terms. The steady streaming which persist in the inner layer finally decays to zero within the outer layer. Wang (1966) has also considered this problem. He has derived the boundary layer equations in the case when the amplitude of oscillation U_0 of the cylinder is small such that $U_0/a\omega << 1$. An excellent review of this problem has been given by Riley (1967).

In the present paper we have used the technique as developed by Wang for the corresponding forced convection problem. The boundary layer equations are developed on the assumption that $\nu/a^2\omega << 1$. The extra effects arising out of the interaction of the curvature, viscosity and thermal conductivity are studied.

2. Formulation of the Problem

The coordinate system is shown in Fig. 1 where a is the radius of the cylinder, T_0, is the ambient temperature and $T_w = T_0(1 + A \cos \omega t)$, $A \ll 1$, is the surface temperature. We take $(g\beta AT_0/\omega)$, $1/\omega$, a and AT_0 as the characteristic velocity, time, length and temperature respectively.

The complete equations of balance of mass, momentum and energy, using Boussinesq approximation, in non-dimensional form are

$$\frac{\partial}{\partial r}(rv) + \frac{\partial u}{\partial \theta} = 0 \qquad \ldots(2.1)$$

$$\frac{\partial u}{\partial t} + \epsilon\left\{v\frac{\partial u}{\partial r} + \frac{u}{r}\frac{\partial u}{\partial \theta} + \frac{uv}{r}\right\} = -\epsilon\frac{1}{r}\frac{\partial p}{\partial \theta} + \phi\cos\theta$$
$$+\frac{1}{M^2}\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{u}{r^2} + \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right\} \qquad \ldots(2.2)$$

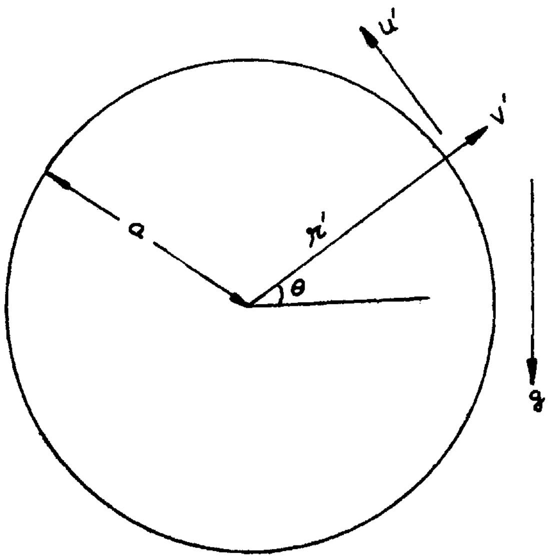

Fig. 1. The coordinate system.

$$\frac{\partial v}{\partial t} + \epsilon \left\{ v \frac{\partial v}{\partial r} + \frac{u}{r}\frac{\partial v}{\partial \theta} - \frac{u^2}{r} \right\} = -\epsilon \frac{\partial P}{\partial r} + \phi \sin\theta + \frac{1}{M^2}\left\{ \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} - \frac{v}{r^2} - \frac{2}{r^2}\frac{\partial u}{\partial \theta} \right\} \quad ...(2.3)$$

$$\frac{\partial \phi}{\partial t} + \epsilon \left(v \frac{\partial \phi}{\partial r} + \frac{u}{r}\frac{\partial \phi}{\partial \theta} \right\} = \frac{1}{\sigma}\frac{1}{M^2}\left\{ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \right\} \quad ...(2.4)$$

where $P = \dfrac{p' + \rho_0 g r' \sin\theta}{\rho(g\beta A T_0/\omega)^2}$ is the modified dimensionless pressure,

$$\phi = \frac{T - T_0}{A T_0}, \quad \epsilon = \frac{g\beta A T_0}{a\omega^2}, \quad M^2 = \frac{a^2\omega}{\nu} \text{ and } \sigma = \frac{\nu}{k}.$$

The boundary conditions are

$$\left.\begin{array}{llll} u = 0 & v = 0 & \phi = e^{it} & \text{at } r = 1 \\ u \to 0 & \phi \to 0 & \text{as } r \to \infty. & \end{array}\right\} \quad ...(2.5)$$

In addition to the Prandtl number σ, the problem is characterised by two parameters viz., ϵ and M^2. The highest order terms in equations (2.2) to (2.4) are multiplied by $1/M^2$. Consequently we can expect boundary layer type of flow for $1/M^2 << 1$. We shall assume this to be so and seek a solution of the problem for small values of ϵ. The situation corresponds to the case of a cylinder whose temperature oscillates harmonically with small amplitude and high frequency.

3. Solution in the Inner Layer

A solution is developed in the inner layer on the assumption that the non-linear inertia terms are of smaller order than the linear terms in the boundary layer equations.

We now define an inner variable as

$$\zeta = \frac{(r-1)}{\epsilon} \sigma^{1/2}$$

and seek perturbation solution of the form

$$\begin{aligned} u(\zeta, \theta, t) &= u_0(\zeta, \theta, t) + \epsilon^{1/2} u_1(\zeta, \theta, t) + \epsilon\, u_2(\zeta, \theta, t) + \ldots \\ v(\zeta, \theta, t) &= \epsilon\, v_0(\zeta, \theta, t) + \epsilon^{3/2} v_1(\zeta, \theta, t) + \epsilon^2 v_2(\zeta, \theta, t) + \ldots \\ P(\zeta, \theta, t) &= \epsilon^{-1} P_0(\zeta, \theta, t) + \epsilon^{-1/2} P_1(\zeta, \theta, t) + P_2(\zeta, \theta, t) + \ldots \\ \phi(\zeta, \theta, t) &= \phi_0(\zeta, \theta, t) + \epsilon^{1/2} \phi_1(\zeta, \theta, t) + \epsilon\, \phi_2(\zeta, \theta, t) + \ldots \end{aligned} \quad ...(3.1)$$

Substituting these expressions in the equations (2.1) to (2.4) and comparing like powers of ϵ we get

$$\frac{\partial u_0}{\partial t} = -\frac{\partial P_0}{\partial \theta} + \phi_0 \cos\theta + \sigma \frac{\partial^2 u_0}{\partial \zeta^2}$$

$$\frac{\partial P_0}{\partial \zeta} = 0, \quad \frac{\partial \phi_0}{\partial t} = \frac{\partial^2 \phi_0}{\partial \zeta^2}, \qquad \ldots(3.2)$$

$$\sigma^{1/2} \frac{\partial v_0}{\partial \zeta} + \frac{\partial u_0}{\partial \theta} = 0.$$

$$\frac{\partial u_1}{\partial t} = -\frac{\partial P_1}{\partial \theta} + \phi_1 \cos\theta + \sigma \frac{\partial^2 u_1}{\partial \zeta^2}$$

$$\frac{\partial P_1}{\partial \zeta} = 0, \quad \frac{\partial \phi_1}{\partial t} = \frac{\partial^2 \phi_1}{\partial \zeta^2}$$

$$\sigma^{1/2} \frac{\partial v_1}{\partial \zeta} + \frac{\partial u_1}{\partial \theta} = 0. \qquad \ldots(3.3)$$

$$\frac{\partial u_2}{\partial t} + \sigma^{1/2} v_0 \frac{\partial u_0}{\partial \zeta} + u_0 \frac{\partial u_0}{\partial \theta} = -\frac{\partial P_2}{\partial \theta} + \sigma^{-1/2} \zeta \frac{\partial P_0}{\partial \theta} + \phi_2 \cos\theta + \sigma \frac{\partial^2 u_2}{\partial \zeta^2} + \sigma^{1/2} \frac{\partial u_0}{\partial \zeta}$$

$$- \sigma^{1/2} \frac{\partial P_2}{\partial \zeta} + \phi_0 \sin\theta = 0$$

$$\frac{\partial \phi_2}{\partial t} + \sigma^{1/2} v_0 \frac{\partial \phi_0}{\partial \zeta} + u_0 \frac{\partial \phi_0}{\partial \theta} = \frac{\partial^2 \phi_2}{\partial \zeta^2} + \sigma^{-1/2} \frac{\partial \phi_0}{\partial \zeta}$$

and

$$\sigma^{1/2} \frac{\partial}{\partial \zeta} (v_2 + \sigma^{-1/2} \zeta v_0) + \frac{\partial u_2}{\partial \theta} = 0. \qquad \ldots(3.4)$$

It is to be noted that the terms $\sigma^{1/2}\partial u_0/\partial \zeta$ and $\sigma^{-1/2}\partial \phi_0/\partial \zeta$ in the set of equations (3.4) are due to the curvature of the cylinder. These terms are absent in Markin's analysis. In deriving the above equations we have taken $1/M^2 = 0(\epsilon^2)$. It is to be noted that the zeroth and first order equations do not contain additional terms due to curvature while curvature enters into the second order equations.

4. Solution

In view of the boundary conditions (2.5), the differential sets (3.2), (3.3) and (3.4) can easily be solved in the form, for $\sigma \neq 1$,

$$\phi_0 = f_1(\zeta) e^{it} \qquad \ldots(4.1)$$

$$u_0 = f_2(\zeta) \cos\theta e^{it} \qquad \ldots(4.2)$$

$$\phi_1 = \sin\theta\, A(\theta) \zeta \qquad \ldots(4.3)$$

$$u_1 = \sin\theta \cos\theta \frac{1}{\sigma} \{B(\theta)\zeta - A(\theta)\zeta^3/6\} \qquad \ldots(4.4)$$

$$\phi_2 = \tfrac{1}{4}\sin\theta\{H(\zeta) + 2\,G(\zeta)e^{2it}\} + \sigma^{-1/2}F(\zeta)e^{it} \qquad \ldots(4.5)$$

$$u_2 = \tfrac{1}{4}\sin 2\theta\{R(\zeta) + P(\zeta)e^{2it}\} + \sigma^{-1/2}\cos\theta\, Q(\zeta)e^{it} \qquad \ldots(4.6)$$

where

$$f_1(\zeta) = e^{-\frac{1+i}{\sqrt{2}}\zeta}$$

$$f_2(\zeta) = \frac{i}{\sigma-1}\left\{ e^{-\frac{1+i}{\sqrt{2}}\zeta} - e^{-\frac{1+i}{\sqrt{2\sigma}}\zeta} \right\}$$

$$F(\zeta) = -\frac{\zeta}{2}\, e^{-\frac{1+i}{\sqrt{2}}\zeta}$$

$$G(\zeta) = \frac{7\sigma - 4\sqrt{\sigma} - 3}{2(\sigma-1)(1+2\sqrt{\sigma}-\sigma)}\, e^{-(1+i)\zeta} - \frac{1}{1+\sqrt{\sigma}}\, e^{-\frac{1+i}{\sqrt{2}}\zeta}$$

$$- \frac{\sqrt{\sigma}}{(\sigma-1)(\sigma_1^2-2)}\, e^{-\sigma_1\frac{1+i}{\sqrt{2}}\zeta} + \frac{1}{2(\sigma-1)}\, e^{-\sqrt{2}(1+i)\zeta}$$

$$H(\zeta) = \frac{2\sigma^{3/2}}{(\sigma+1)^2}\sin\sigma_2\frac{\zeta}{\sqrt{2}}\, e^{-\sigma_1\frac{\zeta}{\sqrt{2}}}$$

$$- \frac{4\sigma^2}{(\sigma-1)(\sigma+1)^2}\cos\sigma_2\frac{\zeta}{\sqrt{2}}\, e^{-\sigma_1\frac{\zeta}{\sqrt{2}}}$$

$$- \frac{2}{1+\sqrt{\sigma}}\sin\frac{\zeta}{\sqrt{2}}\, e^{-\frac{\zeta}{\sqrt{2}}}$$

$$+ \frac{1}{\sigma-1}\, e^{-\sqrt{2}\zeta} + \frac{3\sigma+1}{(\sigma+1)^2} + C\zeta.$$

$$Q(\zeta) = \frac{i}{\sigma-1}\left\{\frac{\zeta}{2} - \frac{1-i}{\sqrt{2}}\right\}\left\{ e^{-\frac{1+i}{\sqrt{2\sigma}}\zeta} - e^{-\frac{1+i}{\sqrt{2}}\zeta}\right\}$$

$$P(\zeta) = \frac{i(7\sigma - 4\sqrt{\sigma} - 3)}{4(\sigma-1)^2(1+2\sqrt{\sigma}-\sigma)}\, e^{-(1+i)\zeta}$$

$$- \frac{i}{(\sigma-1)(\sqrt{\sigma}+1)}\, e^{-\frac{1+i}{\sqrt{2}}\zeta} + \frac{i}{\sqrt{\sigma}(\sigma-1)(\sqrt{\sigma}+1)}\, e^{-\frac{1+i}{\sqrt{2\sigma}}\zeta}$$

$$+ \frac{i}{4(\sigma-1)(2\sigma-1)}\, e^{-\sqrt{2}(1+i)\zeta}$$

$$+ \frac{i(\sigma^3 - 2\sigma^2 - 4\sigma + 4\sigma^{3/2} + 1)}{\sqrt{\sigma}(\sigma-1)^2(1-6\sigma+\sigma^2)}\, e^{-\sigma_1\frac{1+i}{\sqrt{2}}\zeta}$$

$$+ E_1\, e^{-\frac{1+i}{\sqrt{\sigma}}\zeta},$$

$$R(\zeta) = \frac{-1}{4(\sigma-1)^2} \cos \sqrt{\frac{2}{\sigma}}\, \zeta e^{-\sqrt{\frac{2}{\sigma}}\,\zeta}$$

$$+ \frac{\sigma+1}{\sigma^{3/2}\sigma_1^2(\sigma-1)^2} \cos \sigma_1 \frac{\zeta}{\sqrt{2}}\, e^{-\sigma_1 \frac{\zeta}{\sqrt{2}}}$$

$$- \frac{1}{4\sigma(\sigma-1)^2} e^{-\sqrt{2}\zeta} \cos \sqrt{2}\zeta$$

$$+ \frac{1}{\sqrt{\sigma}(1+\sqrt{\sigma})(\sigma-1)} \cos \frac{\zeta}{\sqrt{2\sigma}}\, e^{-\frac{\zeta}{\sqrt{2\sigma}}}$$

$$+ \frac{\sigma-2}{\sigma(1+\sqrt{\sigma})(\sigma-1)} \cos \frac{\zeta}{\sqrt{2}}\, e^{-\frac{\zeta}{\sqrt{2}}}$$

$$- \frac{1}{2(\sigma-1)^2} e^{-\sqrt{\frac{2}{\sigma}}\,\zeta} - \frac{\sigma+1}{4\sigma(\sigma-1)^2} e^{-\sqrt{2}\zeta}$$

$$+ 2E_2 \cos \sigma_2 \frac{\xi}{\sqrt{2}}\, e^{-\sigma_1 \frac{\zeta}{\sqrt{2}}} - 2E_3 \sin \sigma_2 \frac{\zeta}{\sqrt{2}}\, e^{-\sigma_1 \frac{\zeta}{\sqrt{2}}}$$

$$- \frac{C}{12\sigma} \zeta^3 - \frac{3\sigma+1}{2\sigma(\sigma+1)^2} \frac{\zeta^2}{2} + D\zeta + E_4.$$

with

$$E_1 = \frac{i}{(\sigma-1)^2} \left\{ \frac{\sigma^3 - 2\sigma^2 - 4\sigma + 4\sigma^{3/2} + 1}{\sqrt{\sigma}(1-\sigma+2\sqrt{\sigma})(\sigma-1+2\sqrt{\sigma})} + \frac{8\sigma^2 + 4\sigma - 17\sigma^{3/2} - 4 + 9\sigma^{1/2}}{4\sigma^{1/2}(2\sigma-1)} + \frac{7\sigma - 4\sigma^{1/2} - 3}{4(1+2\sqrt{\sigma}-\sigma)} \right\}$$

$$E_2 = \frac{2\sigma^{1/2}(\sigma^3+2\sigma+1)}{(\sigma-1)^2(\sigma+1)^4} - \frac{\sigma^{3/2}(\sigma-1)}{2(\sigma+1)^4},$$

$$E_3 = \frac{\sigma^3+2\sigma+1}{(\sigma-1)(\sigma+1)^4} + \frac{\sigma^2}{(\sigma+1)^4}$$

$$E_4 = \frac{2\sigma - 2\sigma^{3/2} + 6\sigma^{1/2} - 3}{2\sigma(\sigma-1)^2} - \frac{\sigma+1}{\sigma^{3/2}\sigma_1^2(\sigma-1)^2} + \frac{\sigma^{3/2}(\sigma-1)}{(\sigma+1)^4} - \frac{4\sigma^{1/2}(\sigma^3+2\sigma+1)}{(\sigma-1)^2(\sigma+1)^4}$$

$$\sigma_1 = 1 + \sigma^{-1/2}; \quad \sigma_2 = 1 - \sigma^{-1/2}$$

The above solutions contain $A(\theta)$, $B(\theta)$, C and D as arbitrary constants which remain undetermined at this stage even by using the given boundary conditions (2.5).

We write the asymptotic form for ϕ and u as

$$\phi = \sin\theta\left\{\epsilon^{1/2}A(\theta)\zeta + \epsilon\left(\frac{C}{4}\zeta + \frac{3\sigma+1}{4(\sigma+1)^2}\right)\right\} + O(\epsilon^{3/2}) \qquad \ldots(4.7)$$

$$u = \epsilon^{1/2}\sin\theta\cos\theta\left\{\left(\frac{1}{\sigma}\right)B(\theta)\zeta - \left(\frac{1}{6\sigma}\right)A(\theta)\zeta^3\right\}$$
$$+\epsilon\left\{(\tfrac{1}{4})\sin 2\theta\left(-\frac{C}{12\sigma}\zeta^3 - \frac{3\sigma+1}{2\sigma(\sigma+1)^2}\frac{\zeta^2}{2} + D\zeta + E_4\right)\right\} \qquad \ldots(4.8)$$

From (4.7) and (4.8), it is clear that the boundary conditions $u \to 0$ and $\phi \to 0$ when $\zeta \to \infty$ cannot be satisfied even by setting $A(\theta) = B(\theta) = C = D = 0$. We find that a steady tangential component of velocity of $O(\epsilon g\beta A T_0/\omega)$ and a steady temperature difference of $O(\epsilon A T_0)$ persist outside this inner layer. A similar situation arises in the analogous forced convection problem of flow due to an oscillating cylinder. Schlichting (1932) found that only the boundary conditions at the wall could be satisfied. He stipulated that the steady velocity must remain finite outside the "shear wave" layer. In the present problem we shall use the technique due to Stuart and introduce an outer layer and it is at the outer edge of this boundary layer that $u \to 0$ and $\phi \to 0$.

5. Solution in the Outer Layer

In the outer layer the convective terms are of the same order as the diffusive and buoyancy terms. Accordingly we introduce an outer variable as,

$$\bar{\zeta} = (r-1)\frac{\sigma^{1/2}}{\epsilon^{1/2}}$$

and the following expansions for flow and temperature as

$$u(\bar{\zeta},\theta,t) = u_0(\bar{\zeta},\theta,t) + \epsilon^{1/2}u_1(\bar{\zeta},\theta,t) + \epsilon u_2(\bar{\zeta},\theta,t) + \ldots$$
$$v(\bar{\zeta},\theta,t) = \epsilon^{1/2}v_0(\bar{\zeta},\theta,t) + \epsilon v_1(\bar{\zeta},\theta,t) + \epsilon^{3/2}v_2(\bar{\zeta},\theta,t) + \ldots \qquad \ldots(5.1)$$
$$P(\bar{\zeta},\theta,t) = \epsilon^{-1}P_0(\bar{\zeta},\theta,t) + \epsilon^{-1/2}P_1(\bar{\zeta},\theta,t) + P_2(\bar{\zeta},\theta,t) + \ldots$$
$$\phi(\bar{\zeta},\theta,t) = \epsilon\phi_0(\bar{\zeta},\theta,t) + \epsilon^{3/2}\phi_1(\bar{\zeta},\theta,t) + \epsilon^2\phi_2(\bar{\zeta},\theta,t) + \ldots$$

Substituting these expressions in the equations (2.1) to (2.4) and comparing like powers of ϵ we get,

$$\frac{\partial u_0}{\partial t} = -\frac{\partial P_0}{\partial\theta}$$

$$\frac{\partial P_0}{\partial\bar{\zeta}} = 0,\ \frac{\partial\phi_0}{\partial t} = 0$$

$$\sigma^{-1/2}\frac{\partial u_0}{\partial\theta} + \frac{\partial v_0}{\partial\bar{\zeta}} = 0. \qquad \ldots(5.2)$$

$$\frac{\partial u_1}{\partial t} = \sigma^{-1/2}\bar{\zeta}\,\frac{\partial P_0}{\partial \theta} - \frac{\partial P_1}{\partial \theta}$$

$$\frac{\partial P_1}{\partial \bar{\xi}} = 0,\ \frac{\partial \phi_1}{\partial t} = 0$$

$$\sigma^{-1/2}\,\frac{\partial u_1}{\partial \theta} + \frac{\partial}{\partial \bar{\zeta}}\,(v_1 + \sigma^{-1/2}\bar{\zeta}v_0) = 0. \qquad ...(5.3)$$

$$\frac{\partial u_2}{\partial t} = -\,\sigma^{1/2}v_0\,\frac{\partial u_0}{\partial \bar{\zeta}} - u_0\,\frac{\partial u_0}{\partial \theta} - \frac{\partial P_2}{\partial \theta} + \sigma^{-1/2}\bar{\zeta}\,\frac{\partial P_1}{\partial \theta}$$

$$+\,\sigma\,\frac{\partial^2 u_0}{\partial \bar{\zeta}^2} + \phi_0 \cos\theta \qquad ...(5.4)$$

$$\frac{\partial v_0}{\partial t} = -\,\sigma^{1/2}\,\frac{\partial P_2}{\partial \bar{\zeta}},$$

$$\frac{\partial \phi_2}{\partial t} = -\,\sigma^{1/2}\,v_0\,\frac{\partial \phi_0}{\partial \bar{\zeta}} - u_0\,\frac{\partial \phi_0}{\partial \theta} + \frac{\partial^2 \phi_0}{\partial \bar{\zeta}^2}$$

$$\sigma^{-1/2}\,\frac{\partial u_2}{\partial \theta} + \frac{\partial}{\partial \bar{\zeta}}\,(v_2 + \sigma^{-1/2}\bar{\zeta}v_1) = 0.$$

It is to be noted that the equations in the outer layer are not affected by the curvature of the cylinder as expected. Since we are interested only in the steady streaming the differential set (5.4) reduces to

$$\sigma\,\frac{\partial^2 u_0}{\partial \bar{\zeta}^2} + \phi_0 \cos\theta - u_0\,\frac{\partial u_0}{\partial \theta} - \sigma^{1/2}v_0\,\frac{\partial u_0}{\partial \bar{\zeta}} - \frac{\partial P_2}{\partial \theta} = 0,$$

$$\frac{\partial P_2}{\partial \bar{\zeta}} = 0, \qquad ...(5.5)$$

$$\frac{\partial^2 \phi_0}{\partial \bar{\zeta}^2} - u_0\,\frac{\partial \phi_0}{\partial \theta} - \sigma^{1/2}v_0\,\frac{\partial \phi_0}{\partial \bar{\zeta}} = 0.$$

Eliminating pressure by cross differentiation, integrating once with respect to $\bar{\zeta}$ and introducing

$$u_0 = \frac{\partial f}{\partial \bar{\zeta}},\ v_0 = -\,\sigma^{-1/2}\,\frac{\partial f}{\partial \theta} \qquad ...(5.6)$$

we get

$$\sigma\,\frac{\partial^3 f}{\partial \bar{\zeta}^3} + \phi_0 \cos\theta - \frac{\partial f}{\partial \bar{\zeta}}\cdot\frac{\partial^2 f}{\partial \bar{\zeta}\partial \theta} + \frac{\partial f}{\partial \theta}\cdot\frac{\partial^2 f}{\partial \bar{\zeta}^2} = 0 \qquad ...(5.7)$$

$$\frac{\partial^2 \phi_0}{\partial \bar{\zeta}^2} - \frac{\partial f}{\partial \bar{\zeta}}\cdot\frac{\partial \phi_0}{\partial \theta} + \frac{\partial f}{\partial \theta}\cdot\frac{\partial \phi_0}{\partial \bar{\zeta}} = 0 \qquad ...(5.8)$$

with the boundary conditions

$$\frac{\partial f}{\partial \bar{\zeta}} \to 0, \quad \phi_0 \to 0 \quad \text{as } \bar{\zeta} \to \infty. \qquad \text{...(5.9).}$$

The boundary conditions at $\bar{\zeta} = 0$ is obtained through matching with the inner layer solution. Writing the asymptotic form of the inner solution given by eqns. (4.7) and (4.8) in terms of the outer variable $\bar{\zeta}$ and then comparing with the outer solution yield the conditions

$$f = \sin\theta \cos\theta \left\{ B(\theta) \frac{\bar{\zeta}^2}{2\sigma} - \frac{3\sigma + 1}{24\sigma(\sigma+1)^2} \bar{\zeta}^3 \right\} \qquad \text{...(5.10)}$$

$$\phi_0 = \sin\theta \frac{3\sigma + 1}{4(\sigma+1)^2} \qquad \text{...(5.11)}$$

near $\bar{\zeta} = 0$.

In deriving these boundary conditions it is assumed, following Merkin, that $A(\theta) = 0$ and $C = 0$.

We expand f and ϕ as

$$\begin{aligned} f(\eta) &= \lambda\{\theta h_0(\eta) - (a_2/a_1)\theta^3 h_1(\eta) - a_1\theta^3 h_{10}(\eta) + \ldots\} \\ \phi_0(\eta) &= a_1\theta g_0(\eta) - a_2\theta^3 g_1(\eta) - a_1^2\theta^3 g_{10}(\eta) + \ldots \\ B(\theta) &= B_0 + B_1\theta^2 + \ldots \\ \cos\theta &= 1 - (a_{1/2})\,\theta^2 + a_2\theta^4/4 + \ldots, \end{aligned} \qquad \text{...(5.12)}$$

where

$$\lambda\bar{\zeta} = \eta$$

and

$$\lambda^4 = a_1.$$

Introducing the series expansions (5.12) into eqns. (5.7) to (5.11) and equating various powers of θ, we get

$$\left.\begin{aligned} \sigma h_0''' + g_0 + h_0 h_0'' - h_0'^2 &= 0 \\ g_0'' + h_0 g_0' - h_0' g_0 &= 0 \end{aligned}\right\} \qquad \text{...(5.13)}$$

$$\left.\begin{aligned} \sigma h_1''' + g_1 + h_0 h_1'' - 4h_0' h_1' + 3h_0'' h_1 &= 0 \\ g_1'' + 3g_0' h_1 + h_0 g_1' - g_0 h_1' - 3h_0' g_1 &= 0 \end{aligned}\right\} \qquad \text{...(5.14)}$$

$$\left.\begin{aligned} \sigma h_{10}''' + \tfrac{1}{2}g_0 + g_{10} + h_0 h_{10}'' - 4h_0' h_{10}' + 3h_{10} h_0'' &= 0 \\ g_{10}'' + 3g_0' h_{10} + h_0 g_{10}' - g_0 h_{10}' - 3h_0' g_{10} &= 0. \end{aligned}\right\} \qquad \text{(5.15)}$$

with the bondary conditions

$$h_0(0) = h_0'(0) = h_0'(\infty) = 0, \qquad g_1(0) = \frac{3\sigma + 1}{4(1+\sigma)^2}\ g_1(\infty) = 0$$

$$g_0(0) = \frac{1 + 3\sigma}{4(1+\sigma)^2}\ g_0(\infty) = 0, \qquad h_{10}(0) = h_{10}'(0) = h_{10}'(\infty) = 0 \qquad \ldots(5.16)$$

$$h_1(0) = h_1'(0) = h_1'(\infty) = 0, \qquad g_{10}(0) = 0 \quad \text{and} \quad g_{10}(\infty) = 0$$

Dashes denote differentiation with respect to η.

These are the same equations as obtained by Merkin. He has given the numerical solution to these equations for various values of σ. It is to be noted that if we neglect the terms of order $O(\sqrt{\epsilon})$ in the inner layer expansion the matching condition on f would be

$$f = \sin\theta \cos\theta \left\{ -\frac{3\sigma + 1}{24\sigma(\sigma+1)^2}\ \frac{\eta^3}{\lambda^3} \right\}$$

near $\eta = 0$. This gives $h_0''(O) = 0$. With this extra condition on $h_0(\eta)$ the eqns. (5.13) cannot be solved.

The two constants B_0 and B_1 in the expansion of $B(\theta)$ are computed by Merkin from

$$B_0 = -\frac{\sigma}{\lambda} h_0''(0)$$

$$B_1 = -\frac{\sigma}{\lambda}\left[\left(\frac{a_2}{a_1}\right)\left\{ h_0''(0) - h_1''(0) \right\} + a_1 \left\{ \frac{h_0''}{2}(0) - h_{10}''(0) \right\}\right]$$

For the case of a circular cylinder with $a_1 = 1$, $a_2 = \frac{1}{6}$ and $\lambda^4 = a_1$ he gets, for $\sigma = 1$,

$$B(\theta) = -0.1849 - 0.0360\,\theta^2 + \ldots$$

6. Heat Transfer and Shear Stress

In a free convection problem the quantities of engineering interest are the heat transfer and the shear stress from and on the boundary. The heat transfer coefficient $-K(\partial T'/\partial r')_{r'=a}$ can be expressed in terms of the Nusselt Number as

$$N = -\frac{a}{T_w - T_0} \cdot \left(\frac{\partial T'}{\partial r'}\right)_{r'=a}$$

$$= -\epsilon^{-1}\sigma^{1/2}\left[f_1'(0) + \epsilon\sigma^{-1/2}F'(0) + \frac{\epsilon}{4}\left\{ 2G'(0)\,e^{it} + H'(0)\,e^{-it} \right\}\right] \qquad \ldots(6.1)$$

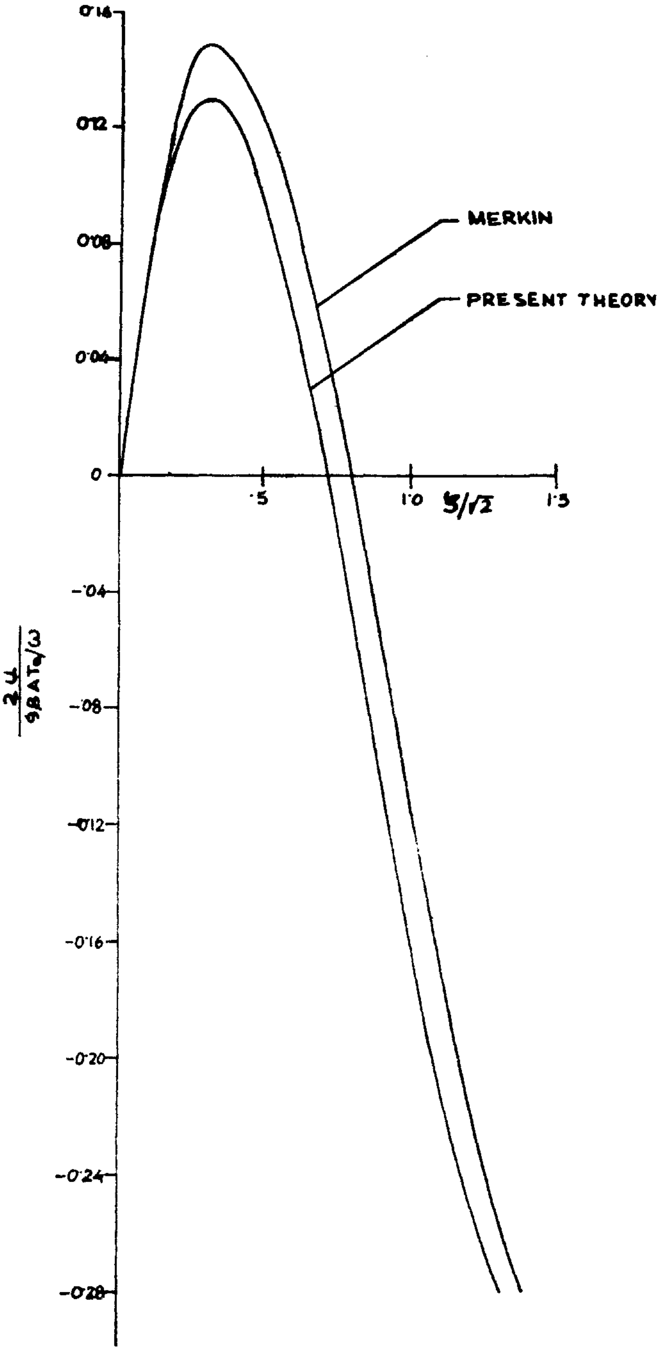

FIG. 2. Tangential velocity profile at $\theta = 0$ and $t = 0$ with $\in = 0.1$ and $\sigma = 1$.

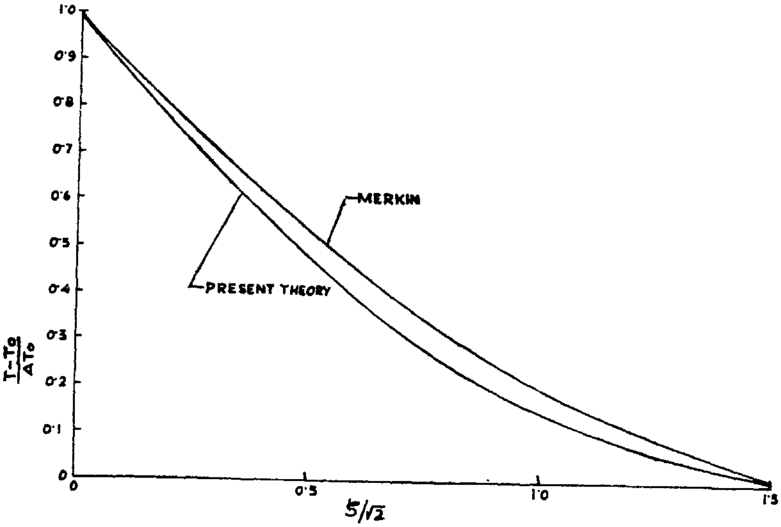

FIG. 3. Temperature distribution at $\theta = 0$ and $t = 0$ with $\epsilon = 0.1$ and $\sigma = 1$.

An average Nusselt Number can be defined as

$$N_{av} = \int_0^{2\pi} N d\theta = \epsilon^{-1}\sigma^{1/2}\left[2\pi \cos\frac{\pi}{4} + \epsilon\sigma^{-1/2}\pi\right] \qquad ...(6.2)$$

The average Nusselt number contains an additional term proportional to π and is of $O(1)$. This term may be attributed to the interaction between the curvature and thermal conductivity.

The shear stress at the surface is

$$\tau_s = \mu\left[r'\frac{\partial}{\partial r'}\left(\frac{u'}{r'}\right) + \frac{1}{r'}\frac{\partial v'}{\partial\theta'}\right]_{r'=a}$$

$$= \rho\left(\frac{g\beta A T_0}{\omega}\right)^2\left[\sigma^{1/2}\frac{\sigma_2}{\sigma-1}\cos\theta\left\{\cos\left(t-\frac{\pi}{4}\right) + \epsilon\sigma^{-1/2}\sin t\right\}\right.$$

$$\left. + \sin 2\theta\left\{\frac{\epsilon^{1/2}}{2\sigma}B(\theta) + \frac{\epsilon}{4}C_3\cos\left(2t-\frac{\pi}{4}\right) + \frac{\epsilon}{4}C_4\right\}\right] \qquad ...(6.3)$$

where

$$B(\theta) = 0.2615 + 0.0509\,\theta^2 + \ldots$$

$$C_3 = \frac{(\sigma^3 - 2\sigma^2 - 4\sigma + 4\sigma^{3/2} + 1)(2^{1/2} - 1 - \sigma^{1/2})}{\sigma(\sigma-1)^2(1-\sigma+2\sigma^{1/2})(\sigma-1+2\sigma^{1/2})}$$

$$+ \frac{-4.2^{1/2}.\sigma^2 + (8+2^{1/2})\sigma^{3/2} + (7.2^{1/2}-9)\sigma - 5\sigma^{1/2} + (4-2.2^{1/2})}{2.2^{1/2}\sigma(\sigma-1)(2\sigma-1)(\sigma^{1/2}-1)}$$

$$+ \frac{\sigma_2(7\sigma - 4\sigma^{1/2} - 3)}{2.2^{1/2}(\sigma-1)^2(1+2\sigma^{1/2}-\sigma)},$$

$$C_4 = \frac{-2\sigma^{5/2} + 7\sigma^{3/2} + 3\sigma^{1/2} - \sigma^2 + 11\sigma - 2}{2.2^{1/2}(\sigma - 1)^2(\sigma + 1 - 2\sigma^{1/2})\sigma}$$

$$- \frac{2^{1/2}\sigma_1(\sigma^3 + 2\sigma + 1)(1 + 2\sigma^{1/2} - \sigma)}{(\sigma - 1)^2(\sigma + 1)^4}$$

$$+ \frac{\sigma(1 + \sigma^{1/2})(\sigma + 2\sigma^{1/2} - 1)}{2^{1/2}(\sigma + 1)^4} .$$

The velocity and temperature profiles at $\theta = 0$ and $\epsilon\sigma^{-1/2} = 0.1$ are plotted in Figs. 2 and 3, and compared with the corresponding profiles obtained from Merkin. The figures show that surface curvature has a noticeable influence on the temperature and velocity distribution.

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