PHASE-AMPLITUDE METHOD FOR A SEPARABLE NON-LOCAL OPTICAL POTENTIAL

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The phase-amplitude method (PAM) is used with the conventional theory of integral equations to construct exact analytical expressions for phase shift, scattering length and effective range for scattering on a separable non-local optical potential. In contrast to earlier approaches to the problem, the present treatment does not involve the evaluation of any contour integral. Although non-local effects tend to obscure the intuitive meaning of the phase function, the PAM may provide a convenient framework to compute low-energy scattering parameters for non-local potentials by using the phase equation derived by two of the present authors (Talukdar *et al.*, 1977).

INTRODUCTION

IN the phase-amplitude method (PAM) the value $\delta(s)$ of the phase function at a distance s from the origin represents the scattering phase shift induced by a local potential $V(r)$, if it is truncated at a point s. This interpretation is consistent with the initial condition $\delta(0) = 0$ for solving the phase equation. The vanishing of $\delta(0)$ corresponds to the fact that a completely amputed potential does not produce any phase shift. Thus the phase function is built up by the potential as one moves away from the origin and it reaches its asymptotic value as soon as one gets out of the range of the potential. Obviously, scattering phase shift is obtained from the asymptotic value of the phase function.

In contrast to the local case, the phase function $\delta(r)$ for a non-local potential does not have such a simple physical meaning with regard to the accumulation of phases. A non-local potential couples the wavefunction at one point with its values at all the other points. This implies that accumulation of phase depends on the wavefunction for all values of r. Looking from this point of view, the PAM seems to lose all its advantages, if the potential is non-local. It is, therefore, of some interest to explore the possibility of extending the phase approach to the case of scattering on non-local potentials.

In a recent paper, two of the present authors (Talukdar *et al.*, 1977) initiated development of the PAM for a general non-local optical potential and constructed a complex non-linear integro-differential equation for $\delta(r)$. This equation has a very complicated mathematical structure. It should, however, be solvable on a digital computer by the method of successive approximations. Even granted this; no physical conclusions can be drawn without extensive investigation. To that end, it may be useful to obtain the scattering phase shift for a soluble model of the non-local potential by employing concepts of the PAM. The present paper is an effort in this direction. The present authors deal with a non-local optical potential with separable

form factors and contemplate deriving exact expressions for low-energy scattering parameters in a relatively non-complicated manner. One may be interested to know if the traditional momentum space approach could be extended to deal with the problem. In principle, there should not be any difficulty. But one would then come across a large number of contour integrals, which are really tedious to perform (Bagchi & Mulligan, 1974). Fortunately, the present method does not involve any contour integral.

Separable non-local potentials have been widely used to represent basic interactions in part, because their use provides computational simplification of the non-relativistic few-body problem and, in part because separable potentials permit simple analytical solutions of the inverse scattering problem (Omnes, 1958; Gourdin & Martin, 1958; Londergan & Moniz 1973). Non-local optical potentials with separable form factors have been used in various contexts with some success. For example, Bencze and Doleschall (1970) investigated a three-body model of the deuteron nucleus collision with absorptive s-wave interaction. Alberg *et al.* (1973) developed a K-nucleus non-local optical potential.

In the next section, the conventional theory of integral equations is tied with the PAM to construct an exact expression for the phase function. The result for the phase shift derived thereby is used in the third section to obtain the low-energy expansion parameters like the scattering length and effective range. In conclusions some observations on the method have been presented. For simplicity only the s-wave scattering is considered and the subscript $l = 0$ is omitted. However, generalization of the present results to higher partial wave is almost trivial. $\hbar^2/2m$ is assumed to be unity here.

Phase Equation for Non-Local Optical Potential

For a non-local optical potential $\lambda_1 V(r, s) + i\lambda_2 W(r, s)$, the s-wave part of the radial Schrödinger equation is

$$u''(r) + k^2 u(r) = \int_0^\infty ds\, [\lambda_1 V(r, s) + i\lambda_2 W(r, s)]\, u(s), \qquad ...(1)$$

where

$$V(r, s) = 2\pi r s \int_{-1}^{1} d(\cos\theta)\, V(r, s) \qquad ...(2a)$$

and

$$W(r, s) = 2\pi r s \int_{1}^{1} d(\cos\theta)\, W(r, s). \qquad ...(2b)$$

Specializing eqn. (1) to the separable class of potentials, we can write $u(r)$ in the form

$$u(r) = \sin kr \left[1 + \frac{1}{k} \int_0^r \cos kr'\, dr' (\lambda_1 v(r') x_1 + i\lambda_2\ \omega(r')\ x_2) \right]$$

$$-\frac{\cos kr}{k}\left[\int_0^r \mathrm{d}r' \sin kr' \left(\lambda_1 v(r') x_1 + i\lambda_2 \omega(r') x_2\right)\right], \qquad \ldots(3)$$

where

$$x_1 = \int_0^\infty \mathrm{d}s\, v(s)\, u(s), \qquad \ldots(4a)$$

and

$$x_2 = \int_0^\infty \mathrm{d}s \omega(s)\, u(s). \qquad \ldots(4b)$$

In writing eqn. (3), the following expressions

$$V(r, s) = \omega(r)\, \omega(s) \qquad \ldots(5a)$$

and

$$W(r, s) = \omega(r)\, \omega(s) \qquad \ldots(5b)$$

are used. Clearly, the radial function $u(r)$ satisfies the regular boundary condition (Newton, 1966), To develop the PAM, the phase and amplitude functions $\delta(r)$ and $\alpha(r)$ respectively are introduced by the following ansatz :

$$\alpha(r) \cos \delta(r) = 1 + \frac{1}{k}\int_0^r \cos kr' \,\mathrm{d}r' \left(\lambda_1 v(r') x_1 = i\lambda_2 \omega(r') x_2\right) \qquad \ldots(6a)$$

and

$$\alpha(r) \sin \delta(r) = -\frac{1}{k}\int_0^r \sin kr' \left(\lambda_1 v(r') x_1 + i\lambda_2 \omega(r') \omega_2\right) \mathrm{d}r'. \qquad \ldots(6b)$$

We may now differentiate eqns. (6*a*) and (6*b*) with respect to r and eliminate $\frac{\mathrm{d}\alpha(r)}{\mathrm{d}r}$ to derive the phase equation i.e., the equation satisfied by $\delta(r)$. It is already pointed out that for a general non-local potential, the equation for $\delta(r)$ is extremely complicated to admit simple numerical solutions. The phase equation is about equally complicated even if the form factors of the potential are chosen separably. Fortunately, for a separable potential a simpler approach can be used to obtain the phase function. Dividin eequation (6*b*) by (6*a*), we get

$$\cot \delta(r) = -\frac{1 + \frac{1}{k}\int_0^r \cos kr' \,\mathrm{d}r' \left(\lambda_1 v(r') x_1 + i\lambda_2 \omega(r') x_2\, r'\right)}{\frac{1}{k}\int_0^r \sin kr' \,\mathrm{d}r' \left(\lambda_1 v(r') x_1 + i\lambda_2 \omega(r') x_2\right)}, \qquad \ldots(7)$$

where X_1 and X_2 are transforms of the form factors of the potential by the solution of eqn. (3).

Thus knowing X_1 and X_2, we can construct an exact expression for $\delta(r)$. It is of interest to note that the solution of eqn. (3) can be obtained analytically by using the standard technique for solving the integral equation with a degenerate kernel. Thus

$$X_1 = \frac{b_1 a_{22} - b_2 a_{12}}{C(k)} \qquad \ldots(8a)$$

and

$$X_2 = \frac{b_2 a_{11} - b_1 a_{21}}{D(k)}. \qquad \ldots(8b)$$

In eqns. (8*a*) & (8*b*) $D(k)$ stands for the Fredholm determinant $a_{11}\, a_{22} - a_{21}\, a_{12}$. The elements a_{ij} and b_k's are given by

$$a_{11} = 1 - \frac{\lambda_1}{k} \int_0^\infty \int_0^r dr\, dr' \sin k(r-r')\, v(r)\, v(r'), \qquad \ldots(9a)$$

$$a_{12} = - \frac{i\lambda_2}{k} \int_0^\infty \int_0^r dr\, dr' \sin k(r-r')\, v(r)\, \omega(r'), \qquad \ldots(9b)$$

$$a_{21} = - \frac{\lambda_1}{k} \int_0^\infty \int_0^r dr\, dr' \sin k(r-r')\, \omega(r)\, v(r'), \qquad \ldots(9c)$$

$$a_{32} = 1 - \frac{i\lambda_2}{k} \int_0^\infty \int_0^r dr\, dr' \sin k(r-r')\, \omega(r)\, \omega(r'), \qquad \ldots(9d)$$

and

$$b_1 = \int_0^\infty v(r) \sin kr\, dr \qquad \ldots(10a)$$

and

$$b_2 = \int_0^\infty \omega(r) \sin kr\, dr. \qquad \ldots(10b)$$

Eqns. (7)—(10) can now be combined to get an exact expression for $\delta(r)$, which asymptotically yields the phase shift.

For the Yamaguchi form factor

$$V(r, s) = e^{-\alpha_1(r+s)}, \qquad \ldots(11a)$$

and

$$W(r, s) = e^{-\alpha_2(r+s)}. \qquad \ldots(11b)$$

The expression for $k \cot \delta$ is obtained in the form

$$k \cot \delta = - \frac{B(k) + D(k)}{C(k)} , \qquad \ldots(12)$$

where

$$B(k) = \frac{\lambda_1 \alpha_1}{(\alpha_1^2+k^2)^2} + \frac{i\lambda_1 \alpha_2}{(\alpha_2^2+k^2)^2} + \frac{i\lambda_1 \lambda_2 \alpha_1 (\alpha_2 - \alpha_1)}{2\alpha_2 (\alpha_1+\alpha_2) (\alpha_1^2+k^2) (\alpha_2^2+k^2)}$$

$$+ \frac{i\lambda_1 \lambda_2 \alpha_2 (\alpha_1 - \alpha_2)}{2\alpha_1 (\alpha_1+\alpha_2) (\alpha_1^2+k^2) (\alpha_2^2+k^2)} , \qquad \ldots(13a)$$

$$C(k) = \frac{\lambda_1}{(\alpha_1^2+k^2)^2} + \frac{i\lambda_2}{(\alpha_2^2+k^2)^2} + \frac{i\lambda_1 \lambda_2 (\alpha_2 - \alpha_1)}{2\alpha_2 (\alpha_1+\alpha_2) (\alpha_1^2+k^2) (\alpha_2^2+k^2)}$$

$$+ \frac{i\lambda_1 \lambda_2 (\alpha_1 - \alpha_2)}{2\alpha_1 (\alpha_1+\alpha_2) (\alpha_1^2+k^2) (\alpha_2^2+k^2)} \qquad \ldots(13b)$$

and the Fredhölm determinant

$$D(k) = 1 - \frac{\lambda_1}{2\alpha_1 (\alpha_1^2+k^2)} - \frac{i\lambda_2}{2\alpha_2 (\alpha_2^2+k^2)}$$

$$+ \frac{i\lambda_1 \lambda_2 (\alpha_1 - \alpha_2)^2}{4\alpha_1 \alpha_2 (\alpha_1+\alpha_2)^2 (\alpha_1^2+k^2) (\alpha_2^2+k^2)} . \qquad \ldots(14)$$

In the limit $\lambda_2 \to -i\lambda_2$, the expression for $D(k)$ in eqn. (14) goes over to that for the Fredhölm determinant associated with the regular solution of the case IV Mongan potential (Mongan, 1969; Mulligan *et al.*, 1976). Eqn. (14) has no local analog since for the local potential $D(k) = 1$. The integral equation with which $D(k)$ is connected possesses a Fredhölm rather than a Volterra Kernel. Thus $D(k)$ may have zeros for any real values of k. A zero of $D(k)$ for real $k \neq 0$ has been called a spurious state (Coz *et al.*, 1970; Arnold & Mac Kellar, 1971). Obviously, a spurious state does not occur in the theories of local potentials.

In view of this discussion, it appears that the PAM may possibly be used to initiate development of description of phenomena characteristic of non-local potentials. In the next section, expressions for low energy scattering parameters namely, the scattering length and effective range for scattering on a non-local optical potential are derived.

Low-Energy Expansion Parameters

In the PAM, the customary scattering length and the effective range are obtained by expressing the so called tangent function $t(r)$ $(=\tan \delta(r))$ in terms of interpolating scattering length $a(r)$ and effective range $b(r)$ (Spruch, 1962). It is seen that $a(r)$ satisfies a non-linear differential equation while the equation satisfied by $b(r)$ is a linear one. However, in the present case, this procedure can be avoided since an

exact expression has been constructed for k cot δ. In the effective range theory, k cot δ is written as

$$k \cot \delta = -\frac{1}{a_s} + \frac{1}{2} k^2 r_{eff} \qquad ...(15)$$

where a_s and r_{eff} denote the scattering length and effective range respectively. If we now expand the right hand side of eqn. (12) in powers of k and compare the resulting expression with eqn. (15), the following results are obtained.

$$\frac{1}{a_s} = \frac{M'}{M}, \qquad ...(16a)$$

where

$$\frac{1}{2} r_{eff} = -\frac{1}{M}\left(N' + M' \frac{N}{M} \right), \qquad ...(16b)$$

$$M = \frac{\lambda_1}{\alpha_1^4} + \frac{i\lambda_2}{\alpha_2^4} + \frac{i\lambda_1 \lambda_2 (\alpha_2 - \alpha_1)^2}{2\alpha_1^4 \alpha_2^4 (\alpha_1 + \alpha_2)}, \qquad ...(17a)$$

$$N = \frac{2\lambda_1}{\alpha_1^6} + \frac{2i\lambda_2}{\alpha_2^6} + \frac{i\lambda_1 \lambda_2 (\alpha_2 - \alpha_1)^2 (2\alpha_1^2 + 2\alpha_2^2 + \alpha_1 \alpha_2)}{2\alpha_1^6 \alpha_2^6 (\alpha_1 + \alpha_2)}, \qquad ...(17b)$$

$$M' = 1 + \frac{\lambda_1}{2\alpha_1^3} + \frac{i\lambda_2}{2\alpha_2^4}\left[1 + \frac{\lambda_1 (\alpha_1 - \alpha_2)^2}{2\alpha_1^3 (\alpha_1 + \alpha_2)_2} \right], \qquad ...(17c)$$

and

$$N' = -\frac{3\lambda_1}{2\alpha_1^5} - i\lambda_2 \left[\frac{3}{2\alpha_2^5} + \frac{\lambda_1 (\alpha_1 - \alpha_2)^2 (3\alpha_1^2 + 3\alpha_2^2 + 4\alpha_1 \alpha_2)}{4\alpha_1^5 \alpha_2^6 (\alpha_1 + \alpha_2)_2} \right]. \qquad ...(17d)$$

One can perform a couple of checks on the fairly complicated results given in eqns. (16) and (17).

(*i*) For $\lambda_2 = 0$, the results in eqns. (16*a*) and (16*b*) go over to the corresponding quantities for the Yamaguchi potential.

(*ii*) In the limit $\lambda_2 \to -i\lambda_2$, we obtain the scattering length and effective range that can be obtained for the Mongan case IV potential by using an entirely different technique (Englefield, 1974).

In our treatment both scattering length and effective range are complex quantities. A complex scattering length refers to an absorptive interaction. With regard to the effective range one needs some clarification. Alberg *et al.* (1973) used a single separable potential with complex strength and range to fit the K^--nucleus scattering data in the isospin $I = 1$ state. The use of a complex range parameter with small imaginary part makes the decaying form factors of the potential slightly oscillatory. One would expect this oscillatory nature to be reflected back in the low energy expansion parameters.

Conclusion

The phase-amplitude method is particularly suitable for dealing with scattering on local potentials. For a non-local potential, the phase equation has a very compli-

cated mathematical structure, which tends to obscure the intuitive meaning of the phase function presumably because the non-locality couples the wavefunction at one point with its values at other points in the configuration space. Here it is shown that if the form factors of the potentials are chosen to be separable, the PAM can be used to construct exact analytical expressions for the phase shift, scattering length and effective range even for an optical potential. Based on this, the authors would venture to suggest that for a general non-local potential however complicated the structure of the phase equation might be, the phase equation of Talukdar *et al.* (1977) is expected to yield numbers which are quite reliable within the framework of the chosen iterative technique used for solving it.

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