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Physics

PERIODIC HEAT TRANSFER TO AN INFINITE CYLINDRICAL CAVITY IN GROUND HAVING AIR AT CONSTANT TEMPERATURE

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This communication presents an analysis of the periodic heat transfer between earth and a cylindrical cavity, in which air is kept at constant temperature. The analysis is based on the appropriate boundary conditions viz., energy balance at the surface of earth and the walls of the cavity. We take into account the periodicity of solar radiation and atmospheric temperature. Explicit expression for the temperature as a function of time, depth and distance from the axis of the cavity has been derived; the expression has been used to investigate the daily variations for Kuwait. Such an analysis is useful in evaluating the thermal flux into underground structures (e.g., sunken greenhouses and cold storages).

INTRODUCTION

FOR efficient and economical heating and cooling of a building, it is essential to evaluate the hourly variation of the thermal flux into the building from the hourly variation of solar insolation and atmospheric temperature. At present, there is no available theory for the prediction of hourly variation of heat transfer between a building and the surrounding earth; an estimate for constant insolation and atmospheric temperature may be made by using empirical relations (*ASHRAE Hbk. Fundamentals*, 1974), based on data obtained in very cold climates for a specific terrain. These relations can at best give an order of magnitude estimate, when the atmospheric temperature and solar insolation are not appreciably different from the conditions for which the data were obtained; in fact, solar insolation and atmospheric temperature do not figure in these empirical relations.

This communication presents an analysis of the periodic heat transfer between earth (subjected to periodic solar radiation and thermal contact with atmospheric air having a periodic temperature) and a cavity, in which air is kept at a constant temperature. Such analyses are applicable to buildings, sunken greenhouses and fully/partially underground cold storages. For the sake of mathematical convenience, the cavity has been assumed to be an infinite cylinder with vertical axis. More realistic geometries present much more difficult computational problems and may be considered later.

The solair temperature (which represents the combined effect of solar insolation and atmospheric air temperature) has been expanded as a Fourier series; the first three terms are adequate to represent the variation but we have included seven terms for better results. The temperature distribution in the ground has been evaluated by obtaining a Fourier series solution, satisfying the appropriate energy balance conditions on the surface of the ground and the surface of the cavity. For computational purposes, the condition that the radial heat flux is zero at infinite distance from the axis of the cavity has been relaxed to the condition that the radial heat flux is zero at a distance from the axis such that the difference between this distance and the radius of the cavity is many times the characteristic lengths of the problem.

Numerical calculations for Kuwait, using the atmospheric and solar insolation data for March 21, 1975 have been made and a discussion of results presented.

Nomenclature

- a radius of cylindrical cavity, ft.
- c specific heat of the ground, BTU/lb °F.
- h heat transfer co-efficient at the surface BTU/hr ft^2 °F.
- h_c heat transfer co-efficient at the walls of the cavity, BTU/hr ft^2 °F.
- k thermal conductivity of the ground, BTU/hr ft °F.
- t time, hours.
- $\triangle R$ difference between the long-wave radiation, incident on the surface from the sky and surroundings, and the radiation emitted by a black body at atmospheric air temperature (BTU/hr. ft^2).
- r radial co-ordinate of cylindrical co-ordinate system.
- $S(t)$ intensity of solar radiation, BTU/hr ft^2.
- ρ_0 density of the ground, BTU/lb °F.
- θ temperature at any point inside the ground, °F.
- θ_{atm} atmospheric air temperature, °F.
- θ_A solair temperature, °F.
- θ_c temperature of the air inside the cavity, °F
- α_0 absorptivity of solar radiation at the surface.
- ϵ long-wave emissivity of the surface.

Analysis

Let us consider a semi-infinite medium (earth) having a cylindrical cavity of radius a as shown in Fig. 1. The surface $z = 0$ is open to atmosphere and the temperature of the cavity is maintained constant at θ_c. The temperature θ at any point, inside the ground is described by the Fourier conduction equation, which in the cylindrical systems of coordinates, is given by

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} = \frac{\rho_0 C}{K}\frac{\partial\theta}{\partial t}, \qquad \ldots (1)$$

where ρ_0 is the density of the ground (in lb/ft^3), C is the specific heat of the ground (in BTU/lb °F) and K is the thermal conductivity of the ground (in BTU/hr ft °F). In writing eqn. (1), we have assumed azimuthal symmetry of the problem. The

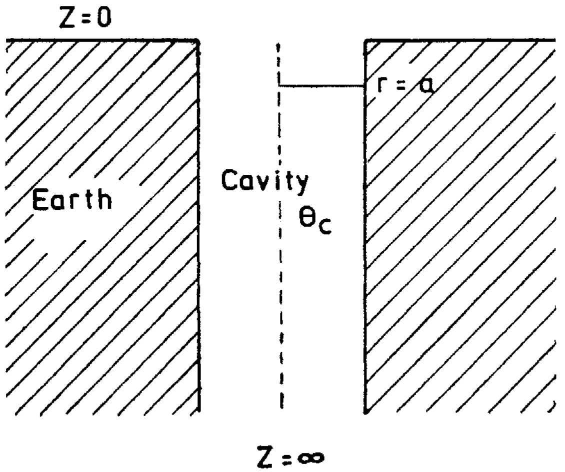

FIG. 1. Sketch of cylindrical cavity.

energy balance at the ground surface (outside the cavity) can be expressed as

$$-K \frac{\partial \theta}{\partial z}\bigg|_{z=0} = h\,\{\theta_A - \theta\,(z=0)\}, \qquad \dots (2)$$

where :

$\theta_A = \theta_{atm} + (\alpha_0 S/h) - (\epsilon \triangle R/h)$ is commonly known as the solair temperature, h is the heat transfer coefficient at the surface (in BTU/hr ft^2 °F), θ_{atm} is atmospheric air temperature (in °F), α_0 is absorptivity of solar radiation at the surface, $S(t)$ is the intensity of solar radiation (in BTU/hr ft^2), ϵ is long-wave emissivity of the surface and $\triangle R$ is difference between the long-wave radiation, incident on the surface from the sky and surroundings, and the radiation emitted by a black body at atmospheric air temperature ($\triangle R$ = 20 BTU/hr ft^2 for horizontal surface).

Similarly, the energy balance condition at the cylindrical surface of the cavity is

$$-K \frac{\partial \theta}{\partial r}\bigg|_{r=a} = h_c\,\{\theta_c - \theta\,(r=a)\} \qquad \dots (3)$$

where h_c is the heat transfer coefficient at the walls of the cavity, BTU/hr ft^2 °F, and θ_c is the temperature of air inside the cavity.

Another boundary condition which should obviously be satisfied is

$$\frac{\partial \theta}{\partial r}\bigg|_{r\to\infty} = 0, \qquad \dots (4)$$

which implies that the thermal flux tends to zero as $r \to \infty$.

We know that the atmospheric temperature and the solar radiations falling on the earth's surface are periodic in time; therefore θ_A and θ should also be periodic and one can express them in the form of Fourier series; i.e.,

$$\theta_A = \text{Re} \sum_{m=0}^{\infty} a_m \exp(im\omega t) \quad \dots (5)$$

and

$$\theta = \text{Re} \sum_{m=0}^{\infty} \theta_m(r, z) \exp(im\omega t) \quad \dots (6)$$

where

$$\omega = 2\pi/\text{period}$$

Let us now introduce dimensionless variables defined as :

$$Z = \left(\frac{\rho_0 C\omega}{2k}\right)^{1/2} z \, , \; \rho = \left(\frac{\rho_0 C\omega}{2k}\right)^{1/2} r$$

$$H = \frac{h}{k}\left(\frac{\rho_0 C\omega}{2k}\right)^{-1/2} \, , \; \rho_a = \left(\frac{\rho_0 C\omega}{2k}\right)^{1/2} a$$

and

$$H_c = \frac{h_c}{k}\left(\frac{\rho_0 C\omega}{2k}\right)^{-1/2} \quad \dots (7)$$

Substituting for θ from eqn. (6) in eqn. (1) and equating the coefficients of $\exp(im\omega t)$ on both sides of the resulting equation, we obtain

$$\frac{\partial^2 \theta_m}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \theta_m}{\partial \rho} + \frac{\partial^2 \theta_m}{\partial Z^2} = 2im\theta_m \quad \dots (8)$$

The corresponding boundary conditions (2) and (3) reduce to

$$-\left(\frac{\partial \theta}{\partial Z}\right)\bigg|_{Z=0} = H\{a_m - \theta_m(Z=0)\} \quad \dots (9a)$$

$$-\left(\frac{\partial \theta_0}{\partial \rho}\right)\bigg|_{\rho=\rho_a} = H_c\{\theta_c - \theta_0(\rho=\rho_a)\} \quad \dots (9b)$$

and

$$-\left(\frac{\partial \theta_m}{\partial \rho}\right)\bigg|_{\rho=\rho_a} = -H_c\theta_m(\rho=\rho_a) \quad \dots (9c)$$

The use of the obvious boundary condition at $r \to \infty$ (see eqn. 4) leads to considerable computational difficulties, hence we replace this boundary condition by

$$\left(\frac{\partial \theta}{\partial \rho}\right)_{\rho=\rho_b} = 0, \quad \dots (9d)$$

where

$$\rho_b = \left(\frac{\rho_0 \omega C}{2k}\right)^{1/2} b,$$

b is an arbitrary parameter such that $(b-a)$ is many times the scale length of the problem viz., $(2k/\rho_0\omega c)^{1/2}$ and k/h_c.

Solution for θ_m

Case I : $m=0$ — Let us now solve eqn. (8) for θ_0, the time independent part of the temperature distribution ($m = 0$). We assume a solution of the form :

$$\theta_0 = \theta_c + R_0(\rho)\, Z_0(z) \quad \text{... (10)}$$

Substituting for θ_0 from eqn. (10) in eqn. (8) we obtain

$$\frac{1}{R_0}\left(\frac{\partial^2 R_0}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial R_0}{\partial \rho}\right) = -\frac{1}{Z_0}\frac{\partial^2 Z_0}{\partial Z^2} = -\alpha^2 \quad \text{... (11)}$$

where α is a constant to be determined from boundary conditions. The expressions for R_0 and Z_0 can be readily obtained. If we now apply the boundary condition (9*b*) at $\rho=\rho_a$ and (9*d*) at $\rho=\rho_b$, we would get

$$\theta_0 = \theta_c + \sum_j A_{0j}\, \psi(\alpha_j\rho) \exp(-\alpha_j z), \quad \text{... (12)}$$

where

$$\psi(\alpha_j\rho) = J_0(\alpha_j\rho) - \frac{J_0'(\alpha_j\rho_b)}{y_0'(\alpha_j\rho_b)}\, y_0(\alpha_j\rho) \quad \text{... (13)}$$

and α_j's are the roots of the following transcendental equation

$$\alpha\,[y_0'(\alpha\rho_b)\, J_0'(\alpha\rho_a) - J_0'(\alpha\rho_b)\, y_0'(\alpha\rho_a)]$$
$$= H_c\,[J_0(\alpha\rho_a)\, y_0'(\alpha\rho_b) - y_0(\alpha\rho_a)\, J_0'(\alpha\rho_b)] \quad \text{... (14)}$$

Now, applying the boundary condition at $z=0$ (eqn. 9*a*), we get

$$[H(a_0-\theta_c)]^{-1} \sum_j A_{0j}(\alpha_j+H)\,\psi(\alpha_j\rho) = 1 \quad \text{... (15)}$$

Using the orthogonality of $\psi(\alpha_j\rho)$, we obtain

$$A_{0j} = \frac{\int_{\rho a}^{\rho b} H(a_0-\theta_c)\,\psi(\alpha_j\rho)\,\rho\, d\rho}{(\alpha_j + H)\int_{\rho a}^{\rho b} \psi^2(\alpha_j\rho)\,\rho\, d\rho} \quad \text{... (16)}$$

Thus one can determine θ_0 with the help of eqns. (12), (13) and (16)

Case II : $m \neq 0$ — Assuming

$$\theta_m = R_m(\rho)\, Z_m(z) \quad \text{.. (17)}$$

we get from eqn. (8)

$$\frac{1}{R_m}\left\{\frac{d^2 R_m}{d\rho^2} + \frac{1}{\rho}\frac{d R_m}{d\rho}\right\} = -\alpha'^2 = -\frac{1}{Z_m}\frac{d^2 Z_m}{dZ^2} + 2im \quad \text{... (18)}$$

Proceeding as in Case I, we obtain

$$\theta_m = \sum_j A_{mj}\, \psi(\alpha_j \rho) \exp[-\{\gamma_{mj} + i\delta_{mj}\}z] \quad \ldots (19)$$

where*

$$(\gamma_m + i\delta_m)^2 \equiv \alpha'^2 + 2im \quad \ldots (20)$$

and the ψ function has been defined earlier (see eqn. 13). The transcendental equation determining α' is the same as given by eqn. (14). Applying the boundary condition at $z = 0$ (eqn. 9*a*) and proceeding as in case I, we get :

$$A_{mj} = \frac{\int_{\rho_a}^{\rho_b} \rho H a_m\, \psi(\alpha_j \rho)\, d\rho}{(r_{mj} + r\delta_{mj} + H) \int_{\rho_a}^{\rho_b} \rho \psi^2 (\alpha_j \rho)\, d\rho} \quad \ldots (21)$$

Thus θ_m can be determined from eqns. (19) and (21). With the help of eqns. (6), (12) and (19), one can find $\theta(\rho, Z, t)$.

Numerical Results and Discussion

In making calculations, the following parameters have been chosen :

$h = 4.0$ BTU/hr ft^2 °F $\qquad \rho_0 = 128$ lb/ft^3
(gray surface, wind speed 7.5 mph)
$k = 0.3$ BTU/hr ft °F $\qquad c = 0.44$ BTU/lb °F
$h_c = 1.46$ BTU/hr ft^2 °F

which corresponds to a characteristic length $L = \left(\frac{\rho_0 \omega c}{2K}\right)^{-1/2} = 0.2$ ft and $(k/h_c) = 0.2$ ft; values of K, ρ_0, C correspond to that of Kuwait. We have chosen all lengths in terms of L. The values of ρ_a and ρ_b are 25 and 40 respectively ($a = 5$ ft and $b = 8$ ft). It can be noted that the difference $(\rho_b - \rho_a) = 15$, satisfies the requirement that $(b - a)$ should be many times the characteristic length L and K/h_c.

Using the hourly data of solar radiation on a horizontal surface and atmospheric temperature for a typical day (March 21, 1975), available from Kuwait International Airport records, the solair temperature was evaluated and expressed as a Fourier series (with a period of 24 hrs) i.e., in the form of eqn. (5). The resulting parameters $[a_m = A_m \exp(-i\, \sigma_m)]$ are given in Table I (Khatri *et al.* 1977.

It is evident that the first two harmonics are enough for most of the calculations; however, we have carried out calculations taking into account the first six harmonics. Very good convergence of the results occurs after the third harmonic.

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*Thus

$$\gamma_{mj}^2 - \delta_{mj}^2 = \alpha_j'^2$$

and

$$\gamma_{mj}\, \delta_{mj} = m.$$

The above equations determine γ_{mj} and δ_{mj}

TABLE I

Fourier analysis of daily variation of solair temperature on ground in Kuwait (March 21, 1975)

m	0	1	2	3	4	5	6
A_m °F	83.41214	42.93614	18.37944	2.47132	2.62090	1.42922	0.64891
σ_m radians		3.40778	0.12662	2.78537	4.13958	1.44181	3.29172

We solve eqn. (14) numerically to find the values of α. There is an infinite discrete set of values of α which satisfies this equation. Therefore, summation in eqns. (12) and (19) generally extends from $j = 0$ to ∞. To test the convergence of this summation and the accuracy of these calculations, we have calculated LHS of eqn. (15) for various values of ρ taking the summation of only first $n(=30, 40, 50)$ terms. We find that at all the values of ρ (between ρ_a and ρ_b) except at $\rho = \rho_a$, the sum converges to unity within an accuracy of 0.5 per cent and the choice of n from 40 to 50 also does not make any significant difference (less than 0.5 per cent). However, at $\rho = \rho_a$ the sum does not come out to be unity. It seems to be sensitive with ρ_b. With $\rho_b = 40$, it comes out 0.50, 0.93 and 0.94 with $n = 30$, 40 and 50 respectively. At $\rho_b = 50$, it comes out to be 0.84, 0.88 and 0.90 with $n = 30$, 40 and 50 respectively. Therefore, it seems that the convergence is poor for large value of ρ_b and also after $n = 50$ the convergence is very slow. We have chosen therefore $\rho_b = 40$ and $n = 50$ in our calculations.

In Fig. 2, we have plotted θ *vs.* ωt for various values of Z at $\rho = 25$ (inner surface of the cavity). It is seen that the peak of the curve shifts as Z increases. This is because of the phase lag introducd in various harmonics. As Z increases temperature variations with respect to time die out; this is because the amplitude of various harmonics decay exponentially with Z (see eqn. 19). Consequently, for distances $Z > 5$ only the steady part of temperature i.e., θ_0, is predominant.

In Fig. 3, the variation of θ with ρ is shown at $Z = 1$, for different values of ωt. It can be seen that for $\rho > 30$, θ varies very slowly with ρ. This is consistant with the fact that variation of θ with ρ for large value of ρ should be

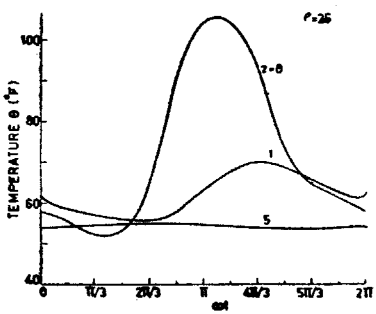

FIG. 2. Temperature of the wall of the cavity as a function of ωt for various Z.

negligible. In the present calculations, ρ_b has been chosen 40 so that $(\rho_b - \rho_a) = 15 \gg 1$; thus eqn. (9*d*) holds good.

Fig. 4 shows the variation of θ *vs.* Z at $\rho = 25$ for various values of ωt. For $\omega t = \pi$ and $3\pi/2$, θ decreases monotonically with Z, but for $\omega t = \pi/2$ and 2π it increases initially and then decreases. This is because of the phase lag introduced in harmonics. This can be verified from Fig. 2, where at $\omega t = \pi/2$ and 2π, the temperature at $Z = 0$ is less than the temperature at $Z = 1$. At all times, for $Z > 2$ the temperature decreases monotonically with Z. This is because the harmonics now become negligible and major contribution to temperature is from θ_0, the time independent part of the temperature. At a large value of Z, θ approaches the temperature of the cavity i.e., $\theta_c(=50\ °F$ in the present calculations). This is expected because away from the ground surface, the surface of the cavity should have nearly the same temperature as the temperature of the air in the cavity.

The fact that θ at $\rho = 25$ for $Z > 5$ is nearly equal to θ_c is very important because it means that the flux of heat coming from the walls of the cavity will be very small at these depths as it is proportional to difference of temperature of the wall and the air inside the cavity.

If one has to find out the total flux of heat integrated over one complete period of time ($\omega t = 0$ to 2π), the contribution of θ_1, θ_2 etc. will be zero and therefore $(\theta_0 - \theta_c)$ only will determine the flux of heat from the walls of the cavity.

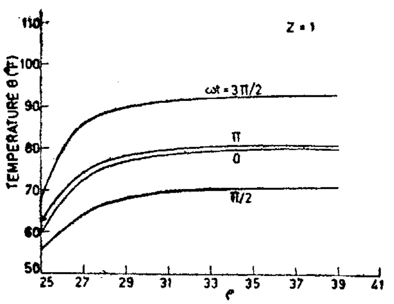

FIG. 3. Temperature as a function of ρ at $Z = 1$ for various values of ωt.

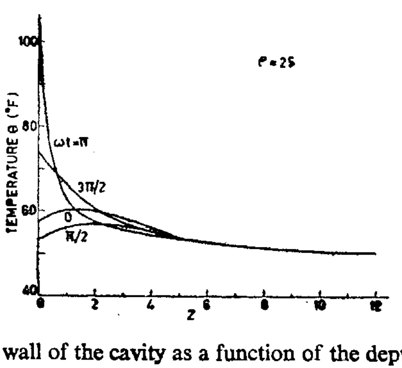

FIG. 4. Temperature of the wall of the cavity as a function of the depth Z for various values of ωt.

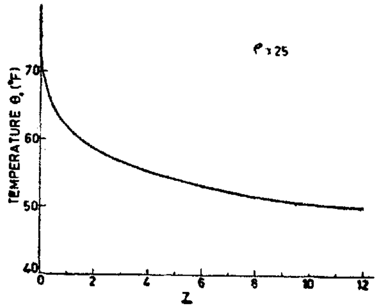

FIG. 5. Time independent part of temperature θ_0, at the wall of the cavity as a function of depth Z.

In Fig. 5, the present authors have plotted θ_0 *vs.* Z ($\rho = 25$). This decreases monotonically and approaches θ_c. (θ_0—θ_c) apart from a multiplying constant is the measure of the total heat flux (integrated over time) per unit length of the cavity. The fact that θ_0 is very close to θ_c for $Z > 5$ means that for maintaining the temperature of a deep cavity at θ_c, the thermal load is very small; this shows the desirability of underground cold storages.

If we consider annual variation of solar insolation and atmospheric temperature, we should get similar results; the main difference will be the scale length. Thus to even out annual variations further underground (scale length 4 ft) needs to be made.

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