AN ERROR-FREE INTRODUCTION TO ELECTROMAGNETISM FROM CONSIDERATIONS OF DIMENSIONS AND SYMMETRY

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The confusing situation regarding the magnetic field vectors and their true nature and characteristics have been discussed. The correct magnetostatic equations—including Coulomb's law—have been deduced from a consideration of dimensions and symmetry, utilizing the duality transformations. It is suggested that an exclusive use of SI units will help avoid unnecessary confusions in the study and teaching of physics, specially in the field of electromagnetism.

INTRODUCTION

THE method of Physics is basically experimental, based on observation and measurement. The result of a measurement is expressed by a dimensionless number—the measure—followed by a unit. The unit of measurement of any physical quantity has the same nature and characteristics (dimensions) as the physical quantity itself. It is a fact of basic importance that *all* units used to measure the same physical quantity must be of the same physical nature, and that the same unit can be used to measure *all* quantitites of the same physical nature. The measure of a physical quantity does not reveal the true character and reality of the quantity measured. It can only tell us how big or small this quantity is, compared with the size of the standard, somewhat arbitrarily chosen as its unit. If one wants to know something about the reality and character of any physical entity, he should search for it in its units, more specifically *in the dimensions* of its units, and *not in the size* of its chosen unit, as has often been done very unfortunately (Sommerfeld, 1952).

This enquiry about the character and reality of physical quantities takes us into the realm of theories. Physicists build up their theories by synthesizing the data obtained from observations and measurements. This synthesis helps them see in their theories, the basic simplicity, symmetry and beauty of Nature. In this beauty of Nature scientists have a near-complete faith, and one can utilize this faith for making valid deductions, as has been done in this paper.

The first great synthesis in Physics was the Newtonian synthesis of motion, leading to the three laws of motion, and the law of universal gravitation. Newton also showed how this synthesis provides a basis for the construction of a coherent system of units of measurement of all dynamical quantities, based on three primaries : length (L), mass (M), and time (T). The FPS, the CGS, the MKS as well as the present SI units are all illustrations of coherent systems of units. The SI, or the international system of units, now adopted throughout the world, is in some way better than all the others because of the fact, among others, that it has six primaries, whereas the others have only three. Obviously a dimensional analysis enquiring into the real nature of physical entities will be more successful when larger number of the independent primaries are at its disposal—there being an upper limit to the useful number of these primary dimensions. In electromagnetism, the gaussian system with three primaries, obscures the nature and characteristics of the field vectors, specially the magnetic field vectors **H** and **B**. Generations of students have been frightened by these confusions and inaccurate statements. This paper shows how from considerations of dimensions and symmetry one can remove these mistakes and clear up the confusions, provided that four primaries are accepted for all electromagnetic quantites, as has been done in the SI system.

Correctness of the Conventional Treatment of the Electric Field

The conventional treatment of electrostatics based on Coulomb's law of interaction between electric charges (q) is validated by experiments, and can therefore, be accepted as correct. We thus have, in the SI system, the force (**F**) between two point charges (q, q') separated by a distance (**r**) in vacuum, given by

$$\mathbf{F} = \frac{1}{4\pi\mu_0} \frac{qq'}{r^2} \hat{\mathbf{r}} \qquad \ldots(1)$$

where $\hat{\mathbf{r}}$ is the unit vector in the direction of **r** joining the two point charges. This gives the electric field intensity (**E**) defined in terms of a small positive test charge :

$$\mathbf{E} = \mathrm{Lt}_{q' \to 0} \frac{\mathbf{F}}{q'} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} \qquad \ldots(2)$$

This equation explains correctly why the capacitance of a parallel plate capacitor increases when one puts a dielectric between its plates [ϵ_0 for vacuum changing to a larger value ϵ for dielectric, E drops, the potential difference $V = Ex$ drops, and one can put more charge on the plate].

It follows directly from the above equations that the force **F** on a charge q placed in an electric field **E** is given by

$$\mathbf{F} = q\mathbf{E}. \qquad \ldots(3)$$

Taking an atomic view of the polarisation (**P**) taking place in the dielectric when it is placed in the electric field of intensity **E** one defines the electric flux density (**D**) as

$$\mathbf{D} \equiv \epsilon\, \mathbf{E} = \epsilon_r \epsilon_0 \mathbf{E} = \epsilon_0\, \mathbf{E} + \mathbf{P}, \qquad \ldots(4)$$

where the relative permittivity ϵ_r is a dimensionless number. In linear substances where (4) also holds in the scalar form (ϵ is constant), there is no difficulty in replacing **D** by ϵ**E** and working with only one field vector **E**. In non-linear ferroelectrics, however, the polarisation vector **P** must also be considered.

Incorrectness of the Conventional Treatment of the Magnetic Field

In the conventional treatment of the magnetic field, using three primaries and with rationalization, the Coulomb's law of force (**F**) between two point poles of strengths q_m and q'_m placed at a distance **r** in vacuum, is written as

$$\mathbf{F} = \frac{1}{4\pi\mu_0} \frac{q_m q'_m}{r^2} \hat{\mathbf{r}}, \qquad \ldots(5)$$

where $\hat{\mathbf{r}}$ is the unit vector along $\mathbf{r}$ joining the two point poles, and μ_0 is *assumed to be* the permeability of free space. This law *is considered to follow* from the corresponding experimentally verified law in electrostatics by the substitution of μ_0 for ϵ_0, and q_m, q'_m for q, q' in eqn. (1). These substitutions are *claimed to be just and fair* on the basis of the "symmetry" observed in the phenomena of electricity and magnetism. Conventionally, this equation (5) is never questioned seriously, although it goes directly against experimental facts like the one that the strength of the magnetic force *increases* very definitely when iron is introduced inside a current-carrying solenoid. Perhaps this is an example of our unlimited respect for the old, the honoured and the obsolete.

Once the Coulomb law is assumed in the form (5), the conventional treatment proceeds to define the magnetic field strength **H** in terms of a positive test pole, thus

$$\mathbf{H} = \mathrm{Lt}_{q'_m \to 0} \quad \frac{\mathbf{F}}{q_m'} = \frac{1}{4\pi\mu_0} \frac{q_m}{r^2}\hat{\mathbf{r}}, \qquad \ldots(6)$$

which is the magnetic field strength produced by a magnetic point pole of strength q_m at a distance **r** from it. This gives the magnetic flux density

$$\mathbf{B} \equiv \mu_0 \mathbf{H} = \frac{1}{4\pi} \frac{q_m}{r^2} \hat{r} \qquad \ldots(7)$$

and the force on magnetic monopole of strength q_m in a magnetic field of strength **H** as

$$\mathbf{F} = q_m \mathbf{H}. \qquad \ldots(8)$$

It follows from this that the torque on a dipole of moment p_m is given by

$$\Gamma = \mathbf{p}_m \times \mathbf{H} \qquad \ldots(9)$$

All these five equations (5) to (9) are incorrect. The uncritical and continued use of these wrong equations have confused generations of students and unguarded physicists. Even James Clerk Maxwell was misguided to put the force exerted by the field **H** on a magnetic pole q_m to be $\mathbf{F} = q_m \mathbf{H}$ (Maxwell 1873).

This confusion has arisen because μ_0 was taken as the magnetic counterpart of ϵ_0, which is incorrect. The counterpart of ϵ_0 can be obtained from a study of the correct dimensions of the electric and magnetic quantities, by first positively giving up all hopes of a mechanistic explanation of the electromagnetic phenomena, and then demanding at least four dimensions for all electromagnetic quantities. The gaussian system with three primaries (L, M, T) obscures the true nature and character of the field vectors. Here we are working in a plane of lower dimensions, and therefore arrive at confusing ideas about the field vectors like the proverbial blind men assessing the elephant.

The Symmetrical Form of Maxwell's Field Equations

The fact that **B** and **E** (also **H** and **D**) belong together follows unambiguously from the theory of relativity in which quantities **B** and $-i\mathbf{E}/c$ (also **H** and $-ic\mathbf{D}$) are coupled together in the anti-symmetrical field-strength four-tensor of the second rank. The wonderful simplicity and beauty of Maxwell's field equations are enhanced in their relativistic formulation for vacuum when symmetry is also added by introducing magnetic monopoles in the formalism. In this paper, however, we shall treat the topic at a more elementary introductory level.

The maxwellian synthesis of electromagnetic phenomena, including optics, was the second great synthesis in physics. Maxwell's field equations express everything that is there to be known about classical electromagnetism. In SI units Maxwell's field equations *for vacuum and without free magnetic monopoles* are written as :

$$\left.\begin{aligned} &(\mathrm{MI}) \ : \nabla . \mathbf{E} = \rho / \epsilon_0 \\ &(\mathrm{MII}) \ : \nabla . \mathbf{B} = 0 \\ &(\mathrm{MIII}) : \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ &(\mathrm{M\,IV}) ; \ c^2 \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \frac{\mathbf{J}}{\epsilon_0} \end{aligned}\right\} \quad \ldots(10)$$

These equations are asymmetric with respect to electric and magnetic fields, because in arriving at these forms it has been assumed as an experimental fact that no free magnetic monopole exists. This is in accordance with Heinrich Hertz's conclusion that "*true magnetism does not exist in Nature; a moving electric charge generates the magnetic field*".

However, it was P. A. M. Dirac who first showed that quantization of electric charges—an experimental fact— demands the existence of free magnetic monopoles. Dirac's papers (Dirac 1931, 1948) show a symmetry between electricity and magnetism quite foreign to the conventional views, and actually gives a relation $e_m = e/2\alpha$ between the smallest quantum of electric charge (e) and the smallest quantum of magnetic pole or magnetic charge (e_m), where $e = 4.803 \times 10^{-10}$ esu is the electronic charge, and $\alpha = 1/137$ is the fine structure constant. J. J. Thomson (Thomson, 1900), M. N. Saha (Saha, 1936, 1949) and H. A. Wilson (Wilson 1949) have separately arrived at the same value ($e/2\alpha$) of the elementary magnetic charge from more elementary considerations, assuming the existence of the magnetic monopoles. Dirac specifically concluded *"The theoretical reciprocity between electricity and magnetism is perfect."* It is therefore, felt that, pending their experimental detection, magnetic monopoles must be assumed to exist. In this respect, the field equations in their usual form (10) may be considered as *incomplete,* although not incorrect. By removing the asymmetry these equations may be made more complete if we properly introduce the magnetic monopole charge density ρ_m and magnetic current density J_m in (MII) and (MIII) respectively.

To do this let us look into the dimensions of the relevant electromagnetic quantities. In SI system we derive the following dimensional formulae in the usual way from definitions and experimentally established relations :

(definition) $[q] = [TA]$

(force/charge) $[E] = [LMT^{-3}A^{-1}]$

(Coulomb law) $[\epsilon_0] = [L^{-3}M^{-1}T^4A^2]$

($D \equiv \epsilon_0 E$) $[D] = [L^{-2}TA]$

(Hertz's result) $[\mu_0] = [LMT^{-2}A^{-2}]$

(Lorentz force Eq.) $[B] = [MT^{-2}A^{-1}]$

$(B = \mu_0 H)$ $[H] = [L^{-1}A]$

(Ampere rule) $[q_m] = [LA]$

These formulae clearly show that μ_0 of the magnetic field having altogether different dimensions, cannot be considered as the counterpart of ϵ_0 of the electric field. It is also seen from these dimensional formulae that E/c and B has the same dimensions (also H and cD). Similarly cq and q_m are also dimensionally equivalent. From symmetry considerations, therefore, we expect that a complete and symmetrical form of Maxwell's second equation (MII) should result from the replacement of the electric field intensity E and electric charge (volume) density ρ in (MI) by their respective equivalents $-cB$ and $-\rho_m/c$ in the magnetic field. The prefixed negative sign is obviously borrowed from relativity theory; however, substitution of $+cB$ for E and $+\rho_m/c$ for ρ should also do, because that simply means the reversal of the chosen positive direction of the magnetic flux.

The above considerations indicate that the symmetry of the field equations (10) with respect to electric and magnetic fields may be restored by demanding their invariance under the duality transformation :

$$\mathbf{E} \to \mp c\mathbf{B} \text{ with } q \to \mp q_m/c \qquad \text{...(11A)}$$

By reversing the direction of the arrow i.e., of the change of field, and the sign of the equivalent quantity in the transformed field simultaneously, we get the alternative form of the duality transformation :

$$\mathbf{B} \to \pm \mathbf{E}/c \text{ with } q_m \to \pm cq \qquad \text{...(11B)}$$

By actual substitution of these duality transformations

$$\mathbf{E} \to -c\mathbf{B} \text{ with } q \to -q_m/c \text{ i.e., } \rho \to -\rho_m/c$$

in (MI), we get

$$-c\,\nabla.\mathbf{B} = -\rho_m/\epsilon_0 c \text{ or } \nabla.\mathbf{B}. = \frac{1}{\epsilon_0 c^2}\rho_m \quad \text{or } \nabla.\mathbf{B} = \mu_0\rho_m$$

The same equation results from the substitution

$$\mathbf{E} \to +c\mathbf{B} \text{ with } q \to +q_m/c \text{ i.e., } \rho \to +\rho_m/c.$$

We thus arrive at the complete and symmetrical form of (MII), namely $\nabla.\mathbf{B} = \mu_0\rho_m$, in presence of free magnetic monopoles. In situations where monopoles do not exist $\rho_m = 0$, and we go back to (MII) as the correct equation representing that situation.

In the same way (MIII) is to be considered *incomplete,* although not incorrect. It may be made *symmetrical and complete* by the introduction of the magnetic current (surface) density $\mathbf{J}_m$ in (MIV) through the duality transformations. Thus by actual substitution of the transformations :

$$\mathbf{E} \to -c\mathbf{B} \text{ with } \rho \to -\rho_m/c \text{ i.e., } \mathbf{J}_m \to -\mathbf{J}_m/c \;\&\; \mathbf{B} \to +\mathbf{E}/c$$

in (MIV), we get

$$c^2 \nabla \times \mathbf{E}/c = \frac{\partial}{\partial t}\left(-c\mathbf{B}\right) + \frac{1}{\epsilon_0}\left(-\mathbf{J}_m/c\right)$$

or $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \frac{\mathbf{J}_m}{\epsilon_0 c^2}$$

or $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mu_0 \mathbf{J}_m$$

The same equation results from the substitution

$\mathbf{E} \to + c\mathbf{B}$ with $\rho \to + \rho_m/c$ i.e., $\mathbf{J} \to - \mathbf{J}_m/c$ & $\mathbf{B} \to - \mathbf{E}/c$

We thus arrive at the complete and symmetrical form of (MIII), namely $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mu_0 \mathbf{J}_m$ in presence of free magnetic monopoles. In situations where monopoles do not exist, $J_m = 0$ and we go back to (MIII) as the correct equation representing that situation.

Thus from considerations of dimensions and symmetry we arrive at the following *complete and symmetrical* form of the electromagnetic field equations *for vacuum* :

$$\left.\begin{aligned} &\text{(SMI)} &&: \nabla . \mathbf{E} = \rho/\epsilon_0 \\ &\text{(SMII)} &&: \nabla . \mathbf{B} = \mu_0 \rho_m \\ &\text{(SMIII)} &&: \nabla \times E = -\frac{\partial \mathbf{B}}{\partial t} - \mu_0 \mathbf{J}_m \\ &\text{(SMIV)} &&: c^2 \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}/\epsilon_0 \end{aligned}\right\} \quad \ldots(12)$$

Coulomb's Law in Magnetostatics

Consider a point magnetic charge of monopole strength q_m situated at the centre of a sphere of volume τ completely enclosed by the surface S, and radiating out its flux. Then, integrating (SMII) over the closed volume τ containing the source q_m at its centre, we get

$$\oint_\tau \nabla \cdot \mathbf{B}\, d\tau = \oint \tau\, \mu_0\, \rho_m\, d\, \tau = \mu_0 q_m$$

Using Gauss' Divergence Theorem, this gives

$$\oint_s \mathbf{B} . d\mathbf{S} = \mu_0 q_m,$$

where the surface integral is taken over the surface S completely enclosing the volume τ of integration.

From the spherical symmetry of the problem, we get

$$B . 4\pi\, r^2 = \mu_0\, q_m \text{ or } B = \frac{\mu_0}{4\pi}\, \frac{q_m}{r^2}.$$

Writing in the vector notation this equation is

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{q_m}{r^2} \hat{\mathbf{r}}. \qquad \ldots(13)$$

This shows that magnetic monopoles produce a **B**-field and not a **H**-field, so that in monopole interactions the **B**-field is fundamental and not the **H**-field. Therefore, the force on a magnetic monopole of strength q_m placed in a magnetic field is to be taken as

$$\mathbf{F} = q_m\mathbf{B}. \qquad \ldots(14)$$

These equations (13) and (14) lead to the correct form of Coulomb's law in magnetostatics :

$$\mathbf{F} = \frac{\mu_0}{4\pi} \frac{q_m q'_m}{r^2} \hat{\mathbf{r}}. \qquad \ldots(15)$$

The above equations correctly indicate that when iron is introduced inside a solenoid, the **B**-field increases, and with it the magnetic force also increases (μ_0 for vacuum being replaced by the larger value μ for iron).

As the magnetic monopole produces a **B**-field, so does a dipole layer or a magnetic shell, as well as a closed current loop. Their equivalence leads to the equation (Ampere rule) :

$$q_m . l \equiv p_m = SI \qquad \ldots(16)$$

giving the correct relationship between the moment of the shell (p_m) of area S, and a current loop of the same area and carrying an electric current I amperes.

The torque (Γ) experienced by a magnetic dipole of moment $p_m \equiv q_m l$ when placed in an external magnetic field **B** is given by

$$\Gamma = \mathbf{p}_m \times \mathbf{B} \qquad \ldots(17)$$

Although experiments are not possible with magnetic monopoles directly, the eqn. (17) for dipoles can be verified experimentally, and is found to stand the test.

CONCLUSIONS

It is concluded that the dogma of the scientific superiority of the "absolute" system with three purely mechanical primary units of length, mass and time, should be given up. A fourth dimension is necessary to understand the electromagnetic phenomena better. Max Planck's position saying that the question of the real dimensions of a physical entity *"has no more meaning than that of the real name of an object"*, is not fair. The conventional magnetostatic equations should be replaced by their correct forms in SI units so that the confusions which frightened generations of students may be given a descent burial.

I humbly suggest that SI units may be preferred uniformly to clear up the unnecessary confusions in the study and teaching of physics, specially in the field of electromagnetism.

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