AXISYMMETRIC VIBRATIONS OF ORTHOTROPIC SHELLS IN A MAGNETIC FIELD

by N. C. DAS *and* S. K. BHATTACHARYA, *Department of Mathematics, Jadavpur University, Calcutta 700032*

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In this paper, asymptotic solutions have been obtained for the problem of wave propagations in an infinite, perfectly conducting cylindrical shell when it is placed in a magnetic field.

INTRODUCTION

THE studies on magneto-elasticity in recent years can be described as the attempts to accomodate a suitable magnetic field in elastic problems and thereby to study their influence on the mechanical phenomena. In this connection, the papers of Kaliski (1960a, b; 1963), Knopoff (1955), Suhubi (1966), Paria (1961), Murthy (1968), Bakshi (1969), Das (1967), are relevant to be cited. In these problems the materials were considered to be homogeneous and isotropic. However, recent advances in material science require that the anisotrophy of materials be taken into account. In particular, missile designers, geophysicists, and solid state physicists are confronted with a variety of problems related to anisotropic materials. It is well known that for the problems of thick-shell and plate like structures—the theories which are based on two-dimensional simplifications of three dimensional theories, are quite unsuitable and must be replaced by the solutions to the three dimensional equations of elasticity.

Mirsky (1964, 1965, 1966) has considered the problem of axisymmetric vibrations of orthotropic cylinders and cylindrical shells, and in the following note an attempt has been made to investigate the solutions for axisymmetric vibrations of an orthotropic shell permeated by a magnetic field. In addition, asymptotic solutions are developed, since the power series solutions are not suitable for all frequencies.

EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

A circular cylindrical shell of orthotropic elastic material of inner and outer radii a and b respectively and subjected to an axial magnetic field is considered. It is treated as a perfect conductor and the region inside and outside the elastic material are assumed to be vacuum.

With reference to the cylindrical co-ordinates (r, θ, z), the displacement field for the case of axisymmetric motions is characterized by

$$u = u(r, z, t),\ v = 0,\ w = w(r, z, t), \qquad ...(1)$$

where u, v, w are, respectively, the components in the radial, circumferential and axial directions and all other quantities involved are functions of r, z and t only.

In the absence of displacement current and charge density, the Maxwell's equations reduce to (in C.G.S. Unit)

$$\left.\begin{array}{ll} \text{Curl } \mathbf{H} = \dfrac{4\pi}{C}\mathbf{J}, & \text{Curl } \mathbf{E} = -\dfrac{1}{C_1}\dfrac{\partial \mathbf{B}}{\partial t} \\ \text{div } \mathbf{B} = 0 \quad , & \mathbf{B} = \mu \mathbf{H} \end{array}\right\} \qquad \ldots(1a)$$

while Ohm's law is given by

$$\mathbf{J} = \sigma\left[\mathbf{E} + \frac{1}{c_1}\left(\frac{\partial \mathbf{u}}{\partial t} \times \mathbf{B}\right)\right]. \qquad \ldots(1b)$$

Taking into account of Lorentz body force, the stress equations of motion become (cf. Love, 1944; and Kaliski, 1963)

$$\left.\begin{array}{l} \dfrac{\partial \sigma_{rr}}{\partial r} - \dfrac{\partial \tau_{rz}}{\partial z} + \dfrac{1}{r}\sigma_{rr} + \dfrac{1}{C_1}(\mathbf{J} \times \mathbf{B})_r = \rho\dfrac{\partial^2 u}{\partial t^2} \\ \text{and} \\ \dfrac{\partial \sigma_{zz}}{\partial z} + \dfrac{1}{r}\tau_{rz} + \dfrac{\partial \tau_{rz}}{\partial r} + \dfrac{1}{C_1}(\mathbf{J} \times \mathbf{B})_z = \rho\dfrac{\partial^2 w}{\partial t^2} \end{array}\right\} \qquad \ldots(1c)$$

For three mutually orthogonal planes of elastic symmetry, the stress-strain relations are given in terms of nine independent elastic constants c_{ij} as follows (cf. Love, 1944)

$$\left.\begin{array}{l} \sigma_{rr} = c_{11}e_{rr} + c_{12}e_{\theta\theta} + c_{13}e_{zz}, \\ \sigma_{\theta\theta} = c_{12}e_{rr} + c_{22}e_{\theta\theta} + c_{23}e_{zz}, \\ \sigma_{zz} = c_{13}e_{rr} + c_{23}e_{\theta\theta} + c_{33}e_{zz} \\ \text{and} \\ \tau_{rz} = c_{44}e_{rz}, \quad \tau_{\theta z} = c_{55}e_{\theta z}, \quad \tau_{r\theta} = c_{66}e_{r\theta} \end{array}\right\} \qquad \ldots(1d)$$

where the strain components are given by

$$e_{rr} = \frac{\partial u}{\partial r}, \; e_{\theta\theta} = \frac{u}{r}, \; e_{zz} = \frac{\partial w}{\partial z}, \; e_{rz} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}, \; e_{r\theta} = e_{\theta z} = 0.$$

The equations in vacuum are

$$\left.\begin{array}{l} \left(\nabla^2 - \dfrac{1}{C_1^2}\dfrac{\partial^2}{\partial t^2}\right)\mathbf{H}^*, \mathbf{E}^* = 0 \\ \text{and} \\ \text{Curl } \mathbf{E}^* = -\dfrac{1}{C_1}\dfrac{\partial \mathbf{H}^*}{\partial t}, \; \text{Curl } \mathbf{H}^* = \dfrac{1}{C_1}\dfrac{\partial \mathbf{E}^*}{\partial t}, \end{array}\right\} \qquad \ldots(1e)$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

and starred quantities represent the value of **H** and **B** in vacuum.

Substituting the values of the stresses from (1*d*) in the equations of motion (1*c*) and using (1*a*) and (1*b*), the equations of magneto-elasticity for an orthotropic perfectly conducting elastic medium are

$$c_{11}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)-c_{22}\frac{u}{r^2}+c_{44}\frac{\partial^2 u}{\partial z^2}+(c_{44}+c_{13})\frac{\partial^2 w}{\partial r\partial z}+$$
$$\frac{1}{r}(c_{13}-c_{23})\frac{\partial w}{\partial z}+\frac{\mu}{4\pi}\{[\text{ rot rot }(\mathbf{u}\times\mathbf{H}_0)]\times\mathbf{H}_0\}_r$$
$$=\rho\frac{\partial^2 u}{\partial t^2}, \qquad \ldots(2)$$

and

$$(c_{44}+c_{13})\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}(c_{44}+c_{23})\frac{\partial u}{\partial z}+c_{44}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)$$
$$+c_{33}\frac{\partial^2 w}{\partial z^2}+\frac{\mu}{4\pi}\{[\text{rot rot }(\mathbf{u}\times\mathbf{H}_0)]\times\mathbf{H}_0\}_z=\rho\frac{\partial^2 w}{\partial t^2} \qquad \ldots(3)$$

where

$$\mathbf{E}=-\frac{1}{c_1}\left(\frac{\partial \mathbf{u}}{\partial t}\times\mathbf{H}_0\right),\ \mathbf{h}=\text{rot }(\mathbf{u}\times\mathbf{H}_0),$$
$$\mathbf{H}=\mathbf{H}_0+\mathbf{h}\qquad ,\ \mathbf{H}_0=(0,0,H),\ (\mu\approx 1), \qquad \ldots(4)$$

c_{ij} are the elastic constants, ρ the mass density, t time, $\mathbf{E}$ the electric intensity vector, $\mathbf{h}$, the perturbed component of the original field $\mathbf{H}$, C_1 the velocity of light in vacuum, $\mu\, h$ the permeability of the medium and the subscripts r and z indicate the components in r and z directions.

The equations in vacuum are [from eqn. (1*e*)]

$$\left.\begin{aligned}&\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)h_r^*=0\\&\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)h_z^*=0\\&\text{rot }\mathbf{E}^*=-\frac{1}{C_1}\frac{\partial\mathbf{h}^*}{\partial t},\ \text{rot }\mathbf{h}^*=\frac{1}{C_1}\frac{\partial\mathbf{E}^*}{\partial t},\end{aligned}\right\} \qquad \ldots(5)$$

where $\mathbf{h}^*=(h_r^*, 0, h_z^*)$ and asterisk denotes quantities in vacuum.

As the magnetic field is directed axially, eqns. (2) and (3) reduce to

$$\left(c_{11}+\frac{H^2}{4\pi}\right)\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)-\left(c_{22}+\frac{H^2}{4\pi}\right)\frac{u}{r^2}+$$
$$\left(c_{44}+\frac{H^2}{4\pi}\right)\frac{\partial^2 u}{\partial z^2}+(c_{44}+c_{13})\frac{\partial^2 w}{\partial r\partial z}+$$
$$\frac{1}{r}(c_{13}-c_{23})\frac{\partial w}{\partial z}=\rho\frac{\partial^2 u}{\partial t^2} \qquad \ldots(6)$$

and

$$(c_{44} + c_{13}) \frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r} (c_{44} + c_{23}) \frac{\partial u}{\partial z} + c_{44} \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) + c_{33} \frac{\partial^2 w}{\partial z^2} = \rho \frac{\partial^2 w}{\partial t^2}. \quad \ldots(7)$$

To obtain the propagation of free harmonic waves in a hollow cylinder of infinite extent, we seek solutions to the equations of motion (6) and (7) in the form

$$\left. \begin{aligned} u(\xi, z, t) &= U(\xi) \cos(\omega t + \alpha z) \\ \text{and} \quad w(\xi, z, t) &= W(\xi) \sin(\omega t + az). \end{aligned} \right\} \quad \ldots(8)$$

where ω is the angular frequency, α is the wave number, $\xi = \alpha r$ is the dimensionless radial co-ordinate.

Again we take the solutions of the eqn. (5) in the form

$$\left. \begin{aligned} h_r^* &= \bar{h}_r^* (\xi) \sin(\omega t + \alpha z) \\ \text{and} \quad h_z^* &= \bar{h}_z^* (\xi) \cos(\omega t + \alpha z) \end{aligned} \right\} \quad \ldots(9)$$

Using (8) and (9) in the set of eqns. (5), (6) and (7), we get

$$\left. \begin{aligned} \frac{d^2 \bar{h}_r^*}{d\xi^2} + \frac{1}{\xi} \frac{d\bar{h}_r^*}{d\xi} - k^2 \bar{h}_r^* &= 0 \\ \frac{d^2 \bar{h}_z^*}{d\xi^2} + \frac{1}{\xi} \frac{d\bar{h}_z^*}{d\xi} - k^2 \bar{h}_z^* &= 0 \end{aligned} \right\} \quad \ldots(10)$$

$$\left. \begin{aligned} \xi^2 U'' + \xi U' + (\beta^2 \xi^2 - m^2) U + c\xi^2 W' + d\xi W &= 0 \\ \xi W'' + W' + Y^2 \xi W - e\xi U' - fU &= 0 \end{aligned} \right\} \quad \ldots(11)$$

where primes denote differentiation with respect to ξ and

$$\left. \begin{aligned} k^2 &= \left(1 - \frac{\omega^2}{\alpha^2 C_1^2}\right), \quad m^2 = \frac{c_{22} + \frac{H^2}{4\pi}}{c_{11} + \frac{H^2}{4\pi}}, \\ \beta^2 &= \frac{\rho\omega^2 - \alpha^2 \left(c_{44} + \frac{\mu H^2}{4\pi}\right)}{\alpha^2 \left(c_{11} + \frac{\mu H^2}{4\pi}\right)}, \quad c = \frac{c_{44} + c_{13}}{c_{11} + \frac{\mu H^2}{4\pi}} \end{aligned} \right\} \quad \ldots(12)$$

$$\gamma^2 = \frac{\rho\omega^2 - \alpha^2 c_{33}}{\alpha^2 c_{44}};\ e = \frac{c_{44} + c_{13}}{c_{44}},$$

$$d = \frac{c_{13} - c_{23}}{c_{11} + \frac{H^2}{4\pi}};\ f = \frac{c_{44} + c_{23}}{c_{44}} = e\left(1 - \frac{d}{c}\right).$$

For free motions, the boundary conditions require that the total stresses vanish on the surfaces $r = a, b$ i.e.,

$$\left.\begin{aligned} &\sigma_{rr} + T_{rr} - T^*_{rr} = 0 \quad \text{on } \xi = \alpha a, \alpha b \\ \text{and} \\ &\tau_{rz} + T_{rz} - T^*_{rz} = 0 \quad \text{on } \xi = \alpha a, \alpha b, \end{aligned}\right\} \qquad \ldots(13)$$

where σ_{rr}, τ_{rz} are the components of the mechanical stresses, T_{rr}, T_{rz} are the components of the Maxwell's stresses in the medium and T^*_{rr}, T^*_{rz} are the Maxwell's stresses in vacuum and the continuity of the electric and magnetic field on the surfaces $r = a, b$ i.e.,

$$h_z = h^*_z,\ E_\theta = E^*_\theta\ (\mu \approx 1). \qquad \ldots(14)$$

Solution

The appropriate solutions of the set of eqn. (10) are as follows :

$$\left.\begin{aligned} \bar{h}^*_r &= M I_0(k\xi) \\ \bar{h}^*_z &= P I_0(k\xi) \end{aligned}\right\} \text{ for } \xi < \alpha a$$

and

$$\left.\begin{aligned} \bar{h}^*_r &= N K_0(k\xi) \\ \bar{h}^*_z &= Q K_0(k\xi) \end{aligned}\right\} \text{ for } \xi > \alpha b,$$

where M, P, N and Q are arbitrary constants and I_0 and K_0 are the modified Bessel's function of first and second kind of order zero respectively. Neglecting terms of the order $O(\xi^{-3})$, the asymptotic representations for $\bar{h}^*_r$ and $\bar{h}^*_z$ may be written as

$$\left.\begin{aligned} &\bar{h}^*_r \simeq M \frac{e^{k\xi}}{\sqrt{2\pi k\xi}}\left\{1 + \frac{1}{8k\xi} + \frac{9}{128k^2\xi^2}\right\} \\ \text{and} \\ &\bar{h}^*_z \simeq P \frac{e^{k\xi}}{\sqrt{2\pi k\xi}}\left\{1 + \frac{1}{8k\xi} + \frac{9}{128k^2\xi^2}\right\} \end{aligned}\right\} \text{ for } \xi < \alpha a \qquad \ldots(15)$$

$$\left.\begin{aligned} \bar{h}_r^* &\simeq N\sqrt{\frac{\pi}{2k\xi}}\, e^{-k\xi}\left\{1-\frac{1}{8k\xi}+\frac{9}{128k^2\xi^2}\right\} \\ \text{and}\qquad & \qquad\qquad\qquad\qquad \text{for } \xi > \alpha b \\ \bar{h}_z^* &\simeq Q\sqrt{\frac{\pi}{2k\xi}}\, e^{-k\xi}\left\{1-\frac{1}{8k\xi}+\frac{9}{128k^2\xi^2}\right\} \end{aligned}\right\} \qquad \ldots(16)$$

Although the solutions of the eqns. (11) can be obtained by the Frobinious power series method, *vide* Mirsky (1964), the asymptotic solutions will be obtained here since the power series solution are not suitable for all frequencies.

Let us assume U of the form

$$U = e^{\lambda\xi} \sum_{n=0}^{\infty} c_n \xi^{\sigma-n} \qquad \ldots(17)$$

Substitution of this expression for U into the first of the eqns. (11) yields on integration

$$W = \xi^{-d/c} \sum_{n=0}^{\infty} c_n \int g_n(\xi)\, d\xi, \qquad \ldots(18)$$

where

$$g_n(\xi) = -\frac{1}{c}\,[(\beta^2+\lambda^2)+\lambda(2\sigma-2n+1)\,\xi^{-1} + \{(\sigma-n)^2-m^2\}\,\xi^{-2}]\, e^{\lambda\xi}\xi^{\sigma-n+d/c} \qquad \ldots(19)$$

If (17) and (18) are introduced into the second part of eqns. (11), the constants λ, σ and the co-efficients c_n are determined by equating the various powers of ξ. In particular we find that for $\sigma = -\frac{1}{2}$, λ satisfy the quadratic equation

$$\lambda^4 + (\beta^2+\gamma^2+ec)\,\lambda^2 + \beta^2\gamma^2 = 0 \qquad \ldots(20)$$

and the co-efficient c_1 and c_2 in terms of the arbitrary constants c_0 are given by

$$\left.\begin{aligned} &\alpha_{00}c_0 + \alpha_{01}c_1 = 0 \\ \text{and}\qquad &\alpha_{02}c_0 + \alpha_{03}c_1 + \alpha_{04}c_2 = 0, \end{aligned}\right\} \qquad \ldots(21)$$

where

$$\alpha_{00} = \frac{1}{\lambda}\left(\tfrac{1}{4}-m^2\right)(\lambda^2+\gamma^2) + \frac{1}{\lambda^3}(\beta^2+\lambda^2)\left[\lambda^2\frac{d^2}{c^2}+\gamma^2\left(\tfrac{1}{2}-\frac{d}{c}\right)\left(\tfrac{3}{2}-\frac{d}{c}\right)\right]$$

$$\alpha_{01} = c\left(f-\tfrac{3}{2}\,e\right) - 2(\lambda^2+\gamma^2) + \frac{1}{\lambda^2}(\beta^2+\lambda^2)\left[\gamma^2\left(\tfrac{3}{2}-\frac{d}{c}\right)-\lambda^2\left(\tfrac{1}{2}+\frac{d}{c}\right)\right]$$

(*equation continued on next page*)

$$\alpha_{02} = (\tfrac{1}{4} - m^2)\left[\frac{\gamma^2}{\lambda^2}\left(\tfrac{5}{2} - \frac{d}{c}\right) - \left(\tfrac{3}{2} + \frac{d}{c}\right)\right] +$$

$$\frac{1}{\lambda^2}\left(\tfrac{1}{2} - \frac{d}{c}\right)(\beta^2 + \lambda^2)\left[\frac{d^2}{c^2} + \frac{\gamma^2}{\lambda^2}\left(\tfrac{3}{2} - \frac{d}{c}\right)\left(\tfrac{5}{2} - \frac{d}{c}\right)\right]$$

$$\alpha_{03} = \lambda\left(3 + \frac{2d}{c}\right) + \frac{1}{\lambda}(\tfrac{9}{4} - m^2)(\lambda^2 + \gamma^2) -$$

$$2\frac{\gamma^2}{\lambda}\left(\tfrac{5}{2} - \frac{d}{c}\right) + \frac{1}{\lambda}(\beta^2 + \lambda^2)\left[\frac{d^2}{c^2} + \right.$$

$$\left.\frac{\gamma^2}{\lambda^2}\left(\tfrac{3}{2} - \frac{d}{c}\right)\left(\tfrac{5}{2} - \frac{d}{c}\right)\right]$$

$$\alpha_{04} = c(f - \tfrac{5}{2}e) - 4(\lambda^2 + \gamma^2) +$$

$$\frac{1}{\lambda^2}(\beta^2 + \lambda^2)\left[\gamma^2\left(\tfrac{5}{2} - \frac{d}{c}\right) - \lambda^2\left(\tfrac{3}{2} + \frac{d}{c}\right)\right] \quad \ldots(22)$$

The form of the solution depends on the nature of the roots of the eqn. (20) for λ. For most orthotropic materials,

$$\rho\omega^2 > \alpha^2 c_{33} \Rightarrow \rho\omega^2 > \alpha^2 c_{44},$$

which implies $\gamma^2 > 0$. Now since

$$\beta^2 = \frac{\rho\omega^2 - \alpha^2\left(c_{44} + \dfrac{H^2}{4\pi}\right)}{\alpha^2\left(c_{11} + \dfrac{H^2}{4\pi}\right)}$$

therefore imposing conditions on H, the intensity of the magnetic field, we shall consider the following three cases.

Case I: If we choose H, in such a way that

$$\rho\omega^2 > \alpha^2\left(c_{44} + \frac{H^2}{4\pi}\right),$$

then $\beta^2 > 0$. Therefore in this case all the roots of eqn. (20) of λ take the form,

$$\lambda = \pm i\lambda_1, \pm i\lambda_2 \quad (i^2 = -1),$$

where

$$\lambda_{1,2}^2 = \tfrac{1}{2}(\beta^2 + \gamma^2 + ec) \pm \{(\beta^2 + \gamma^2 + ec)^2 - 4\beta^2\gamma^2\}^{1/2}]$$

Neglecting terms of the order $O(\xi^{-3})$, the asymptotic representations for U and W may be written as

$$\tilde{U} \sim (A \cos \lambda_1\xi + B \sin \lambda_1\xi)(\xi^{-1/2} + c_2^{(1)}\xi^{-5/2} + (B \cos \lambda_1\xi - A \sin \lambda_1\xi) c_1^{(1)} \xi^{-3/2} + (C \cos \lambda_2\xi + D \sin \lambda_2\xi)(\xi^{-1/2} + c_2^{(2)}\xi^{-5/2}) + (D \cos \lambda_2\xi - C \sin \lambda_2\xi) c_1^{(2)} \xi^{-3/2} \quad \ldots(23a)$$

$$\tilde{W} \sim (B \cos \lambda_1\xi - A \sin \lambda_1\xi)(a_{11}\xi^{-1/2} + a_{12}\xi^{-5/2}) - (A \cos \lambda_1\xi + B \sin \lambda_1\xi) a_{13}\xi^{-3/2} + (D \cos \lambda_2\xi - C \sin \lambda_2\xi)(\bar{a}_{11}\xi^{-1/2} + \bar{a}_{12}\xi^{-5/2}) - (C \cos \lambda_2\xi + D \sin \lambda_2\xi) \bar{a}_{13}\xi^{-3/2} \quad \ldots(23b)$$

where,

$$a_{11} = \frac{\beta^2 - \lambda_1^2}{\lambda_1 c}$$

$$a_{12} = \frac{1}{c}\left[(2\lambda_1 c_1^{(1)} - \lambda_1^2 c_2^{(1)} + \tfrac{1}{4} - m^2 + \frac{\beta^2}{\lambda_1} c_2^{(1)} - (\beta^2 - \lambda_1^2)\left(\frac{d}{c} - \tfrac{1}{2}\right)\left(\frac{d}{c} - \tfrac{3}{2}\right)\Big/\lambda_1^3 + \left(\frac{d}{c} - \tfrac{3}{2}\right)(\lambda_1^2 - \beta^2)\, c_1^{(1)}/\lambda_1^2\right]$$

$$a_{13} = \frac{1}{c}(\beta^2 - \lambda_1^2)\left[\left(\frac{d}{c} - \tfrac{1}{2}\right)\Big/\lambda_1^2 + \frac{c_1^{(1)}}{\lambda_1}\right]$$

$$\bar{a}_{11} = a_{11}\big]_{\lambda_1=\lambda_2}$$

$$\bar{a}_{12} = a_{12}\big]_{\substack{\lambda_1=\lambda_2\\ c_1^{(1)}=c_1^{(2)}\\ c_2^{(1)}=c_2^{(2)}}}$$

$$\bar{a}_{13} = a_{13}\big]_{\substack{\lambda_1=\lambda_2\\ c_1^{(1)}=c_1^{(2)}}}$$

$$c_0 = 1,\ ic_1^{(2)} = c_1(\lambda_2),\ ic_1^{(1)} = c_1(\lambda_1)$$

$$c_2^{(2)} = c_2(\lambda_2),\ c_2^{(1)} = c_2(\lambda_1)$$

and A, B, C and D are the arbitrary constants of integration.

Case II :

Let H be given by

$$\rho\omega^2 = \alpha^2\left(c_{44} + \frac{H^2}{4\pi}\right),$$

which implies $\beta^2 = 0$. The roots of the eqn. (20) for λ, take the form

$$\lambda = \pm\, i\lambda_3,\ 0,\ 0$$

where,

$$\lambda_3^2 = (\gamma^2 + ec).$$

Again neglecting terms of the order $O(\xi^{-3})$, the asymptotic representation of U and W may be written as

$$\bar{U} \sim (E\cos\lambda_3\xi + F\sin\lambda_3\xi)\,(\xi^{-1/2} + c_2^{(3)}\xi^{-5/2}) + (F\cos\lambda_3\xi - E\sin\lambda_3\xi)\,c_1^{(3)}\,\xi^{-3/2} + G(\xi^{-1/2} + c_1^{(0)}\,\xi^{-3/2} + c_2^{(0)}\,\xi^{-5/2}) \qquad \ldots(24a)$$

and

$$\bar{W} \sim (F\cos\lambda_3\xi - E\sin\lambda_3\xi)\,(a_{11}\xi^{-1/2} + a_{12}\xi^{-5/2}) - (E\cos\lambda_3\xi + F\sin\lambda_3\xi)\,a_{13}\xi^{-3/2} - G(\bar{a}_{11}\xi^{-3/2} + \bar{a}_{12}\xi^{-5/2}), \qquad \ldots(24b)$$

where

$$a_{11} = \frac{\beta^2 - \lambda_3^2}{\lambda_3 c},$$

$$a_{12} = \frac{1}{c}\Big[2\lambda_3 c_1^{(3)} - \lambda_3^2\,c_2^{(3)} + \tfrac{1}{4} - m^2 + \beta^2 c_2^{(3)}/\lambda_3 - (\beta^2 - \lambda_3^2)\left(\frac{d}{c} - \tfrac{1}{2}\right)\left(\frac{d}{c} - \tfrac{3}{2}\right)\Big/\lambda_3^3 + \left(\frac{d}{c} - \tfrac{3}{2}\right)(\lambda_3^2 - \beta^2)\,c_1^{(3)}/\lambda_3^2\Big]$$

$$a_{13} = \frac{1}{c}\,(\beta^2 - \lambda_3^2)\left[\left(\frac{d}{c} - \tfrac{1}{2}\right)\Big/\lambda_3^2 + c_1^{(3)}/\lambda_3\right]$$

$$\bar{a}_{11} = \frac{1}{c}\,(\tfrac{1}{4} - m^2)\Big/\left(\frac{d}{c} - \tfrac{3}{2}\right)$$

$$\bar{a}_{12} = \frac{1}{c}\,(\tfrac{9}{4} - m^2)\Big/\left(\frac{d}{c} - \tfrac{5}{2}\right)$$

$$c_0 = 1,\ ic_1^{(3)} = c_1(\lambda_3),\ c_2^{(3)} = c_2(\lambda_3),\ c_1^{(0)} = c_1(0),\ c_2^{(0)} = c_2(0)$$

and E, F, G are the arbitrary constants of integration.

Case III : If the intensity of the magnetic field **H** be such that,

$$\rho\omega^2 < \alpha^2\left(c_{44} + \frac{H^2}{4\pi}\right)$$

then $\beta^2 < 0$ and consequently the roots of eqn. (20) for λ will be of the form

$$\lambda \doteq \pm\, i\lambda_4,\ \lambda_5,\ -\lambda_6,$$

where

$$\lambda_4^2 = \tfrac{1}{2}\,[(\beta^2 + \gamma^2 + ec) + \{(\beta^2 + \gamma^2 + ec)^2 - 4\beta^2\gamma^2\}^{1/2}]$$

$$\lambda_{5,6} = \pm\frac{1}{\sqrt{2}}\,[-(\beta^2 + \gamma^2 + ec) + \sqrt{(\beta^2 + \gamma^2 + ce)^2 - 4\beta^2\gamma^2}]^{1/2}$$

Similarly, neglecting terms of the order $O(\xi^{-3})$, the asymptotic representation for U and W may be written as

$$\begin{aligned}\bar{U} \sim\ & (I\cos\lambda_4\xi + J\sin\lambda_4\xi)\,(\xi^{-1/2} + c_2^{(4)}\,\xi^{-5/2}) + \\ & (J\cos\lambda_4\xi - I\sin\lambda_4\xi)\,c_1^{(4)}\,\xi^{-3/2} + \\ & Ke^{\lambda_5\xi}(\xi^{-1/2} + c_1^{(5)}\,\xi^{-3/2} + c_2^{(5)}\,\xi^{-5/2}) + \\ & Le^{-\lambda_6\xi}(\xi^{-1/2} + c_1^{(6)}\,\xi^{-3/2} + c_2^{(6)}\,\xi^{-5/2})\end{aligned} \qquad \text{...(25}a\text{)}$$

$$\begin{aligned}\bar{W} \sim\ & (J\cos\lambda_4\xi - I\sin\lambda_4\xi)\,(a_{11}\xi^{-1/2} + a_{12}\xi^{-5/2}) - \\ & (I\cos\lambda_4\xi + J\sin\lambda_4\xi)\,a_{13}\xi^{-3/2} + \\ & Ke^{\lambda_5\xi}(\bar{a}_{11}\xi^{-1/2} + \bar{a}_{12}\xi^{-3/2} + \bar{a}_{13}\xi^{-5/2}) + \\ & Le^{-\lambda_6\xi}(\bar{b}_{11}\xi^{-1/2} + \bar{b}_{12}\xi^{-3/2} + \bar{b}_{13}\xi^{-5/2}),\end{aligned} \qquad \text{...(25}b\text{)}$$

where

$$\bar{a}_{11} = -\frac{1}{c}\,\frac{\beta^2 + \lambda_5^2}{\lambda_5},$$

$$\bar{a}_{12} = -\frac{1}{c}\,\frac{\beta^2 + \lambda_5^2}{\lambda_5}\,c_1^{(5)} + \frac{1}{c\lambda_5}\left\{\frac{\beta^2 + \lambda_5^2}{\lambda_5}\left(\frac{d}{c} - \frac{1}{2}\right)\right\},$$

$$\bar{a}_{13} = -\frac{1}{c}\left[\frac{\beta^2 + \lambda_5^2}{\lambda_5}\,c_2^{(5)} + \frac{1}{\lambda_5}\left\{\frac{\beta^2 + \lambda_5^2}{\lambda_5}\left(\frac{d}{c} - \frac{3}{2}\right) + 2\lambda_5\right\}c_1^{(5)} + \frac{1}{\lambda_5}\left\{\frac{\left(\frac{d}{c} - \frac{3}{2}\right)}{\lambda_5} - \frac{1}{4} + m^2\right\}\right],$$

$$\bar{b}_{11} = \frac{1}{c}\,\frac{\beta^2 + \lambda_6^2}{\lambda_6},$$

$$\bar{b}_{12} = \frac{1}{c}\left[\frac{\beta^2 + \lambda_6^2}{\lambda_6}\,c_1^{(6)} + \frac{1}{\lambda_6}\left\{\frac{\beta^2 + \lambda_6^2}{\lambda_6}\left(\frac{d}{c} - \frac{1}{2}\right)\right\}\right]$$

$$\bar{b}_{13} = \frac{1}{c}\left[\frac{\beta^2 + \lambda_6^2}{\lambda_6}\,c_2^{(6)} - \frac{c_1^{(6)}}{\lambda_6}\left\{\frac{(\beta^2 + \lambda_6^2)\left(\frac{d}{c} - \frac{3}{2}\right)}{\lambda_6} + 2\lambda_6\right\} - \frac{1}{\lambda_6}\left\{\frac{\left(\frac{d}{c}\frac{3}{2} - \right)}{\lambda_6} + \frac{1}{4} - m^2\right\}\right]$$

and a_{11}, a_{12}, a_{13} are the same as case II provided the sub- and superscripts are changed from 3 to 4.

In the above equations,

$$c_0 = 1$$

$$c_1^{(5)} = c_1(\lambda_5) \quad c_1^{(6)} = c_1(\lambda_6)$$

$$c_2^{(5)} = c_2(\lambda_5) \quad \epsilon_2^{(6)} = c_2(\lambda_6)$$

and I, G, K, L are the arbitrary constants of integration.

Frequency Equations

Frequency equations in each case can be obtained *via* the boundary conditions (13) and (14). Here we shall deduce the same for the first case only.

Substituting from (15), (16), (23*a*), and (23*b*) in the boundary conditions (13) and (14) we obtain eight linear homogeneous equations in the constants A, B, C, D, M, N, P and Q. For non-trivial solutions of these constants, the co-efficient determinant vanishes and we get the frequency equation as

$$\det \phi_{ij} = 0,\ i = 1, 2, \ldots 8 \text{ and } j = 1, 2, \ldots 8. \qquad \ldots(26)$$

and ϕ_{ij} for different values of i and j are given by,

$$\phi_{11} = \alpha\left(c_{11} + \frac{H^2}{4\pi}\right)(-\lambda_1 R_1 \sin \lambda_1\bar{a} + R_2 \cos \lambda_1\bar{a} - \lambda_1 c_1^{(1)}\,\bar{a}^{-3/2} \cos \lambda_1\bar{a} + \tfrac{3}{2}\,c_1^{(1)}\,\bar{a}^{-5/2} \sin \lambda_1\bar{a}) + \frac{\alpha}{\bar{a}}\left(c_{12} + \frac{H^2}{4\pi}\right)(R_1 \cos \lambda_1\bar{a} - c_1^{(1)}\,\bar{a}^{-3/2} \sin \lambda_1\bar{a}) - \alpha c_{23}(R_3 \sin \lambda_1\bar{a} + a_{13}\bar{a}^{-3/2} \cos \lambda_1\bar{a}).$$

$$\phi_{12} = \alpha\left(c_{11} + \frac{H^2}{4\pi}\right)(\lambda_1 R_1 \cos \lambda_1 \bar{a} + R_2 \sin \lambda_1 \bar{a} -$$
$$\lambda_1 c_1^{(1)} \bar{a}^{-3/2} \sin \lambda_1 \bar{a} - \tfrac{3}{2} c_1^{(1)} \bar{a}^{-5/2} \cos \lambda_1 \bar{a}) +$$
$$\frac{\alpha}{\bar{a}}\left(c_{12} + \frac{H^2}{4\pi}\right)(R_1 \sin \lambda_1 \bar{a} + c_1^{(1)} \bar{a}^{-3/2} \cos \lambda_1 \bar{a}) +$$
$$\alpha c_{23}(R_3 \cos \lambda_1 \bar{a} - a_{13}\bar{a}^{-3/2} \sin \lambda_1 \bar{a}).$$

$$\phi_{15} = -\frac{H}{4\pi}\frac{e^{k\bar{a}}}{\sqrt{2\pi k\bar{a}}}\left(1 + \frac{1}{8k\bar{a}} + \frac{9}{128k^2\bar{a}^2}\right).$$

$$\phi_{26} = -\frac{H}{4\pi}\sqrt{\frac{\pi}{2k\bar{b}}}\, e^{-k\bar{b}}\left(1 - \frac{1}{8k\bar{b}} + \frac{9}{128k^2\bar{b}^2}\right).$$

$$\phi_{31} = \alpha\left(\frac{c_{66}}{2} + \frac{H^2}{4\pi}\right)(R_1 \cos \lambda_1 \bar{a} - c_1^{(1)} \bar{a}^{-3/2} \sin \lambda_1 \bar{a}) -$$
$$\tfrac{1}{2}\alpha c_{66}(-\lambda_1 R_3 \cos \lambda_1 \bar{a} - R_4 \sin \lambda_1 \bar{a} +$$
$$\lambda_1 a_{13}\bar{a}^{-3/2} \sin \lambda_1 \bar{a} + \tfrac{3}{2} a_{13}\bar{a}^{-5/2} \cos \lambda_1 \bar{a}).$$

$$\phi_{32} = \alpha\left(\frac{c_{66}}{2} + \frac{H^2}{4\pi}\right)(R_1 \sin \lambda_1 \bar{a} + c_1^{(1)} \bar{a}^{-3/2} \cos \lambda_1 \bar{a}) -$$
$$\tfrac{1}{2}\alpha c_{66}(-\lambda_1 R_3 \sin \lambda_1 \bar{a} + R_4 \cos \lambda_1 \bar{a} -$$
$$\lambda_1 a_{13}\bar{a}^{-3/2} \cos \lambda_1 \bar{a} + \tfrac{3}{2} a_{13}\bar{a}^{-5/2} \sin \lambda_1 \bar{a}).$$

$$\phi_{37} = \frac{H}{4\pi}\frac{e^{k\bar{a}}}{\sqrt{2\pi k\bar{a}}}\left(1 + \frac{1}{8k\bar{a}} + \frac{9}{128k^2\bar{a}^2}\right).$$

$$\phi_{48} = \frac{H}{4\pi}\sqrt{\frac{\pi}{2k\bar{b}}}\, e^{-k\bar{b}}\left(1 - \frac{1}{8k\bar{b}} + \frac{9}{128k^2\bar{b}^2}\right)$$

$$\phi_{51} = \frac{H\omega^2}{c_1^2\,\alpha}(R_1 \cos \lambda_1 \bar{a} - c_1^{(1)} \bar{a}^{-3/2} \sin \lambda_1 \bar{a}).$$

$$\phi_{52} = \frac{H\omega^2}{c_1^2\,\alpha}(R_1 \sin \lambda_1 \bar{a} + c_1^{(1)} \bar{a}^{-3/2} \cos \lambda_1 \bar{a}).$$

$$\phi_{55} = -\frac{ke^{k\bar{a}}}{\sqrt{2\pi k\bar{a}}}\left(1 + \frac{1}{8k\bar{a}} + \frac{9}{128k^2\bar{a}^2}\right) +$$
$$\frac{e^{k\bar{a}}}{\sqrt{2\pi k}}\left(\frac{1}{2(\bar{a})^{3/2}} + \frac{3}{16k(\bar{a})^{5/2}} + \frac{45}{256k^2(\bar{a})^{7/3}}\right).$$

$$\phi_{57} = \frac{e^{k\bar{a}}}{\sqrt{2\pi k\bar{a}}}\left(1 + \frac{1}{8k\bar{a}} + \frac{9}{128k^2\bar{a}^2}\right).$$

$$\phi_{66} = \sqrt{\frac{k\pi}{2\bar{b}}}\, e^{-k\bar{b}}\left(1 - \frac{1}{8kb} + \frac{9}{128k^2\bar{b}^2}\right) - \sqrt{\frac{\pi}{2k}}\, e^{-k\bar{b}}\left(-\frac{1}{2(\bar{b})^{3/2}} + \frac{3}{16k(\bar{b})^{5/2}} - \frac{45}{256k^2(b)^{7/2}}\right).$$

$$\phi_{68} = \sqrt{\frac{\pi}{2k\bar{b}}}\, e^{-k\bar{b}}\left(1 - \frac{1}{8k\bar{b}} + \frac{9}{128k^2\bar{b}^2}\right).$$

$$\phi_{71} = \alpha H(-R_1\lambda_1 \sin \lambda_1\bar{a} + R_2 \cos \lambda_1\bar{a} - \lambda_1 c_1^{(1)}\, \bar{a}^{-3/2} \cos \lambda_1\bar{a} + \tfrac{3}{2}\, c_1^{(1)}\, \bar{a}^{-5/2} \sin \lambda_1\bar{a}) + \frac{H\alpha}{\bar{a}}\,(R_1 \cos \lambda_1\bar{a} - c_1^{(1)}\, \bar{a}^{-3/2} \sin \lambda_1\bar{a}).$$

$$\phi_{72} = \alpha H(R_1\lambda_1 \cos \lambda_1\bar{a} + R_2 \sin \lambda_1\bar{a} - \lambda_1 c_1^{(1)}\, \bar{a}^{-3/2} \sin \lambda_1\bar{a} - \tfrac{3}{2}\, c_1^{(1)}\, \bar{a}^{-5/2} \cos\lambda_1\bar{a}) + \frac{\alpha H}{\bar{a}}\,(R_1 \sin \lambda_1\bar{a} + c_1^{(1)}\, \bar{a}^{-3/2} \cos \lambda_1\bar{a}).$$

$$\phi_{75} = \frac{e^{k\bar{a}}}{\sqrt{2\pi k\bar{a}}}\left(1 + \frac{1}{8k\bar{a}} + \frac{9}{128k^2\bar{a}^2}\right).$$

$$\phi_{86} = \sqrt{\frac{\pi}{2k\bar{b}}}\, e^{-k\bar{b}}\left(1 - \frac{1}{8k\bar{b}} + \frac{9}{128k^2\bar{b}^2}\right).$$

$$\phi_{13} = \phi_{11}]_{\lambda_1=\lambda_2} \qquad \phi_{14} = \phi_{12}]_{\lambda_1=\lambda_2} \qquad \phi_{21} = \phi_{11}]_{\bar{a}=\bar{b}}$$

$$\phi_{22} = \phi_{12}]_{\bar{a}=\bar{b}} \qquad \phi_{23} = \phi_{13}]_{\bar{a}=\bar{b}} \qquad \phi_{24} = \phi_{14}]_{\bar{a}=\bar{b}}$$

$$\phi_{33} = \phi_{31}]_{\lambda_1=\lambda_2} \qquad \phi_{34} = \phi_{32}]_{\lambda_1=\lambda_2} \qquad \phi_{41} = \phi_{31}]_{\bar{a}=\bar{b}}$$

$$\phi_{42} = \phi_{32}]_{\bar{a}=\bar{b}} \qquad \phi_{43} = \phi_{33}]_{\bar{a}=\bar{b}} \qquad \phi_{44} = \phi_{34}]_{\bar{a}=\bar{b}}$$

$$\phi_{53} = \phi_{51}]_{\lambda_1=\lambda_2} \qquad \phi_{54} = \phi_{52}]_{\lambda_1=\lambda_2} \qquad \phi_{61} = \phi_{51}]_{\bar{a}=\bar{b}}$$

$$\phi_{62} = \phi_{52}]_{\bar{a}=\bar{b}} \qquad \phi_{63} = \phi_{53}]_{\bar{a}=\bar{b}} \qquad \phi_{64} = \phi_{54}]_{\bar{a}=\bar{b}}$$

$$\phi_{73} = \phi_{71}]_{\lambda_1=\lambda_2} \qquad \phi_{74} = \phi_{72}]_{\lambda_1=\lambda_2} \qquad \phi_{81} = \phi_{71}]_{\bar{a}=\bar{b}}$$

$$\phi_{82} = \phi_{72}]_{\bar{a}=\bar{b}} \qquad \phi_{83} = \phi_{73}]_{\bar{a}=\bar{b}} \qquad \phi_{84} = \phi_{74}]_{\bar{a}=\bar{b}}$$

$$\bar{a} = \alpha a, \quad \bar{b} = \alpha b.$$

where

$$R_1 = (\bar{a}^{-1/2} + c_2^{(1)} \bar{a}^{-5/2}).$$

$$R_2 = (-\tfrac{1}{2} \bar{a}^{-3/2} - \tfrac{5}{2} c_2^{(1)} \bar{a}^{-7/2}).$$

$$R_3 = (a_{11}\bar{a}^{-1/2} + a_{12}\bar{a}^{-5/2}).$$

$$R_4 = (-\tfrac{1}{2} a_{11}\bar{a}^{-3/4} - \tfrac{5}{2} a_{12}\bar{a}^{-7/2}).$$

DISCUSSION

From the foregoing analysis, we see that the uniform magnetic field, in the axial direction, does influence the displacement components U, W. The nature of the asymptotic shell solutions depends on the nature of the roots of the eqn. (20) which, in turn, depends on the frequency ranges, elastic constants, and the intensity of the magnetic field. It is clear from the analysis that the nature of the solutions change with the magnetic field characterised by the relations

$$H^2 > = < 4\pi \left(\frac{\rho\omega^2}{\alpha^2} - c_{44}\right).$$

For high frequency range, and in the absence of the magnetic field (i.e. $H = 0$), we get from eqn. (12)

$$\beta^2 > 0, \text{ since } \rho\omega^2 > \alpha^2 c_{33}$$

Also for most of the orthotropic materials,

$$\rho\omega^2 > \alpha^2 c_{33} \Rightarrow \rho\omega^2 > \alpha^2 c_{44}$$

$$\text{i.e. } \beta^2 > 0, \ \gamma^2 > 0.$$

This case corresponds to the solution of the form of eqn. (23). Now, by applying a suitable magnetic field, such that $\beta^2 = 0$, the solutions assume the form of the eqns. (24). For low frequency range, if $\rho\omega^2 < \alpha^2 c_{33}$ and $\rho\omega^2 < \alpha^2 c_{44}$ i.e. $\beta^2 < 0$, $\gamma^2 < 0$, the introduction of the magnetic field does not alter the form of the classical shell solutions.

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